Smooth functions on $\mathbf{c_0}$

Petr Hájek

The space c_0 lies at the heart of many constructions of higher order smooth functions on Banach spaces. To name a few, recall Torunczyk's proof of the existence of C^k-smooth partitions of unity on WCG spaces ([12]) or Haydon's recent constructions of C^∞-bump functions on certain $C(K)$ spaces ([7]).
The crucial property of c_0 that allows for those constructions is a rich supply of C^∞-smooth functions that depend locally on finitely many coordinates.
The main result of the present note (Theorem 6) implies that every C^2-smooth function on c_0 has a locally compact derivative.
This, in turn, means that every C^2-smooth function on c_0 "almost" depends locally on finitely many coordinates, and confirms our intuition of c_0 as being a very "flat" space.
Our work was originally motivated by the question of Jaramillo (that we answer in the negative-see also [4]) whether there exists a C^∞-smooth function on $c_0(\Gamma)$ which attains its minimum at exactly one point.
However, our Corollaries (8-11) generalize to the case of C^2-smooth functions on c_0 some results that Pelczynski [11] and Aron [1] obtained for polynomials and analytic functions on $C(K)$ spaces, and some work of the author [6] on convex functions on c_0. In particular, we show that every C^2-smooth (nonlinear) operator from c_0 into a superreflexive space is locally compact. This implies that there exists no C^2-smooth operator from c_0 onto ℓ_2, answering a question of S. Bates.
The main idea of our work is contained in Lemma 2. Roughly speaking, it claims that a symmetric function with uniformly continuous derivative defined on c_0^n, with zero derivative at the origin, is almost constant along the basic vectors if n is large enough.
Repeated applications of this principle lead to the proof of Theorem 6.

Our notation and terminology is mostly standard, as in [4]. By c_0^n we denote the space of finite sequences of length n with the supremum norm $\|\cdot\|_\infty$. Its dual space ℓ_1^n is equipped with the canonical norm $\|\cdot\|_1$. $B_{c_0^n}$ stands for the closed unit ball of c_0^n.
The canonical basic vectors in c_0^n (resp. ℓ_1^n) are denoted by e_i (resp. f_i).
To simplify the notation, we put $u \cdot v = \sum\limits_{i=1}^n u_i v_i$ for $u, v \in c_0^n$.
By Π_n we denote the group of all permutations of $\{1, \dots, n\}$.
We say that a function f defined on a subset of c_0^n is symmetric provided

$$f\Big(\sum_{i=1}^n a_i e_{\pi(i)}\Big) = f\Big(\sum_{i=1}^n a_i e_i\Big)$$

for every $\pi \in \Pi_n$ and every $a_i \in I\!R$ such that $\sum\limits_{i=1}^n a_i e_{\pi(i)}, \sum\limits_{i=1}^n a_i e_i \in \mathrm{Dom}(f)$.
The modification of this definition to the case of c_0 should be clear.

We will also use a *modulus of continuity* for a given uniformly continuous function f from a metric space $\big(X_1, \mathrm{d}_1\big)$ into a metric space $\big(X_2, \mathrm{d}_2\big)$. It is an increasing real function $\omega(\delta)$, $\delta \geq 0$, $\lim\limits_{\delta \to 0} \omega(\delta) = 0$, such that

$$\mathrm{d}_1(x_1, x_2) \leq \delta \quad \text{implies} \quad \mathrm{d}_2(f(x_1), f(x_2)) \leq \omega(\delta).$$

In order to simplify the terminology, we use the term *function with uniformly continuous derivative on* B_X assuming implicitly that the derivative of the function is in fact uniformly continuous in some open neighbourhood of B_X. The uniform continuity of the derivative is used in the following way: If $\|x - y\|_\infty < \delta$, then

$$f(x) - f(y) = \langle f'(y), x - y\rangle + \xi, \quad \text{where } |\xi| < \delta \cdot \omega(\delta). \tag{1}$$

Indeed,

$$|f(x) - f(y) - \langle f'(y), x - y\rangle| \leq \int_0^1 |\langle f'(y + t(x - y)) - f'(y), x - y\rangle| \mathrm{d}t \leq \omega(\delta) \cdot \delta.$$

We start with a simple auxiliary lemma. Its elegant proof (originally due to Lindenstrauss) has been suggested to us by the referee.

Lemma 1. *Let* $\xi > 0$ *and* $m \in I\!N$. *Given two vectors* $u, v \in c_0^m$ *such that* $\|u\|_\infty, \|v\|_\infty \leq \xi$, *there exists* $A \subset \{1, \ldots, m\}$ *such that*

$$\left| \sum_{i \in A} u_i - \sum_{\substack{1 \leq i \leq m \\ i \notin A}} u_i \right| \leq 2\xi$$

and

$$\left| \sum_{i \in A} v_i - \sum_{\substack{1 \leq i \leq m \\ i \notin A}} v_i \right| \leq 2\xi.$$

Proof: Let $T : c_0^m \to I\!R^2$ be the operator given by $T(x) = (x \cdot u, x \cdot v)$, and let $y = \sum\limits_{i=1}^m y_i e_i$ be an extreme point in the unit ball of $\mathrm{Ker}(T)$. There are at most two coordinates y_i for which $|y_i| < 1$. Indeed, assume $|y_1|, |y_2|, |y_3| < 1$. Since T has rank two, there is a $z \neq 0$ in $\mathrm{span}\{e_1, e_2, e_3\}$ so that $T(z) = 0$. If $\|z\|_\infty$ is small enough, $y \pm z$ is in the unit ball of $\mathrm{Ker}(T)$, contradicting the fact that y is extreme. Replacing the coordinates where $|y_i| < 1$ by the sign of y_i (in case $y_i = 0$ we use 1) changes $y \cdot u$ and $y \cdot v$ at most 2ξ. By putting $A = \{i,\ \mathrm{sign}(y_i) = 1\}$ the Lemma is proved.

$\diamondsuit$

The following Lemma contains the main idea of the proof of Theorem 6. Given a symmetric real function f with uniformly continuous derivative (with modulus of continuity $\omega(\delta)$), $f(0) = 0$, $f'(0) = 0$, defined on $B_{c_0^n}$, it provides us with an estimate on the growth of f along the basic vectors e_i, which depends only on ω and n (not on f).
It turns out that (while keeping ω fixed) letting $n \to \infty$ implies $|f(e_1)| \to 0$.
Later we will find a suitable generalization for the case of nonsymmetric f.
A statement like Lemma 2 is obviously useful for investigation of finite dimensional restrictions of a given f with uniformly continuous derivative on B_{c_0}.

Lemma 2. *Let $\varepsilon > 0$, f be a real symmetric function on $B_{c_0^n}$ with uniformly continuous derivative. Suppose $f(0) = 0$, $f'(0) = 0$ and let $w(\delta)$ be the modulus of continuity of f'. If $n \geq n(\omega, \varepsilon)$, where $n(\omega, \varepsilon) \in I\!R$ depends only on the function $\omega(\cdot)$ and ε, then $|f(e_1)| < \varepsilon$.*

PROOF:
Let $\xi = \frac{\varepsilon}{10}$ and fix k so that $\omega(1/k) < \xi$. We put $n(\omega, \varepsilon) > \frac{3}{2}\big(2 \cdot \frac{\omega(2)}{\xi} + 3\big) \cdot 3^{k-1}$. Starting with $y_0 = e_1$, and $x_0 = 0$, define inductively two sequences, x_j and y_j, for $1 \leq j \leq k-1$, satisfying:

(a) $x_j = y_j$ except in their first coordinate which is 0 for the x_j's and 1 for the y_j's;
(b) for each $1 \leq i \leq j$ there is a coordinate where both x_j and y_j have the value i/k;
(c) $|f(x_j) - f(x_{j-1})|, |f(y_j) - f(y_{j-1})| \leq 3\xi/k$.

Assume, for the moment, that such a construction is possible. Put now $x = x_{k-1}$ and $y = y_{k-1}$, and estimate

$$|f(e_1) - f(0)| \leq \sum_{j=1}^{k-1} |f(y_j) - f(y_{j-1})| + |f(y) - f(x)| + \sum_{j=1}^{k-1} |f(x_j) - f(x_{j-1})|.$$

The terms in the two sums are less than $3\xi/k$ each by (c).
To estimate the middle term, we use the symmetry of f: Let A be the set of cardinality $k-1$ where x and y attain the values $1/k, 2/k, \ldots, (k-1)/k$ and let $B = A \cup \{1\}$. By (a) the coordinates of $x + (1/k)\chi_B$ are just a rearrangement of those of y. The symmetry of f implies that $f(y) = f(x + (1/k)\chi_B)$, and thus by (1)

$$|f(x) - f(y)| = |f(x) - f(x + (1/k)\chi_B)| \leq \omega(1/k) < \xi.$$

Summing all the inequalities the result follows.

The vectors x_j and y_j are constructed by induction. This is done in such a way that they also satisfy the additional condition

(d) there is a subset A_j of coordinates, whose cardinality is at least $n/3^j$, so that x_j and y_j have the constant value j/k on A_j.

Assume that x_{j-1} and y_{j-1} have been chosen. Let

$$B = \{p \in A_{j-1} : \max\big(|f'(x_{j-1})_p|, |f'(y_{j-1})_p|\big) \leq \xi\}.$$

Since $\|f'(x_{j-1})\|_1,\ \|f'(y_{j-1})\|_1 \leq \omega(2)$, it follows (since n is large enough) that $|B| \geq \frac{(2/3)n}{3^{j-1}}$. Moreover, this estimate remains true even if we add the restrictions that $1 \notin B$, and that there are $j-2$ coordinates outside B where the values i/k for $i \leq j-2$ (as in condition (b)) are attained by x_{j-1} and y_{j-1}.
By Lemma 1 there is a subset C of B so that if we put $z = \chi_C - \chi_{B \setminus C}$, then $\max\left(|\langle f'(x_{j-1}), z\rangle|, |\langle f'(y_{j-1}), z\rangle|\right) \leq 2\xi$. Assuming (as we may) that $|C| \geq |B \setminus C|$, we take $A_j = C$, $x_j = x_{j-1} + \frac{z}{k}$ and $y_j = y_{j-1} + \frac{z}{k}$.
With this choice (a), (b) and (d) are clearly satisfied. To check (c), we set

$$f(x_j) - f(x_{j-1}) = \langle f'(x_{j-1}), x_j - x_{j-1}\rangle + E$$

where, by the choice of C, $|\langle f'(x_{j-1}), x_j - x_{j-1}\rangle| \leq \frac{2\xi}{k}$, and where $|E| \leq \omega(\|x_j - x_{j-1}\|)\|x_j - x_{j-1}\| \leq \omega(\frac{1}{k}) \cdot (\frac{1}{k}) < \frac{\xi}{k}$. Similar estimate works for $f(y_j) - f(y_{j-1})$.
The proof is finished.

$\diamondsuit$

A brief examination of the proof of Lemma 2 shows that throughout we worked only with points in $B_{c_0^n}$ which have all coordinates larger than or equal to $-1/k$. Given a real symmetric function f on $B^+_{c_0^n} = B_{c_0^n} \cap \{x,\ x_i \geq 0,\ 1 \leq i \leq n\}$, with uniformly continuous derivative (with modulus $\omega(\delta)$), $f(0) = 0$, $f'(0) = 0$, we may pass to a symmetric function $\tilde{f}(x) = f(\frac{1}{k}\sum_{i=1}^{n} e_i + x) - \langle f'(\frac{1}{k}\sum_{i=1}^{n} e_i), x\rangle - f(\frac{1}{k}\sum_{i=1}^{n} e_i)$ defined on $\frac{k-1}{k}B^+_{c_0^n}$. Clearly, $\tilde{f}(0) = 0$, $\tilde{f}'(0) = 0$, so the method of the proof of Lemma 2 applies and we obtain that if $n \geq n(\omega, \varepsilon)$ then $|\tilde{f}(\frac{k-1}{k}e_1)| < \varepsilon$. Consequently, using (1) we get

$$\begin{aligned} f(e_1) =& f(e_1 + \frac{1}{k}\sum_{i=2}^{n} e_i) + \xi, \quad |\xi| < \frac{1}{k}\omega(\frac{1}{k}), \\ |f(e_1)| \leq& |\tilde{f}(\frac{k-1}{k}e_1)| + \omega(\frac{1}{k}) + \frac{2}{k}\omega(\frac{1}{k}) \leq \varepsilon + 3\omega(\frac{1}{k}). \end{aligned}$$

Since we can choose k to be arbitrary small, we have the following slight improvement of Lemma 2:

Lemma 3. *Let $\varepsilon > 0$, f be a real symmetric function defined on $B^+_{c_0^n}$ with uniformly continuous derivative with modulus ω. Suppose $f(0) = 0$, $f'(0) = 0$. If $n \geq \tilde{n}(\omega, \varepsilon)$, then $|f(e_1)| < \varepsilon$.*

Although Lemma 3 follows immediately from Lemma 2, we do state it explicitly because its application leads to a somewhat more elegant formulation of the result below.

Proposition 4. *Let f be a real symmetric Fréchet differentiable function on B_{c_0} with uniformly continuous derivative. Then f is constant on B_{c_0}.*

PROOF: By contradiction. Suppose f is not constant on B_{c_0}. We may assume, without loss of generality, that $f(0) = 0$, $f'(0) = 0$ and $f(x) = \varepsilon > 0$ for $x = \sum_{i=1}^{n} x_i e_i$.
Put $x^k = \sum_{i=1}^{n} x_i e_{kn+i}$. The sequence $\{x^k\}_{k=1}^{\infty}$ forms a block basis in c_0 (equivalent to the original basis), and from the symmetry of f we obtain $f(x^k) = f(x) = \varepsilon$ for every $k \in \mathbb{N}$. This is a contradiction with Lemma 2.

$\diamondsuit$

Lemma 5. *Let $\varepsilon > 0$, f be a real function on $B_{c_0^m}$ with uniformly continuous derivative (with modulus of continuity $\omega(\delta)$) and such that $\sup_{B_{c_0^m}} \|f'\|_1 \leq \omega(2)$. Let $v \in B_{c_0^m}$ and $\{u_i\}_{i=1}^n$ be a block sequence such that $v + u_i \in B_{c_0^m}$. If n is large enough, then $\min_{1\leq i\leq n} |f(v + u_i) - f(v)| < \varepsilon$.*

PROOF: We proceed by contradiction. We define a bounded affine operator $\phi : c_0^n \to c_0^m$ by $\phi(\sum_{i=1}^{n} a_i e_i) = v + \sum_{i=1}^{n} a_i u_i$. Since ϕ is 2-Lipschitz, the real function $\tilde{f} = f \circ \phi$ defined on $B^+_{c_0^n}$ has a uniformly continuous derivative with modulus $2\omega(\delta)$. We may assume without loss of generality that $\tilde{f}(0) = 0$, and also $\|\tilde{f}'(0)\|_1 < \frac{\varepsilon}{2}$, $\tilde{f}(e_i) \geq \varepsilon$ for $1 \leq i \leq n$. Indeed, given $\{u_i\}_{i=1}^n$, where n is large enough, there exists a set $A \subset \{1, \ldots, n\}$, $\mathrm{card}(A) > \frac{\varepsilon}{10} \cdot \frac{n}{\omega(2)}$ such that $\sum_{i\in A} |\tilde{f}'(0)_i| < \frac{\varepsilon}{2}$ and either $\tilde{f}(e_i) \geq \varepsilon$, $i \in A$ or $\tilde{f}(e_i) \leq -\varepsilon$, $i \in A$. Thus, it is sufficient to replace $\{1, \ldots, n\}$ by A. Replacing $\tilde{f}(x)$ by $\mathrm{sign}\, \tilde{f}(e_1) \cdot (\tilde{f}(x) - \langle \tilde{f}'(0), x\rangle)$ and keeping the notations, we may assume using (1) that

$$\tilde{f}'(0) = 0, \quad \tilde{f}(e_i) \geq \frac{\varepsilon}{2}, \ \ 1 \leq i \leq n.$$

We introduce a real symmetric function f_s on $B^+_{c_0^n}$ by

$$f_s\Big(\sum_{i=1}^{n} a_i e_i\Big) = \frac{1}{n!} \sum_{\pi\in\Pi_n} \tilde{f}\Big(\sum_{i=1}^{n} a_{\pi(i)} e_i\Big).$$

It is standard to verify that f_s has a uniformly continuous derivative with modulus $2\omega(\delta)$, $f_s(0) = 0$, $f_s'(0) = 0$, $f_s(e_i) \geq \frac{\varepsilon}{2}$ for $1 \leq i \leq n$ and $\sup_{B_{c_0^n}} \|f_s'\|_1 \leq 2\omega(2)$. (In fact, f_s is a convex combination of functions which satisfy the above set of conditions).
By Lemma 3, we are done.

$\diamondsuit$

Theorem 6. *Let f be a real Fréchet-differentiable function on B_{c_0} with uniformly continuous derivative (with modulus of continuity $\omega(\delta)$). Then $f'(B_{c_0})$ is relatively compact in ℓ_1.*

Proof: By contradiction. We may assume that $f(0) = 0$ and $f'(0) = 0$. Therefore $\sup_{B_{c_0}} \|f'\|_1 \le \omega(1)$ and we assume $u_n \in B_{c_0}$ satisfy $\|f'(u_n) - f'(u_m)\|_1 > 4\rho$ for all $n \ne m$. Let $\varepsilon > 0$. By passing to a subsequence, and a standard "gliding hump" argument, we can assume that there are disjoint finite sets (A_n) so that $\|f'(u_n)|_{A_n}\| > 2\rho$. Let $v_n \in B_{c_0}$ be supported in A_n such that $|\langle f'(u_n), v_n\rangle| \ge \rho$, and in addition $v_n + u_n \in B_{c_0}$. Then, for every $t > 0$, $f(u_n + tv_n) - f(u_n) = \langle f'(u_n), tv_n\rangle + E$ where $|E| \le t\omega(t)$. As the first term is in absolute value bounded below by $t\rho$, and $\omega(t) \to 0$, it follows that we can choose a $t \le 1$ so that for some fixed $\theta > 0$, $|f(u_n + tv_n) - f(u_n)| > \theta$ for all n.

We now set the notation for the next step.
By passing to a subsequence, we can assume that the limits of $f(u_n)$ and $f(u_n + tv_n)$ exist. Adding a constant to f, passing to subsequences, changing notation, and disregarding quantities that can be made arbitrary small, we can assume that there are sequences $\{u_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ in B_{c_0} so that

(i) $f(u_n) = 0$ for all n, $f(w_n) = 2\beta > 0$ for all n;
(ii) the sequences are supported in an increasing sequence of finite intervals $I_n = [1, m_n]$;
(iii) u_n and w_n are equal on I_{n-1};
(iv) all the u_j for $j > n$ are equal on I_n.

Claim 7. *There is an integer k, so that for some infinite subset M of $\mathbb{N}$, and every vector v with $w_n + v \in B_{c_0}$ and having a finite support, starting after k, the set $\{n \in M : |f(w_n + v) - 2\beta| > \beta\}$ is finite.*

Proof: If this is not the case, define inductively decreasing sequence of infinite subsets $\{M_j\}_{j=1}^{\infty}$ of M, and disjoint finitely supported vectors $\{v_j\}_{j=1}^{\infty}$ so that for each j the set $M_{j+1} = \{n \in M_j \,|f(w_n + v_j) - 2\beta| > \beta\}$ is infinite. Given N, we can thus find disjoint blocks $v_1, \ldots, v_N$ and an n, so that $|f(w_n + v_j) - 2\beta| > \beta$ for $j \le N$. Since $f(w_n) = 2\beta$ and N is arbitrary, this contradicts Lemma 5.

By passing to a subsequence of w_n indexed by M and discarding a few first w_n's, assume that $k \in I_1$. It follows from the claim that for every v, supported by $[m_1+1, \infty)$, $f(v + w_n) > \beta$ for all but a finite number of n's (provided also $v + w_n \in B$). Note also that all the w_n's, as well as u_2 have the same values on I_1.
To finish the proof, we now pass to a subsequence of (w_n) and perturb them to be "disjointly supported except for a common u_2 part" using the Claim 7 as follows:
All the w_n's for $n \ge 3$ agree on I_2, let v_2 be this common value. Since u_2 and v_2 agree on I_1, $w_n - v_2 + u_2$ is different from w_n only on $I_2 \setminus I_1$, and it has the form $u_2 + x_n \in B_{c_0}$, where x_n is supported in I_n by $[m_2 + 1, m_n]$. By Claim 7 there is an $n_1 \ge 3$, so that $f(u_2 + x_{n_1}) \ge \beta$.
Inductively, having chosen n_j and x_{n_j}, supported in I_{n_j}, all the w_n's for $n > n_j$ agree on I_{n_j}, and take v_j to be this common value. Since u_2 and v_j agree on I_1, $w_n - v_j + u_2 \in B_{c_0}$ is different from w_n only on $I_n \setminus I_1$, and it has the form $u_2 + x_n \in B_{c_0}$, where x_n is supported in I_n by $[m_{n_j} + 1, m_n]$. By Claim 7 there is an $n_{j+1} > n_j$, so that $f(u_2 + x_{n_{j+1}}) \ge \beta$.
Since the x_{n_j}'s are disjoint blocks, and $f(u_2) = 0$, this contradicts Lemma 5.

$\diamondsuit$

Corollary 8. *Let f be a real Fréchet differentiable function on $c_0(\Gamma)$ with locally uniform continuous derivative. Then f depends locally on countably many coordinates.*

Proof: To prove Corollary 8, it is enough (by the usual shifting and scaling arguments) to show that if f' is uniformly continuous in $B_{c_0(\Gamma)}$, then f depends on countably many coordinates in $B_{c_0(\Gamma)}$.
By Theorem 6, $f'(B_{c_0(\Gamma)})$ is a relatively norm compact set in $\ell_1(\Gamma)$. Thus, there exists a countable set $A \subset \Gamma$ such that $\operatorname{supp} f'(x) \subset A$ for every $x \in B_{c_0(\Gamma)}$.
It is now easy to observe that whenever $x, y \in B_{c_0(\Gamma)}$, $x = \sum\limits_{\gamma\in\Gamma} x_\gamma e_\gamma$, $y = \sum\limits_{\gamma\in\Gamma} y_\gamma e_\gamma$ are such that $x_\gamma = y_\gamma$ for every $\gamma \in A$, then $f(x) = f(y)$. Indeed,

$$\begin{aligned} f(y) &= f(x) + \int_0^1 \langle f'(x + t(y-x)), y - x\rangle \, dt \\ &= f(x) + \int_0^1 \sum_{\gamma\in\Gamma} f'(x+t(y-x))_\gamma \cdot (y-x)_\gamma \, dt. \end{aligned}$$

The last integral is clearly equal to zero because for every $\gamma \in \Gamma$ either $f'(x+t(y-x))_\gamma = 0$ or $(y-x)_\gamma = 0$. This finishes the proof.

$\diamondsuit$

As an immediate consequence, we obtain a negative answer to a question posed by J. Jaramillo (see e.g. [4, p.90]): *Does there exist a C^∞-Fréchet smooth function on $c_0(\Gamma)$ that attains its minimum at exactly one point?*

Corollary 9. *There exists no C^2-Fréchet smooth function on $c_0(\Gamma)$ that attains its minimum at exactly one point.*

For the purpose of this note we will say that a real function f defined on B_{c_0} is *weakly sequentially continuous* if $\lim\limits_{n\to\infty} f(x_n)$ exists for every weakly Cauchy sequence $\{x\}_{n\in\mathbb{N}}$ in B_{c_0}.

Corollary 10. *Let f be a Fréchet differentiable real function with uniformly continuous derivative defined on B_{c_0}. Then f is weakly sequentially continuous on B_{c_0}.*

Proof: Since $K = \overline{f'(B_{c_0})}$ is norm compact by Theorem 6, given a weakly Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ we have:

$$\lim_{n,m\to\infty} \langle \phi, x_n - x_m\rangle = 0 \text{ uniformly in } \phi \in K.$$

By the mean value theorem, for some point x in the interval joining x_n and x_m, we have:

$$|f(x_n) - f(x_m)| = |\langle f'(x), x_n - x_m\rangle| \le \sup_{\phi\in K} |\langle \phi, x_n - x_m\rangle| \to 0 \text{ as } n, m \to \infty.$$

$\diamondsuit$

Using the Bessaga-Pelczynski theorem, it is an easy exercise to show that every linear operator $T : c_0 \to X$, X not containing a copy of c_0, is necessarily compact. We suspect that this statement holds (locally) for general operators with uniformly continuous derivative. However, we are able to prove this only for certain Banach spaces.
Let us recall that a (in general nonlinear) continuous operator $T : X \to Y$ where X, Y are Banach spaces is called *locally compact* if for every $x \in X$ there exists a neighbourhood O, $x \in O$ such that $T(O)$ is relatively compact in Y.

Corollary 11. *Let T be a (nonlinear, in general) Fréchet differentiable operator from c_0 into a Banach space X with nontrivial type. Suppose that T' is locally uniformly continuous. Then T is locally compact.*

PROOF: We may assume, by contradiction, that T' is uniformly continuous in B_{c_0}, $\{x_n\}_{n=1}^{\infty} \subset B_{c_0}$ is weakly Cauchy and, by Rosenthal's theorem (as ℓ_1 does not imbed into X), $\{T(x_n)\}_{n \in I\!N}$ is weakly Cauchy, $\|Tx_n\| > \gamma > 0$.
Define $\tilde{T} : c_0 = c_0 \oplus c_0 \to X$ by $\tilde{T}\big((x, y)\big) = T(x) - T(y)$. A weakly Cauchy sequence $\tilde{x}_n = (x_{2n}, x_{2n+1})$ maps into a weakly null sequence $\tilde{y}_n = T(x_{2n}) - T(x_{2n+1})$. By the proof of Theorem 3.3 and Corollary 3.6 in [5], passing to a subsequence of $\{\tilde{y}_n\}$, there exists $p > 1$, $p \in I\!N$ and a linear operator $L : X \to \ell_p$, $L\tilde{y}_n = e_n$. Put $\phi(v) = \sum_{i=1}^{\infty} (-1)^i v_i^p$ for $v = \sum_{i=1}^{\infty} v_i e_i \in \ell_p$. The real function $\phi \circ \tilde{T}$ is not weakly sequentially continuous, a contradiction with Corollary 10.

$\diamondsuit$

Let us remark that in particular every superreflexive Banach space satisfies the assumptions of Corollary 11. An easy modification of the proof yields the same conclusion for ℓ_1.

S. Bates ([2]) has recently shown that given ℓ_p, $1 < p < \infty$, and a separable Banach space X, there exists a C^{∞}-Fréchet smooth and onto operator $T : \ell_p \to X$. He also showed that given any separable Banach spaces X, Y there exists a C^1-Fréchet smooth, Lipschitz and onto operator $T : X \to Y$. He asked whether the former statement holds true for c_0 instead of ℓ_p. Using the Baire category principle, it is immediate from Corollary 11 that there exists no C^2-Fréchet smooth and onto operator $T : c_0 \to \ell_p$, $1 \le p < \infty$.

A natural question arises as to which Banach spaces satisfy an analogue of Theorem 6. Obviously, this property is preserved by taking a quotient. Although our arguments seem to be mostly finite dimensional, they do involve in a crucial way also the infinite-dimensional structure of the space. Without going into much details, let us point out that $C[0, 1]$ is a π_1^{∞} space (a space built up from isometric copies of c_0^n- for details see [10]) and yet it does not satisfy Theorem 6 (since ℓ_2 is a quotient of $C[0, 1]$).

On the other hand, spaces $C(K)$ where K is scattered seem to be natural candidates for generalizations of Theorem 6. In the case when $K^{(\omega_0)} = \emptyset$ this follows readily from our above results.

Corollary 12. *Let K be a scattered compact, $K^{(\omega_0)} = \emptyset$. Let f be a Fréchet differentiable function on $C(K)$ with locally uniformly continuous derivative. Then f' is locally compact.*

Proof. By contradiction, we may assume that $f'(\lambda B_X)$ is not relatively compact, where X is a separable subspace of $C(K)$ and $\lambda > 0$ is arbitrary. By classical results (using Stone-Weierstrass theorem, for details see e.g. [4] and [9]), there exists a separable subspace Y of $C(K)$ such that $X \subseteq Y$ and Y is isometric to some $C(L)$, where L is scattered, and it is a continuous image of K. Thus, $L^{(\omega_0)} = \emptyset$, L is countable and Y is isomorphic to c_0 by [3]. Consequently, $f'(\lambda B_Y)$ is not relatively compact for any $\lambda > 0$, which is a contradiction with Theorem 6.

$\diamondsuit$

In connection with Corollary 9, it should be noted that in [7] there are examples of non-separable Banach spaces $C(K)$ (where $K^{(\omega_0)} = \emptyset$) that admit C^∞-Fréchet smooth convex functions which attain their minimum at exactly one point.

Acknowledgment. The author would like to thank the referee for several improvements that he suggested. He shortened most of the arguments and supplied an elegant proof of Lemma 1.

References

[1] R. Aron, *Compact polynomials and compact differentiable mappings between Banach spaces* in "Seminaire Pierre Lelong, 1974-1975", Springer-Verlag Lecture Notes in Math. No. 524 (1976).

[2] S.M. Bates, *On smooth, nonlinear surjections of Banach space*, to appear.

[3] C. Bessaga and A. Pelczynski, *Spaces of continuous functions (IV)*, Studia Math. 19 (1960), 53-62.

[4] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Monograph Surveys Pure Appl. Maths. 64 (Pitman, 1993).

[5] J. Farmer and W.B. Johnson, *Polynomial Schur and polynomial Dunford-Pettis properties*, Contemp. Math. 144, (1993), 95–105.

[6] P. Hájek, *On C^2-smooth norms on c_0*, to appear in Mathematica.

[7] P. Hájek, *Analytic renormings of C(K) spaces*, Serdica Math. J. 22, (1996), 25–28.

[8] R.G. Haydon, *Normes et partitions de l'unite indefiniment differentiables sur certains espaces de Banach*, C. R. Acad. Sci. Paris, 315 (1992), 1175-1178.

[9] E. Lacey, *The isometric theory of classical Banach spaces*, Springer-Verlag, (1974).

[10] E. Michael and A. Pelczynski, *Separable Banach spaces which admit ℓ_n^∞ approximations*, Israel J. Math. 4, (1966), 189-198.

[11] A. Pelczynski, *A Theorem of Dunford-Pettis Type for Polynomial Operators*, Bull. Pol. Acad. Sci. XI (1963), 379-386.

[12] H. Torunczyk, *Smooth partitions of unity on some nonseparable Banach spaces*, Studia Math. 46 (1973), 43-51.

Department of Mathematics, University of Alberta, Edmonton, T6G 2G1, Canada.
E-mail address: phajek@vega.math.ualberta.ca