# SMOOTH NONCOMPACT OPERATORS FROM $C(K)$, $K$ SCATTERED

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Abstract. Let $X$ be a Banach space, $K$ be a scattered compact and $T : B_{C(K)} \to X$ be a Fréchet smooth operator whose derivative is uniformly continuous. We introduce the smooth biconjugate $T^{**} : B_{C(K)^{**}} \to X^{**}$ and prove that if $T$ is noncompact, then the derivative of $T^{**}$ at some point is a noncompact linear operator. Using this we conclude, among other things, that either $\overline{T(B_{c_0})}$ is compact or else $\ell_1$ is a complemented subspace of $X^*$. We also give some relevant examples of smooth functions and operators, in particular a $C^{1,u}$-smooth noncompact operator from $B_{c_0}$ which does not fix any (affine) basic sequence.

## Introduction.

The theory of linear operators from $C(K)$ spaces is a vast and important part of Banach space theory. One of the approaches to this subject is through the reduction (or fixing) properties of a given $T \in \mathcal{L}(C(K), X)$. Let us recall the following classical result of Pelczynski, and refer to Rosenthal's article in [JL, Chapter 36] and [DU, Chapter VI] for the history, many more results of this type and references.

**Theorem 0.1**
Let $X$ be a Banach space, $K$ be compact, and $T : C(K) \to X$ be a non-weakly compact linear operator. Then there exists $c_0 \cong Y \hookrightarrow C(K)$ such that $T \restriction_Y$ acts as an isomorphism. Moreover, if $K$ is scattered, the same result holds for $T$ a noncompact linear operator.

In his work on the Dunford-Pettis property, Pelczynski [P1, 2] relying on the use of vector measures, induction by the degree of the polynomial and the use of biconjugates $P^{**}$ to polynomials (which he is able to define for weakly compact polynomials or in case when $c_0$ is not contained in $X$) obtained the following nonlinear extension of Theorem 0.1.

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**Theorem 0.2**
Let $X$ be a Banach space, $K$ be a scattered compact, and $P : C(K) \to X$ be a noncompact polynomial operator. Then $c_0 \hookrightarrow X$.

In the same paper Pelczynski observed that in general the assumption of scatteredness cannot be removed, constructing a homogeneous polynomial $P : C[0,1] \to \ell_1$ for which $P(B_{C[0,1]})$ contains $B_{\ell_1}$. Let us remark that from [H3] and the fact that every Banach space containing $\ell_1$ (a condition characterizing precisely all $C(K)$, where $K$ is a non-scattered compact) has an $\ell_2$ quotient, it follows that for every $C(K)$, $K$ nonscattered and every separable Banach space $X$, there exists a homogeneous polynomial $P : C(K) \to X$ of degree 2, such that $P(B_{C(K)})$ contains $B_X$. This of course means that a structural theory for polynomials from $C(K)$, $K$ nonscattered compact, along the classical lines of Theorem 0.1 is not possible.

Our aim in the present paper is to investigate Pelczynski's-type result for general $C^{1,u}$-smooth operators. Note again that $C^1$-smoothness alone leads only to a trivial theory (due to nontrivial work of Bates [B], [BL, p. 261]), stating that arbitrary separable Banach space is $C^1$-smooth range of every separable Banach space. In our paper we treat the localized version (which is equivalent to the original one for polynomials) when $T : B_{C(K)} \to X$ is Fréchet differentiable, and $T'$ is uniformly continuous. The following question (suggested by our previous work in [H1], [H2], and explicitly asked also in Godefroy's article in the Handbook [JL, p. 799]) is the source of this note.

**Question 0.3**
Let $X$ be a Banach space, $K$ be a scattered compact, and $T : B_{C(K)} \to X$ be a $C^{1,u}$-smooth noncompact operator. Is then $c_0 \hookrightarrow X$?

Keeping in mind the reduction and fixing properties of linear operators, we can propose the following variants of the question.

**Question 0.4 (reduction)**
Let $X$ be a Banach space, $K$ be a scattered compact, and $T : B_{C(K)} \to X$ be a $C^{1,u}$-smooth noncompact operator. Does there exists $c_0 \cong Y \hookrightarrow C(K)$ such that $T\restriction_Y$ is noncompact?

or even
**Question 0.5 (fixing)**
Let $X$ be a Banach space, $K$ be a scattered compact, and $T : B_{C(K)} \to X$ be a $C^{1,u}$-smooth noncompact operator. Does there exists a sequence $\{u_n\}_{n=1}^\infty$ in $B_{C(K)}$, such that both $\{u_n\}_{n=1}^\infty$ and $\{T(u_n)\}_{n=1}^\infty$ are equivalent to the canonical basis of $c_0$?

It is obvious that the condition in Q 0.5 is the strongest and implies the other two, whose mutual relation is not quite clear. In the linear case, the questions are equivalent due to Theorem 0.1, and for polynomial operators Q 0.3 has a positive answer due to Theorem 0.2. In our paper, we develop some basic theory of smooth nonlinear operators, in order to deal with Q 0.3-5. The theory is formulated for $C(K)$, $K$ countable, (or just $c_0$) spaces, but due to the general reduction results (Theorem 1.5), the statements remain valid (with obvious modifications) for all $C(K)$, $K$ scattered. Let us pass to a brief discussion of our results. In section 1 we show that every $C^{1,u}$-smooth operator $T : B_{C(K)} \to X$, $K$ countable, has a canonical $C^{1,u}$-smooth extension $T^{**} : B_{C(K)^{**}} \to X^{**}$ (in the general $C(K)$, $K$ scattered, situation, the biconjugates $T^{**}$ can also be introduced, but their domain will be contained in $Y \hookrightarrow C(K)^{**}$, where $Y$ is the $w^*$-sequential closure of $C(K)$ in $(C(K)^{**}, w^*)$). We prove that Q 0.4 has an affirmative answer, provided we consider a reduction to an affine subspace $Y \cong c_0$ of $C(K)$ (i.e. a subspace not necessarily containing the origin). For a linear subspace $Y \cong c_0$ the answer to Q 0.4 is trivially negative even for polynomial operators. In section 2, we focus on operators from $B_{c_0}$. The main general result (using the reduction) is that if $T : B_{C(K)} \to X$, $K$ countable, is a $C^{1,u}$-smooth noncompact operator, then there exists a point $x^{**} \in B_{C(K)^{**}}$ at which $(T^{**})'(x^{**}) \restriction_{C(K)}$ is a noncompact linear operator. This implies (for all $K$ scattered) in particular that $\ell_\infty \hookrightarrow X^{**}$, a weak answer to Q 0.3 (it also implies that Q 0.5 is true for $T^{**}$). For special classes of $X$, such as duals, weakly sequentially complete spaces, Banach lattices or spaces with PCP (in particular RNP) property the statement in Q 0.3 is indeed true. In section 3 we investigate the summability properties of smooth functions on $c_0$, which are closely connected with Q 0.5. By a result of Aron and Globevnik ([AG], see also an earlier related result [Bo]), $\sum\limits_{i=1}^{\infty} |f(e_i)| < \infty$, for every polynomial $f$ on $c_0$. This type of result imply that the answer to (affine version of) Q 0.5 is affirmative for polynomial operators, improving Theorem 0.2.

As we will show, for $C^{1,1}$-smooth functions this property fails, and this allows us to construct in section 4 a $C^{1,1}$-smooth counterexample to the general statement in Question 0.5. Unfortunately, our results are not strong enough to solve the original Question 0.3. So in fact our paper contains indications going in both directions. It seems, however, that our conditions on $X$ basically exclude all the known examples of $X$ which come in mind while seeking a counterexample to Q 0.3. In particular the Bourgain-Delbaen $\mathcal{L}_\infty$ spaces [B] without $c_0$, Gowers' space [G] without $c_0$ or boundedly complete basic sequence, spaces of JT type (Ghoussoub-Maurey [GM]) all satisfy Q 0.3. Moreover, relying on Bourgain-Pisier results [BP] we know that if there exists $X$ violating Q 0.3, then there also exists such $\mathcal{L}_\infty$ space.

Let us now establish the terminology and notation. Let $X, Y$ be Banach spaces. Let $\omega(t) : I\!R^+ \to I\!R^+$, $\omega(0) = 0$ be a nondecreasing function. We say that a function $f : S \to X$, $S \subset Y$ has *modulus of continuity* $\omega(t)$, whenever $x, y \in S$, $\|x - y\| < \varepsilon$ implies that $\|f(x) - f(y)\| < \omega(\varepsilon)$ (the definition of course makes sense for mappings between general metric spaces). A continuous (nonlinear, in general) operator $T : S \to X$, where $S \subset Y$ is called a $C^{1,u}$-smooth operator if $T$ is Fréchet differentiable on $\mathrm{int}(S)$ and there exists a modulus $\omega(t)$ such that both $T$ and $T'$ have modulus of continuity $\omega(t)$. By $C^{1,1}$-smooth operator we mean an operator for which $T$ and $T'$ are Lipschitz. An operator $T : S \to X$ is called *weakly sequentially continuous (wsc)* if it maps weakly Cauchy sequences $\{x_n\}_{n=1}^\infty \subset S \subset Y$ into norm convergent sequences $\{T(x_n\}_{n=1}^\infty \subset X$. An operator $T : S \to X$ is called *compact* if $\overline{T(S)} \subset X$ is a norm compact set. By results of [H2], a $C^{1,u}$-smooth operator $T : B_{C(K)} \to X$, $K$ scattered, is wsc iff it is compact. For subsets $M, N \subset \mathbb{N}$ we use the notation $M < N$ if $\max(M) < \min(N)$. If one of these sets is a singleton, we may abbreviate this notation by replacing the set with its element. The symbol $X \cong Y$ indicates that the Banach spaces $X, Y$ are isomorphic. Given a scattered compact $K$, and a point $p \in K$, we will use the notation $C_0(K) = \{f : f \in C(K), f(p) = 0\}$. In the statements regarding $C_0(K)$ below, it is understood that $p$ is fixed but arbitrary. We also use the simple fact that $C_0(K) \cong C(K)$ for all infinite scattered compacts. Recall that a Banach space $X$ has the point of continuity property (PCP), if every weakly closed bounded subset of $X$ contains a point of weak-to-norm continuity for the identity mapping.

## 1. Smooth operators from $C(K)$ spaces.

Recall the basic fact that given two Banach spaces $E, F$, for any $T \in \mathcal{L}(E, F)$ there canonically exists a conjugate $T^* \in \mathcal{L}(F^*, E^*)$, and thus also a biconjugate operator $T^{**} \in \mathcal{L}(E^{**}, F^{**})$. Pelczynski [P2] observed in the proof of Theorem 0.2 that notwithstanding the lack of duality, biconjugate operators can be canonically defined also for weakly compact polynomial operators from $\mathcal{P}(C(K), X)$ spaces. In this section we are going to generalize this definition further for all $C^{1,u}$-smooth operators $T : B_{C(K)} \to X$, $K$ countable and arbitrary separable Banach space $X$. For a general $C(K)$ , $K$ scattered, a biconjugate can be defined along the same lines, except that its domain will be the $w^*$-sequential closure of $B_{C(K)}$ in $(B^{**}_{C(K)}, w^*)$. Since our later results rely on a separable reduction argument, we do not treat the general case here. Let us mention in passing that similar generalization is in fact possible for operators acting from spaces of class $\mathcal{C}$, introduced in [H2]. The next lemma is a variation on Lemma 5 from [H2]. We sketch a proof for readers convenience.

**Lemma 1.1**
Let $K$ be a scattered compact, $X = C(K)$ or $C_0(K)$, $f : B_X \to \mathbb{R}$ be $C^{1,u}$-smooth, $\{x_n\}_{n=1}^\infty$ be weakly Cauchy in $B_X$. Then $\{f'(x_n)\}_{n=1}^\infty$ is norm convergent in $X^*$.

**Proof.** By Lemma 5 and the proof of Theorem 10 of [H2], $f'(x_n)$ is norm relatively compact. By a standard argument, it is enough to prove the result under the additional assumption that $\sup\{\|x_n\| : n \in \mathbb{N}\} = r < 1$. If we assume that $\phi = \lim_{n\to\infty} f'(x_{2n})$, $\psi = \lim_{n\to\infty} f'(x_{2n+1})$, and $0 \neq h \in (1-r)B_X$, we have the following:

$$f(x_n + h) = f(x_n) + \int_0^{\|h\|} f'(x_n + \tau \frac{h}{\|h\|})(\frac{h}{\|h\|}) d\tau = f(x_n) + f'(x_n)(h) + R_n,$$

where $|R_n| \le \omega(\|h\|)\|h\|$. So $0 = \lim_{n\to\infty} f(x_{2n} + h) - \lim_{n\to\infty} f(x_{2n+1} + h) = \lim_{n\to\infty} f(x_{2n}) - \lim_{n\to\infty} f(x_{2n+1}) + (\phi - \psi)(h) + \lim_{n\to\infty}(R_{2n} - R_{2n+1}) = (\phi - \psi)(h) + R$, where $|R| \le 2\omega(\|h\|)\|h\|$. Letting $\|h\| \to 0$ we see that $(\phi - \psi)(h) = o(\|h\|)$ and so $\phi = \psi$. ■

**Proposition 1.2**

Let $K$ be a countable compact, $X = C(K)$ or $C_0(K)$, $f : B_X \to I\!R$ be $C^{1,u}$-smooth. Then there exists a canonical $C^{1,u}$-smooth and $w^*$-sequentially continuous extension $f^{**} : B_{X^{**}} \to I\!R$, $f^{**} \restriction_{B_X} = f$. Moreover, $(f^{**})'(x^{**}) \in X^* \hookrightarrow X^{***}$, for all $x^{**} \in B_{X^{**}}$ i.e. the derivatives are $w^*$-continuous functionals.

**Proof.** Since $X$ is $c_0$ saturated ([PS], for class $\mathcal{C}$ we have to invoke [H2] Proposition 6 instead), $\ell_1 \not\hookrightarrow X$. For $0 < \lambda \le 1$ we have by Odell-Rosenthal's theorem ([LT1, p. 101]) that every $x^{**} \in \lambda B_{X^{**}}$ is a $w^*$-limit of a sequence $\{x_n\}_{n=1}^{\infty} \subset \lambda B_X$. (In fact, as the referee of this note has pointed out, a simpler argument using Alaoglu's theorem and $C(K)^* = \ell_1$ can be employed here). We know that $\lim\limits_{n\to\infty} f(x_n)$ exists, so we set $f^{**}(x^{**}) = \lim\limits_{n\to\infty} f(x_n)$. We need to check that this definition is independent of the choice of $\{x_n\}_{n=1}^{\infty}$. However, this is immediate since if $x^{**} = w^* - \lim\limits_{n\to\infty} x_{2n} = w^* - \lim\limits_{n\to\infty} x_{2n+1}$, then $x^{**} = w^* - \lim\limits_{n\to\infty} x_n$ and the result follows due to wsc property of $f$ again ([H2]). Next, we have to verify that $f^{**}$ is $C^{1,u}$-smooth. Let us check first that for $x^{**} \in \lambda B_{X^{**}}$, $(f^{**})'(x^{**}) = \lim\limits_{n\to\infty} f'(x_n) = \phi \in X^*$ (the limit exists due to Lemma 4 of [H2]). For $h^{**} \in (1-\lambda)B_{X^{**}}$, $h = w^* - \lim\limits_{n\to\infty} h_n$, $h_n \in (1-\lambda)B_X$ we have

$$f^{**}(x^{**} + h^{**}) - f^{**}(x^{**}) = \lim_{n\to\infty} (f(x_n + h_n) - f(x_n)) = \lim_{n\to\infty} f'(x_n)(h_n) + R_n$$

where $|R_n| \le \omega(\|h\|)\|h\|$. Thus

$$|f^{**}(x^{**}+h^{**}) - f^{**}(x^{**}) - \lim_{n\to\infty} \phi(h_n)| = |f^{**}(x^{**}+h^{**}) - f^{**}(x^{**}) - \phi(h)| \le \omega(\|h\|)\|h\|,$$

and the conclusion follows. Let us now indicate why $f^{**}$ and $(f^{**})'$ have the modulus of continuity $\omega(\cdot)$. This clearly follows from the following fact. For $x^{**}, y^{**} \in \lambda B_X$ we can find sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \in \lambda B_X$ such that $\|x_n - y_n\| \le \|x^{**} - y^{**}\|$ for every $n \in I\!N$, and moreover $x^{**} = w^* - \lim\limits_{n\to\infty} x_n$, $y^{**} = w^* - \lim\limits_{n\to\infty} y_n$. Indeed, by the Odell-Rosenthal's theorem [LT1, p. 101], choose first $\{x_n\}_{n=1}^{\infty} \in \lambda B_X$ $w^*$-convergent to $x^{**}$, and $\{z_n\}_{n=1}^{\infty} \in \|x^{**} - y^{**}\| B_X$, $w^*$-convergent to $y^{**} - x^{**}$. At this point we surely have that $\tilde{y}_n = x_n + z_n$ is $w^*$-convergent to $y^{**}$, but we still need the norm estimate on $\tilde{y}_n$. Using the fact that we are working in $X = C(K)$ or $C_0(K)$, it suffices to truncate setting $y_n(t) = \min\{\lambda, \max\{-\lambda, \tilde{y}_n(t)\}\}$. Let us remark that in case $\mathcal{C}$, one needs to shrink the domain to get the same modulus. The problem is to generalize Odell-Rosenthal for a pair of sequences as used here. ■

**Proposition 1.3**

Let $K$ be a countable compact, $Y$ be a Banach space, $X = C(K)$ or $C_0(K)$, $T : B_X \to Y$, $T$ be $C^{1,u}$-smooth operator. Then there exists a $C^{1,u}$-smooth and $w^*$-sequentially continuous canonical extension $T^{**} : B_{X^{**}} \to Y^{**}$. Moreover, $(T^{**})'(x^{**}) \in \mathcal{L}^{**}(X, Y^{**}) \subset \mathcal{L}(X^{**}, Y^{**})$, for every $x^{**} \in B_{X^{**}}$, i.e. $(T^{**})'(x^{**})$ are $w^* - w^*$ continuous.

**Proof.** Given $y^* \in B_{Y^*}$, we set $f_{y^*} = y^* \circ T : B_X \to I\!R$. As $f_{y^*}$ is $C^{1,u}$-smooth, and modulus of continuity of $f'_{y^*}$ is $\omega(\cdot)$, it is wsc and by Proposition 1.2 there exists $f^{**}_{y^*} : B_{X^{**}} \to I\!R$ extending $f_{y^*}$, such that $f^{**}_{y^*}(x^{**}) = \lim_{n\to\infty} f_{y^*}(x_n)$. In particular, $T$ maps weakly Cauchy sequences into weakly Cauchy sequences. We can therefore define the extension $T^{**} : B_{X^{**}} \to Y^{**}$ as follows. Let $x^{**} \in \lambda B_{X^{**}}$, $x^{**} = w^* - \lim x_n$, $x_n \in \lambda B_X$. We set

$$T^{**}(x^{**}) = w^* - \lim_{n\to\infty} T(x_n) \in Y^{**}.$$

This formula is independent of the sequence $\{x_n\}_{n=1}^{\infty}$, and the existence and uniqueness of $T^{**}(x^{**})$ is clear. We continue by proving that $T^{**}$ is Fréchet differentiable in its domain. We have for every $y^* \in B_{Y^*}$, $x^{**} = w^* - \lim_{n\to\infty} x_n$ and $z^{**} = w^* - \lim_{n\to\infty} z_n$ from the domain

$$y^*(T^{**}(x^{**} + z^{**}) - T^{**}(x^{**})) = \lim_{n\to\infty} y^*(T(x_n + z_n) - T(x_n)).$$

Also

$$y^*(T(x_n + z_n)) = y^*(T(x_n)) + f'_{y^*}(x_n)(z_n) + R_n, \text{ where } |R_n| \le \omega(\|z_n\|)\|z_n\|.$$

Recall that by Proposition 1.2 and Lemma 1.1

$$\lim_{n\to\infty} f'_{y^*}(x_n) = (f^{**}_{y^*})'(x^{**}) \text{ in norm.}$$

So

$$|y^*(T^{**}(x^{**} + z^{**}) - T^{**}(x^{**})) - (f^{**}_{y^*})'(x^{**})(z^{**})| \le \omega(\|z_n\|)\|z_n\|.$$

In particular,

$$y^*\left(\frac{T^{**}(x^{**} + \lambda z^{**}) - T^{**}(x^{**})}{\lambda} - \frac{T^{**}(x^{**} + \varrho z^{**}) - T^{**}(x^{**})}{\varrho}\right) \le \omega(\lambda) + \omega(\varrho),$$

independently of $y^* \in B_{Y^*}$ and $z^{**} \in B_{X^{**}}$, which implies that $T^{**}(x^{**})$ has uniform directional derivatives. Similarly, we can prove the linear relations between the directional derivatives in order to see that $(T^{**})'(x^{**})$ exists in the Fréchet sense.

Once we have established the differentiability of $T^{**}$, we continue by proving that that $(T^{**})'(x^{**}) = \lim_{n\to\infty} (T'(x_n))^{**}$ in the weak operator topology (note that $(T'(x_n))^{**}$ is just the ordinary linear biconjugate operator to $T'(x_n)$). That is to say we claim that

$$y^*((T^{**})'(x^{**})(z^{**})) = \lim_{n\to\infty} y^*((T'(x_n))^{**}(z^{**}) \text{ for all } y^* \in B_{Y^*} \text{ and } z^{**} \in B_{X^{**}}.$$

Using the notation from above, this follows using standard arithmetic from the following relations.

$$((T'(x_n))^{**}(z^{**}) = w^* - \lim_{k\to\infty} T'(x_n)(z_k).$$

$$y^*(T^{**}(x^{**}+z^{**}) - T^{**}(x^{**})) = \lim_{n\to\infty}\lim_{k\to\infty} y^*(T(x_n+z_k) - T(x_n)).$$

The weak operator topology convergence, together with the trick used in the proof of Proposition 1.2 in order to preserve modulus $\omega(\cdot)$ for the extension, yield the same conclusion here, namely $(T^{**})'(x^{**})$ has modulus of continuity $\omega(\cdot)$ as a function of $x^{**}$. The $w^* - w^*$ continuity of $(T^{**})'(x^{**})$ follows using similar arguments.

■

The previous extension results will be used for a study of smooth operators on $C(K)$ spaces. As one of our corollaries below we prove that if $T$ is noncompact, then there exists $x^{**} \in B_{X^{**}}$ such that $(T^{**})'(x^{**})$ is a noncompact linear operator. This implies in particular that there exists a noncompact linear operator from $X$ to $Y^{**}$, so that by Theorem 0.1 $c_0$ is contained in $Y^{**}$. However, we first need to prove the reduction lemma below, which transfers the problem to the simplest space $c_0$ and gives more information.

**Lemma 1.4**

Given a countable ordinal $\alpha$, let $T : B_{C([0,\alpha])} \to X$ be a noncompact $C^{1,u}$-smooth operator. Then there exists $F \in B_{C([0,\alpha])}$ and a sequence $\{u_n\}_{n\in\mathbb{N}}$ of disjointly supported elements from $C[0,\alpha]$, with $F + u_n \in B_{C([0,\alpha])}$ for all $n \in \mathbb{N}$, and such that $T(F+u_n)$ is a noncompact sequence in $X$.

**Proof.** We may and will assume that $X$ is separable. Suppose that $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in $B_{C[0,\alpha]}$ such that $T(y_n)$ is noncompact. We will WLOG assume that our original sequence has the following additional properties. The sequence $y_n$, and so also $T(y_n)$, are weakly Cauchy. Using the standard argument from the proof of Lemma 12 in [H2], there exists some $\varepsilon > 0$, a sequence $\{f_i\}_{i\in\mathbb{N}} \in B_{X^*}$ so that $f_i(T(y_n)) = 0$ for $n < i$ and $f_i(T(y_i)) > \varepsilon$. Moreover, as $(B_{X^*}, w^*)$ is metrizable, $f_i$ is $w^*$-convergent, and (by replacing $f_i$ by $f_{2i+1} - f_{2i}$ and passing to subsequences) we may assume that in fact $f_i$ is $w^*$-null. Fix a system $\{\varepsilon_\beta\}_{\beta\le\alpha}$ of positive numbers such that $\sum\limits_{\beta\le\alpha} \varepsilon_\beta < \frac{\varepsilon}{2}$.

Using an (necessarily finite) inductive argument in $j$, we are going to construct a system consisting of the following objects:

(i) a decreasing sequence $\beta_j$ of ordinals $\alpha = \beta_1 > \beta_2 > \cdots > \beta_m = 0$,

(ii) a decreasing system $M_{j+1} \subset M_j$ of subsets of $M_1 = \mathbb{N}$, $1 \le j \le m$,

(iii) a function $F \in B_{C([0,\alpha])}$, $F \restriction_{[\beta_j+1,\beta_{j+1}]} = F(\beta_{j+1})$ is constant,

(iv) a system of sequences $\{y_n^j\}_{n\in M_j, 1\le j\le m}$ in $B_{C[0,\alpha]}$, $\{y_n^1\}_{n\in\mathbb{N}} = \{y_n\}_{n\in\mathbb{N}}$, and for every $j < m$, and $n \in M_{j+1}$ we have $y_n^j(\tau) = y_n^{j+1}(\tau)$ for all $\tau \in [0,\beta_{j+1}] \cup [\beta_j+1,\alpha]$. For a fixed $j$, the system $\{\mathrm{supp}(y_n^{j+1} - F) \cap [\beta_{j+1}+1,\alpha]\}_{n\in M_{j+1}}$ is pairwise disjoint.

$|f_n(y_n^{j+1}) - f_n(y_n^j)| < \varepsilon_{\beta_j}$ for all $n \in M_{j+1}$.

We present only the inductive step from $j$ to $j+1$, as the first step requires only minor changes. Suppose we have so far constructed: $\beta_i$, $M_i$ and the sequences $\{y_n^i\}_{n\in M_i}$ for $i \le j$, and $F$ is partially defined on $[\beta_j + 1, \alpha]$.

If $\beta_j$ is nonlimit, the step to a smaller ordinal $\beta_{j+1} = \beta_j - 1$ is really trivial, setting $F(\beta_{j+1} + 1) = F(\beta_j) = \lim_{n\in M_j} y_n(\beta_j)$, and using some standard perturbation arguments together with the inductive assumption we choose appropriate $M_{j+1}$ and $\{y_n^{j+1}\}_{n\in M_{j+1}}$. In this case we will have $y_n^{j+1}(\beta_{j+1} + 1) = F(\beta_{j+1} + 1)$.

So we may assume that $\beta_j$ is a limit ordinal. Put $r = \lim_{n\to\infty} y_n(\beta_j)$.

For $\varrho < \eta < \beta_j$, we define a continuous operator on $C[0,\alpha]$ by $P_\varrho^\eta(x) = x - \chi_{[\varrho+1,\eta]}x + r\chi_{[\varrho+1,\eta]}$ for $x \in C[0,\alpha]$. Similarly, for $\varrho < \eta < \theta < \beta_j$ we define $P_\varrho^{\eta,\theta}(x) = x - \chi_{[\varrho+1,\eta]}x + r\chi_{[\varrho+1,\eta]} - \chi_{[\theta+1,\beta_j]}x + r\chi_{[\theta+1,\beta_j]}$ for $x \in C[0,\alpha]$.

For a fixed $\varrho < \beta_j$, we have the following alternative. Either for every $\varrho < \eta < \beta_j$ there exists an infinite set $\{n \in M_j : |f_n(P_\varrho^\eta(y_n^j)) - f_n(y_n^j)| < \varepsilon_{\beta_j}\}$. In this case we say that $\varrho$ is of type $I$. Or else there exists $\varrho < \eta < \beta_j$ such that $\{n \in M_j : |f_n(P_\varrho^\eta(y_n^j)) - f_n(y_n^j)| < \varepsilon_{\beta_j}\}$ is finite, and we say that $\varrho$ is of type $II$. Given $y_n^j$ is of type $(\varrho,\eta)$ if $|f_n(P_\varrho^\eta(y_n^j)) - f_n(y_n^j)| \ge \varepsilon_{\beta_j}$. We claim that there exists $\varrho < \beta_j$ of type $I$. Assuming, by contradiction, that all $\varrho < \beta_j$ are of type $II$, using the fact that $\beta_j$ is a limit ordinal we obtain for every $N \in I\!N$ a sequence $\varrho_1 < \eta_1 < \varrho_2 < \eta_2 < \cdots < \varrho_N < \eta_N < \beta_j$ and some $y_n^j$, $n \in M_j$ which is of type $(\varrho_i, \eta_i)$ for all $1 \le i \le N$. This is a contradiction with Lemma 5 of [H1]. This allows us to choose $\beta_{j+1} = \varrho < \beta_j$ of type $I$, and extend the definition of $F$ on $[\beta_{j+1} + 1, \beta_j]$ by the constant value $r$. We continue now by defining $M_{j+1}$ and $\{y_n^{j+1}\}_{n\in M_{j+1}}$ by induction. Let $n_1 \in M_j$, and using that $\lim_{\tau\to\beta_j} y_{n_1}^j(\tau) = r$, find $\rho < \eta_1 < \beta_j$ such that

$$y_{n_1}^{j+1}(\tau) = y_{n_1}^j(\tau) \text{ for } \tau \notin [\eta_1 + 1, \beta_j],$$

$$y_{n_1}^{j+1}(\tau) = r \text{ for } \tau \in [\eta_1 + 1, \beta_j].$$

satisfies $|f_{n_1}(y_{n_1}^{j+1}) - f_{n_1}(y_{n_1}^j)| < \varepsilon_{\beta_j}$. Having found $n_1, \ldots, n_i$ and the corresponding $\eta_1 < \cdots < \eta_i < \beta_j$ and $y_{n_1}^{j+1}, \ldots y_{n_i}^{j+1}$ we proceed as follows. Pick $n_{i+1} \in M_j$, $n_{i+1} > n_i$ such that

$$|f_{n_{i+1}}(P_\rho^{\eta_i}(y_{n_{i+1}}^j)) - f_{n_{i+1}}(y_{n_{i+1}}^j)| < \varepsilon_{\beta_j},$$

and set $y_{n_{i+1}}^{j+1} = P_\rho^{\eta_i,\eta_{i+1}}(y_{n_{i+1}}^j)$ for a large enough $\eta_i < \eta_{i+1} < \beta_j$, so that $|f_{n_{i+1}}(y_{n_{i+1}}^{j+1}) - f_{n_{i+1}}(y_{n_{i+1}}^j)| < \varepsilon_{\beta_j}$ remains valid. We have thus described $M_{j+1} = \{n_i\}_{i\in I\!N}$ and $\{y_n^{j+1}\}_{n\in M_{j+1}}$.

The above described inductive procedure ends in finitely many $m$ steps, due to the well-ordering of $\alpha$. The last step provides us with a desired sequence $\{y_n^m\}_{n\in M_m}$ and a function $F$. To conclude, it remains to put $u_n = y_n^m - F$. ■

**Theorem 1.5**

Given a scattered compact $K$ and a Banach space $X$, let $T : B_{C(K)} \to X$ be a noncompact $C^{1,u}$-smooth operator. Then there exists an affine subspace $c_0 \cong Y \subset C(K)$ such that $T \restriction_{Y \cap B_{C(K)}}$ is noncompact. Moreover every $C^{1,u}$-smooth real function on $Y \cap B_{C(K)}$ is wsc.

**Proof.** Let $\{y_n\}_{n\in\mathbb{N}} \in B_{C(K)}$ be such that $\{T(y_n)\}_{n\in\mathbb{N}}$ is not relatively compact. By a standard argument of passing to suitable separable subalgebra of $C(K)$ generated by $\{y_n\}_{n\in\mathbb{N}}$, it suffices to prove the statement for every countable compact. By the classical result of Mazurkiewicz and Sierpinski in [MS] this is equivalent to the case $K = [0, \alpha]$, $\alpha$ a countable ordinal, and $X$ is a separable subspace containing the range of $T(C(K))$. The reduction result now follows from the previous Lemma 1.4. The last fact on wsc property follows from the explicit description of the space $Y$, which satisfies the conditions used in the proof of Theorem 10 of [H2]. ■

Theorem 1.5 gives a general positive answer to an affine version of Q 0.4. In the next section we will investigate noncompact operators from $B_{c_0}$, in the canonical norm. It is standard to check (relying on the mentioned proof of Theorem 10 in [H2]), that all our statements remain valid when the domain is a convex and lattice bounded set with nonempty interior, as is the case in the reduction theorem. So the results of the next section apply to the reduced operators from a general scattered $C(K)$. We have chosen the canonical version for the obvious reason of notational simplicity and clarity.

## 2. Smooth operators from $c_0$

In this section we establish general structural properties of $T$ and $X$, assuming that there exists a noncompact $C^{1,u}$-smooth operator $T : B_{c_0} \to X$. Our results come close to $X$ having a $c_0$ a subspace, but we did not manage to prove this condition in full generality.

Our main structural result is that $(T^{**})'(x^{**}) \in \mathcal{L}^{**}(c_0, X^{**})$ is a noncompact linear operator for some point $x^{**} \in B_{c_0}$, which implies that $X^{**}$ contains a copy of $\ell_\infty$, $X^*$ contains a complemented copy of $\ell_1$ and $X$ has a $c_0$ quotient. When applied to some classes of $X$, such as Banach lattices, duals etc., our results allow to conclude that $X$ indeed contains $c_0$, as conjectured. For spaces with PCP property we show that all $C^{1,u}$-smooth operators are in fact compact. In fact, most known examples of Banach spaces seem to be covered by our criterions. On the other hand, by the result of Bourgain and Pisier [BP], every separable space $X$ not containing $c_0$ is contained in a $\mathcal{L}_\infty$ space not containing $c_0$. This seems to suggest a canonical way to a counterexample, namely constructing a concrete $\mathcal{L}_\infty$ space. However this appears to be a delicate problem, since the classical $\mathcal{L}_\infty$ spaces of Bourgain and Delbaen ([B]) have PCP and cannot help (as was suggested by Haydon in [Hay]). Let us finally recall the fact that due to the reduction Theorem 1.5, all results in this section remain valid (upon obvious modifications) for $C^{1,u}$-smooth operators $T : B_{C(K)} \to X$, where $K$ is countable (or even scattered, if we use the appropriate biconjugate).

**Lemma 2.1**

Let $X$ be a Banach space, $T : B_{c_0} \to X$ be a $C^{1,u}$-smooth operator such that $\overline{T(B_{c_0})}$ is not compact. Then there exist sequences $\{f_n\}_{n=1}^{\infty} \in B_{X^*}$ and $\{T_n\}_{n=1}^{\infty} \in \mathcal{L}(c_0, X)$, $\sup \|T_n\| < \infty$ such that $\langle T_n e_i, f_i\rangle \geq 1$ whenever $i < n$, where $\{e_i\}_{i=1}^{\infty}$ is the canonical basis of $c_0$.

**Proof.** We know that $T$ maps weakly Cauchy sequences from $B_{c_0}$ to weakly Cauchy sequences from $X$. The assumption that $\overline{T(B_{c_0})}$ is non-compact together with Lemma 12 and Proposition 7 of [H2] imply that there exist $u, \{v_n\}_{n=1}^{\infty} \in B_{c_0}$ ($\{v_n\} \sim \{e_n\}$) such that $\lim_{n\to\infty} T(u+v_n) = T(u)$ does not hold and therefore $\{T(u+v_n)\}_{n=1}^{\infty}$ cannot be convergent. By passing to a subsequence we may assume that for some $\delta > 0$

$$\|T(u+v_n) - T(u+v_m)\| > 2\delta \quad \text{if} \quad n \neq m.$$

For the rest of the proof, we may WLOG assume that $u = 0$, $v_n = e_n$, $T(0) = 0$ and $T'(0) = 0$. Indeed, these conditions are easily achieved by replacing $T$ with

$$\tilde{T} : B_{c_0} \to X \quad : \quad \tilde{T}\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = T\Big(u + \sum_{i=1}^{\infty} a_i v_i\Big) - T(u) - T'(u)\Big(\sum_{i=1}^{\infty} a_i v_i\Big).$$

In the above formula $T'(u)$ may be assumed to be a compact linear operator, since otherwise by Theorem 0.1 $c_0 \hookrightarrow X$ and the conclusion of the lemma follows easily. As every compact operator from $c_0$ can be approximated by finite dimensional operators ($l_1$ has the approximation property - see [LT1, p. 33]), compact perturbations cannot violate the conclusion of the lemma. We have $\|T(e_i)\| > \delta$, and $\{T(e_i)\}_{i=1}\infty \subset X$ is weakly null. By passing to a subsequence of $\{e_i\}_{i=1}^{\infty}$, relabelled as $\{e_i\}_{i=1}^{\infty}$ again, $y_i = T(e_i)$ is a seminormalized basic sequence in $X$ ([LT1, p. 5] or [FHHMPZ, p. 173]), with its biorthogonal functionals $\varphi_i \in \frac{1}{\delta} B_{X^*}$ satisfying

$$\varphi_n(y_m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

In case $X$ is a dual space, using standard perturbation arguments together with Goldstine's theorem these functionals can be assumed to be from the predual $X_*$.

**Claim 2.2**
For every $\tau > 0$, there exists a subsequence $\{e_{n_i}\}_{i=1}^{\infty}$ of $\{e_i\}_{i=1}^{\infty}$ such that $N \geq k$ implies

$$\left| \varphi_{n_k} \circ T\Big(\sum_{i=1}^{N} \alpha_i e_{n_i}\Big) - \varphi_{n_k} \circ T\Big(\sum_{i=1}^{k} \alpha_i e_{n_i}\Big) \right| \leq \tau \quad \text{for all} |\alpha_i| \leq 1.$$

**Proof of Claim.** By induction. Set $n_1 = 1$, fix a finite set

$$S = \{-1, -\frac{l-1}{l}, -\frac{l-2}{l}, \dots, \frac{l-1}{l}, 1\} \subset [-1,1] \text{ such that } \omega(\frac{1}{l}) < \frac{\tau}{4}.$$

By Corollary 10 of [H1] there exists $m_1 \in I\!N$ such that $N \geq m_1$ implies

$$\Big|\varphi_{n_1} \circ T\big(\alpha e_{n_1} + \sum_{i=m_1}^{N} \alpha_i e_i\big) - \varphi_{n_1} \circ T(\alpha e_{n_1})\Big| < \frac{\tau}{4}$$

for every $\alpha \in S$, $|\alpha_i| \leq 1$. We choose $n_2 = m_1$ and continue by finding $m_2 \in I\!N$, $m_2 > m_1$, such that $N \geq m_2$ implies

$$\Big|\varphi_{n_2} \circ T\big(\alpha e_{n_1} + \beta e_{n_2} + \sum_{i=m_2}^{N} \alpha_i e_i\big) - \varphi_{n_2} \circ T(\alpha e_{n_1} + \beta e_{n_2})\Big| < \frac{\tau}{4}$$

for every $\alpha, \beta \in S$, $|\alpha_i| \leq 1$.

We set $n_3 = m_2$ and continue in an obvious manner.

Using this inductive procedure, we obtain a sequence $\{e_{n_i}\}_{i=1}^{\infty}$ such that $N \geq k$ implies

$$\Big|\varphi_{n_k} \circ T\big(\sum_{i=1}^{N} \alpha_i e_{n_i}\big) - \varphi_{n_k} \circ T\big(\sum_{i=1}^{k} \alpha_i e_{n_i}\big)\Big| < \frac{\tau}{4}$$

for every $\alpha_i \in S$. In order to pass to arbitrary values of $\alpha_i \in [-1, 1]$ it suffices to recall that $\omega\big(\frac{1}{l}\big) < \frac{\tau}{4}$. ■

Before we proceed, we reindex $\{e_{n_i}\}_{i=1}^{\infty}$ as $\{e_i\}_{i=1}^{\infty}$ again.

**Claim 2.3**

For every $\tau > 0$ there exists a subsequence $\{e_{n_i}\}_{i=1}^{\infty}$ of $\{e_i\}_{i=1}^{\infty}$ such that

$$\Big|\varphi_{n_k} \circ T\big(\sum_{i=1}^{k} \alpha_i e_{n_i}\big) - \varphi_{n_k} \circ T(\alpha_k e_{n_k})\Big| \leq \tau \quad \text{for all } |\alpha_i| \leq 1.$$

**Proof of Claim.** Relies again on Lemma 5 from [H1]. It gives us that for $k$ large enough (and depending only on the modulus of continuity of $T'$), $l > k$ and $\alpha_l \in S$ fixed, there exists $i < k$ such that

$$|\varphi_{n_l} \circ T(\alpha_i e_i + \alpha_l e_l) - \varphi_{n_l} \circ T(\alpha_l e_l)| < \frac{\tau}{4} \quad \text{for } |\alpha_i| \leq 1.$$

In fact, Lemma 5 of [H1] gives an upper bound on the number of $i$ for which the above estimate is not valid. Since $S$ is a finite set, Repeating this argument for each $\alpha_l \in S$, we get that for $k$ large enough but fixed and any $l > k$ there exists some $i_l < k$

$$|\varphi_{n_l} \circ T(\alpha_{i_l} e_{i_l} + \alpha_l e_l) - \varphi_{n_l} \circ T(\alpha_l e_l)| < \frac{\tau}{4}$$

for $\alpha_l \in S$, $|\alpha_{i_l}| \leq 1$.

Clearly, there exists an infinite subsequence $k < M_1 \subset I\!N$ such that $n_1 := i_l = i_m$ for every $l, m \in M_1$. Next, choose a large enough initial segment $I \subset M_1$, so that for every $I < l \in M_1$, there exists some $i_l$, $i_l \neq n_1$, such that for every $\alpha_{n_1}, \alpha_l \in S$, and $|\alpha_{i_l}| \leq 1$

$$|\varphi_{n_l} \circ T(\alpha_{n_1} e_{n_1} + \alpha_{i_l} e_{i_l} + \alpha_l e_l) - \varphi_{n_l} \circ T(\alpha_{n_1} e_{n_1} + \alpha_l e_l)| < \frac{\tau}{8}.$$

Again, there exists an infinite subsequence $I < M_2 \subset M_1$ such that $n_2 := i_l = i_m$ for every $l, m \in M_2$. We continue in an obvious way by induction; after having constructed $n_1, \dots, n_k \in I\!N$ and infinite sequences $M_k \subset M_{k-1} \subset \dots \subset M_1 \subset I\!N$, $M_{i-1} \ni n_i < M_i$, the inductive step consists of choosing a long enough initial sequence $I \subset M_k$ so that $\forall l > I$, $l \in M_k$, $\exists i_l \in I$, $i_l \notin \{n_1, \dots, n_k\}$, such that $\forall |\alpha_{i_l}| \leq 1$ $\forall \alpha_{n_1}, \dots, \alpha_{n_k} \in S$, $\alpha_l \in S$

$$\left|\varphi_{n_l} \circ T\Big(\sum_{i=1}^{k} \alpha_{n_i} e_{n_i} + \alpha_{i_l} e_{i_l} + \alpha_l e_l\Big) - \varphi_{n_l} \circ T\Big(\sum_{i=1}^{k} \alpha_{n_i} e_{n_i} + \alpha_l e_l\Big)\right| < \frac{\tau}{2^{k+2}}.$$

We then find an infinite subsequence $I < M_{k+1} \subset M_k$ such that $I \ni n_{k+1} := i_l = i_m$ for every $l, m \in M_{k+1}$. The sequence $\{e_{n_i}\}_{i=1}^{\infty}$ obtained in this way satisfies

$$\left|\varphi_{n_k} \circ T\Big(\sum_{i=1}^{k} \alpha_i e_{n_i}\Big) - \varphi_{n_k} \circ T(\alpha_k e_{n_k})\right| \leq$$

$$\left|\varphi_{n_k} \circ T\Big(\sum_{i=1}^{k-2} \alpha_i e_{n_i} + \alpha_k e_{n_k}\Big) - \varphi_{n_k} \circ T(\alpha_k e_{n_k})\right| +$$

$$+ \left|\varphi_{n_k} \circ T\Big(\sum_{i=1}^{k} \alpha_i e_{n_i}\Big) - \varphi_{n_k} \circ T\Big(\sum_{i=1}^{k-2} \alpha_i e_{n_i} + \alpha_k e_{n_k}\Big)\right| \leq$$

$$\left|\varphi_{n_k} \circ T\Big(\sum_{i=1}^{k-3} \alpha_i e_{n_i} + \alpha_k e_{n_k}\Big) - \varphi_{n_k} \circ T(\alpha_k e_{n_k})\right| +$$

$$+ \left|\varphi_{n_k} \circ T\Big(\sum_{i=1}^{k-2} \alpha_i e_{n_i} + \alpha_k e_{n_k}\Big) - \varphi_{n_k} \circ T\Big(\sum_{i=1}^{k-3} \alpha_i e_i + \alpha_k e_{n_k}\Big)\right| + \frac{\tau}{2^{k+2}}$$

$$\leq \dots \leq |\varphi_{n_k} \circ T(\alpha_1 e_{n_1} + \alpha_k e_{n_k}) - \varphi_{n_k} \circ T(\alpha_k e_{n_k})| + \tau\Big(\frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^{k+2}}\Big) \leq$$

$$\leq \tau\Big(\frac{1}{2^2} + \dots + \frac{1}{2^{k+2}}\Big) \leq \frac{\tau}{2}$$

whenever $\alpha_i \in S$. Passing to arbitrary $\alpha_i \in [-1, 1]$, at the expense of adding $\frac{\tau}{2}$ on the right hand side, is possible due to $\omega\big(\frac{1}{l}\big) < \frac{1}{4}\tau$.

■

Combining Claim 2 and Claim 3 we obtain that given $\tau > 0$ we may WLOG assume that $T$ satisfies (assuming $n \geq k$):

$$\left|\varphi_k \circ T\big(\sum_{i=1}^{n} \alpha_i e_i\big) - \varphi_k \circ T(\alpha_k e_k)\right| \leq 2\tau.$$

Recall that $\varphi_k \circ T(0) = 0$, $\varphi_k \circ T(e_k) = 1$, $\|\varphi_k\| \leq \frac{1}{\delta}$. Since

$$1 = \varphi_k \circ T(e_k) = \int_0^1 \varphi_k\big(\langle T'(te_k), e_k\rangle\big)\, dt$$

there exists $t_0 \in [0,1]$ where $\varphi_k\big(\langle T'(t_0 e_k), e_k\rangle\big) \geq 1$. Fix $\Delta > 0$ satisfying $\omega(\Delta) < \frac{\delta}{8}$. Then for $t \in [t_0 - \Delta, t_0 + \Delta]$ we have $\|T'(te_k) - T'(t_0 e_k)\| \leq \frac{\delta}{8}$ and thus

$$\begin{aligned}&\varphi_k\big(\langle T'(te_k), e_k\rangle\big) \geq \\ &\varphi_k\big(\langle T'(t_0 e_k), e_k\rangle\big) - \frac{1}{\delta}\|T'(t_0 e_k) - T'(te_k)\| \geq \frac{7}{8}.\end{aligned}$$

Consequently,

$$\varphi_k \circ T\big((t_0 + \frac{\Delta}{2})e_k\big) - \varphi_k \circ T(t_0 e_k) \geq \frac{7}{8}\frac{\Delta}{2}.$$

If, on the other hand, we have for some $r \in [0,1]$

$$\varphi_k \circ T\big((r + \frac{\Delta}{2})e_k\big) - \varphi_k \circ T(re_k) \geq \frac{6}{8}\frac{\Delta}{2},$$

then there exists $s \in [r, r + \frac{\Delta}{2}]$ for which $\varphi_k\left(\langle T'(se_k), e_k\rangle\right) \geq \frac{6}{8}$ and thus for every $t \in [r, r + \frac{\Delta}{2}]$ we have $\|T'(te_k) - T'(se_k)\| \leq \frac{\delta}{8}$ and in particular $\varphi_k\left(\langle T'(te_k), e_k\rangle\right) \geq \frac{6}{8} - \frac{1}{8} = \frac{5}{8}$. We now set the value of $\tau = \frac{\Delta}{64}$, and we suppose that $\{e_n\}_{n=1}^{\infty}$ satisfies both Claim 2 and 3. For every $k \in I\!N$ there exists an interval $J_k \subset [0,1]$ of length $\Delta$ such that

$$\varphi_k\left(\langle T'(te_k), e_k\rangle\right) \geq \frac{7}{8} \text{ for } t \in J_k.$$

There exists an infinite subsequence $\{n_i\}_{i=1}^{\infty}$ of $I\!N$ such that $[a,b] = J \subset \bigcap_{i=1}^{\infty} J_{n_i}$ is an interval of length $\frac{\Delta}{2}$. We may again WLOG assume that $n_i = i$. We have

$$\varphi_k \circ T(be_k) - \varphi_k \circ T(ae_k) \geq \frac{7\Delta}{2^4}.$$

Moreover we have for any $|\alpha_i| \leq 1$

$$\varphi_k \circ T\big(\sum_{i=1}^{k-1} \alpha_i e_i + b e_k + \sum_{i=k+1}^{n} \alpha_i e_i\big) - \varphi_k \circ T\big(\sum_{i=1}^{k-1} \alpha_i e_i + a e_k + \sum_{i=k+1}^{n} \alpha_i e_i\big) \geq$$

$$\varphi_k \circ T(b e_k) - \varphi_k \circ T(a e_k) - 4\tau \geq \frac{7\Delta}{2^4} - \frac{\Delta}{2^4} = \frac{6\Delta}{2^4}.$$

Thus, for every $c \in [a,b]$, $|\alpha_i| \leq 1$

$$\varphi_k\big(\langle T'\big(\sum_{i=1}^{k-1} \alpha_i e_i + c e_k + \sum_{i=1}^{n} \alpha_i e_i\big), e_k\rangle\big) \geq \frac{5}{8}.$$

To finish the proof of Lemma 1 we set for $n \in \mathbb{N}$:

$$T_n : c_0 \to X \text{ to be } T_n = \frac{8}{5\delta} T'\big(\sum_{i=1}^{n} a e_i\big),$$
$$f_n = \delta \varphi_n.$$

■

Our main structural result on noncompact smooth operators is the following.

**Theorem 2.4**
Let $X$ be a Banach space, $T : B_{c_0} \to X$ be a $C^{1,u}$-smooth operator such that $\overline{T(B_{c_0})}$ is not compact. Then there exists a point $x^{**} \in B_{c_0^{**}}$, such that $(T^{**})'(x^{**}) \in \mathcal{L}^{**}(c_0, X^{**})$ is a noncompact linear operator. Moreover, if $X$ is a dual space, we can get in addition $(T^{**})'(x^{**})(c_0) \subset X$, and $(T^{**})'(x^{**}) \restriction_{c_0}$ is noncompact.

**Proof.** In the proof of Lemma 2.1, we have established the existence of $u, \{v_n\}_{n=1}^{\infty} \in B_{c_0}$, such that $v_n$ are disjointly supported vectors ($\{v_n\} \sim \{e_n\}$), and corresponding biorthogonal functionals $\{f_n\}_{n=1}^{\infty} \in B_{X^*}$ (or $B_{X_*}$, if $X$ is a dual space) to $\{T(v_n) - T(u)\}_{n=1}^{\infty}$ in $X$, so that

$$\langle T'\big(\sum_{i=1}^{n} a v_i\big)(v_k), f_k\rangle \geq \mu > 0 \text{ for every } n \geq k.$$

It suffices to put $x^{**} = w^* - \lim_{n\to\infty} \sum_{i=1}^{\infty} a v_i$, since $(T^{**})'(x^{**})$ being a weak operator limit of the sequence $\{T'\big(\sum_{i=1}^{n} a v_i\big)\}_{n=1}^{\infty}$ is, due to the above inequality, clearly a noncompact linear operator. The case when $X$ is a dual space follows by standard $w^*$-compactness argument using the additional information that $f_i \in X_*$. ■

The following are immediate consequences.

**Corollary 2.5**

Let $X$ be a Banach space, $T : B_{c_0} \to X$ be a $C^{1,u}$-smooth operator such that $\overline{T(B_{c_0})}$ is not compact. Then $X$ has the following properties.

(i) $\ell_\infty \hookrightarrow X^{**}$, $\ell_1$ is a complemented subspace of $X^*$ and $X$ has a $c_0$ quotient.
(ii) $X$ does not have nontrivial cotype.
(iii) $X$ is not weakly sequentially complete.

**Proof of** $(i)$**.** $(T^{**})'(x^{**}) \restriction_{c_0}$ is a noncompact operator, so by Theorem 0.1, $c_0 \hookrightarrow X^{**}$. The rest are general consequences of this fact, to be found in [LT1] or [FHHMPZ].

**Proof of** $(ii)$**.** By the principle of local reflexivity [FHHMPZ, p. 292] $c_0^n$ embeds uniformly to $X$, which is equivalent to $X$ lacking nontrivial cotype [DJT, p. 283].

**Proof of** $(iii)$**.** The weak sequential completeness of $X$, together with the $w^*$-to-weak operator topology continuity of the mapping $x^{**} \to (T^{**})'(x^{**})$ implies that $(T^{**})'(x^{**})(c_0) \subset X$, so by (i) we get $c_0 \hookrightarrow X$ which is however a contradiction with the weak sequential completeness of $X$. ■

**Corollary 2.6**

Let $X$ be a Banach space with any of the following properties:
(i) $X$ is a dual space,
(ii) $X$ is a complemented subspace of a Banach lattice,
(iii) $X$ is a subspace of a space with an unconditional basis,
(iv) $X$ has property (u) of Pelczynski.
Suppose that there exists a $C^{1,u}$-smooth operator $T : B_{c_0} \to X$, such that $\overline{T(B_{c_0})}$ is not compact. Then $c_0 \hookrightarrow X$.

**Proof.** (i) follows along the same lines as (iii) of Corollary 2.5, using the functionals from predual. (ii)-(iv) follow from (iii) of Corollary 2.5 and the classical results in [LT1,2], according to which any Banach space from one of these classes is weakly sequentially complete unless it contains $c_0$. ■

Recall that a Banach space $X$ has the point of continuity property (PCP), if every weakly closed bounded subset of $X$ contains a point of weak-to-norm continuity for the identity mapping. Spaces with the PCP property have been extensively studied by many authors. In particular it is known that all RNP spaces belong to this class, and in the following theorem we will use the fundamental description of separable PCP spaces as those admitting a boundedly complete skipped blocking finite dimensional decomposition. The last notion is due to Bourgain and Rosenthal, and its equivalence to the PCP was established by Ghoussoub and Maurey in [GM]. We refer to this paper for the result and further references in this area.

**Theorem 2.7**

Let $X$ be a Banach space with the PCP property. Then every $C^{1,u}$-smooth operator $T : B_{c_0} \to X$ is compact.

**Proof.** Since PCP is a hereditary property, we may WLOG assume that $X$ is separable. We proceed by contradiction, assuming that there exists a $C^{1,u}$-smooth noncompact operator $T : B_{c_0} \to X$, and $X$ has a boundedly complete skipped blocking finite dimensional decomposition. That is to say, there exists a sequence $G_i$ of finite dimensional subspaces of $X$ satisfying

(1) $X = \overline{\text{span}} \bigcup\limits_{i=1}^{\infty} G_i$

(2) $G_k \cap \overline{\text{span}} \bigcup\limits_{i \neq k} G_i = \{0\}$

(3) if $\{m_k\}_{k=1}^{\infty}$, $\{n_k\}_{k=1}^{\infty}$ are sequences from $I\!N$, $m_k < n_k + 1 < m_{k+1}$ then setting $H_k = \overline{\text{span}} \bigcup\limits_{i=m_k}^{n_k}$, $\{H_k\}_{k-1}^{\infty}$ is a boundedly complete FDD for $\overline{\text{span}} \bigcup\limits_{k=1}^{\infty} H_k$.

In our proof we will use the notation from the proof of Lemma 2.1. The starting point of our proof are the results obtained there, in particular, we assume that $\{e_i\}_{i=1}^{\infty} \in B_{c_0}$ is a seminormalized basic sequence equivalent to the canonical basis, $y_i = T(e_i)$ is a seminormalized basic sequence in $X$ with its biorthogonal functionals $\varphi_i \in \frac{1}{\delta} B_{X^*}$ satisfying

$$\varphi_n(y_m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the following relations hold for some $\tau > 0$.

$$\left| \varphi_{n_k} \circ T\big(\sum_{i=1}^{k} \alpha_i e_{n_i}\big) - \varphi_{n_k} \circ T(\alpha_k e_{n_k}) \right| \leq \tau \quad \text{for all } |\alpha_i| \leq 1.$$

$$\left| \varphi_k \circ T\big(\sum_{i=1}^{n} \alpha_i e_i\big) - \varphi_k \circ T(\alpha_k e_k) \right| \leq 2\tau \quad \text{for all } n \geq k, |\alpha_i| \leq 1.$$

We now proceed by constructing sequences of integers $\{m_k\}_{k=1}^{\infty}$, $\{n_k\}_{k=1}^{\infty}$, $\{l_k\}_{k=1}^{\infty}$ as follows:

Fix a sequence $\varepsilon_n \searrow 0$, $\sum\limits_{n=1}^{\infty} \varepsilon_n < 1$, put $m_1 = 1$, $l_1 = 1$. Set $n_1 > m_1$ such that

$$\operatorname{dist}\big(T(e_1), \operatorname{span} \bigcup_{i=1}^{n_1} G_i\big) < \varepsilon_1.$$

Next, put $m_2 = n_1 + 2$ and choose $l_2$ which satisfies for $|\alpha_i| \leq 1$

$$\operatorname{dist}\big(T(e_1 + \sum_{i=l_2}^{N} \alpha_i e_i) - T(e_1), \overline{\text{span}} \bigcup_{i=m_2}^{\infty} G_i\big) < \varepsilon_2.$$

The existence of such $l_2$ follows since $T(e_1 + x) - T(e_1)$ maps weakly null sequences $\{x_n\}$ from $B_{c_0}$ to weakly null sequences in $X$, and span $\bigcup\limits_{i=1}^{m_2-1} G_i$ is finite dimensional. Next choose $n_2$ such that

$$\operatorname{dist}\big(T(e_{l_1} + e_{l_2}) - T(e_{l_1}), \operatorname{span} \bigcup_{i=m_2}^{n_2} G_i\big) < \varepsilon_2.$$

Put $m_3 = n_2 + 2$, and continue by induction as follows. Having constructed $\{n_i\}_{i=1}^k$, $\{m_i\}_{i=1}^k$, $\{l_i\}_{i=1}^k$, we set $m_{k+1} = n_k + 2$. We then find $l_{k+1} > l_k$ for which if $|\alpha_i| \leq 1$ then

$$\operatorname{dist}\big(T(\sum_{i=1}^k e_{l_i} + \sum_{i=l_{k+1}}^N \alpha_i e_i) - T(\sum_{i=1}^k e_{l_i}), \overline{\operatorname{span}} \bigcup_{i=m_{k+1}}^{\infty} G_i\big) < \varepsilon_k.$$

Finally, find $n_{k+1} > m_{k+1}$ for which

$$\operatorname{dist}\big(T(\sum_{i=1}^{k+1} e_{l_i}) - T(\sum_{i=1}^k e_{l_i}), \operatorname{span} \bigcup_{i=m_{k+1}}^{n_{k+1}} G_i\big) < \varepsilon_k.$$

Denote $y_0 = 0$, $y_k = T(\sum\limits_{i=1}^k e_{l_i})$, $H_k = \operatorname{span} \bigcup\limits_{i=m_{k+1}}^{n_{k+1}} G_i$. With this notation, it is clear that for some $z_k$, $\|z_k\| < \varepsilon_k$

$$u_k := y_{k+1} - y_k + z_k \in H_k.$$

Since $\sum\limits_{k=1}^N u_k = y_{N+1} + \sum\limits_{k=1}^N z_k$ is a norm bounded sequence, it is norm convergent. Thus

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} T(\sum_{i=1}^n e_{l_i})$$

exists in norm. However,

$$\varphi_{l_n}(y_n) \geq \varphi_{l_n}\big(T(e_{l_n})\big) - \tau = 1 - \tau$$
$$\varphi_{l_n}(y_{n-1}) \leq \varphi_{l_n}\big(T(0)\big) + \tau = \tau.$$

Thus $\|y_n - y_{n-1}\| \geq (1 - 2\tau)\frac{1}{\|\varphi_{l_n}\|} \geq (1 - 2\tau)\delta$, a contradiction.

■

In particular, and answering a question of Haydon from [Hay] in the negative, we have the following.

**Corollary 2.8**

Let $X$ be a Bourgain-Delbaen $\mathcal{L}_\infty$ space (cf. [B]), $T : B_{c_0} \to X$ be $C^{1,u}$-smooth. Then $\overline{T(B_{c_0})}$ is compact.

**Proof.** Combining the results in [B] and [GM], these spaces have the PCP property.

■

### 3. Summability properties of smooth functions on $c_0$

Given a function $f : B_{c_0} \to I\!R$, we are interested in the value $V = \sum_{n=1}^{\infty} |f(e_n)|$. There are numerous results which give the convergence of the last summation. In the complex scalar case (when $c_0$ is over the complex field and $f$ is complex), Aron and Globevnik [AG] (generalizing K. John's earlier work [J]) showed that if $f$ is a homogeneous polynomial, then $V \le \sup_{x\in B_{c_0}} |f(x)|$. This estimate is independent of the degree of the polynomial. Aron, Beauzamy and Enflo [ABE] treated the corresponding real case. The result is that for a general $k$-homogeneous polynomial $V \le 4k^2 \sup_{x\in B_{c_0}} |f(x)|$, but there exists $k$-homogeneous polynomials for which $V \ge k \sup_{x\in B_{c_0}} |f(x)|$. Thus an upper estimate using the supremum of $f$, independent on the degree, does not exists even for homogeneous polynomials. However, in [H2], we prove the following degree free estimate for every homogeneous polynomial: $V \le 16 \sup_{x\in B_{c_0}} \|f''(x)\|$. The main result of this section is a construction of nonhomogeneous real polynomials for which $V$ cannot be estimated from above using $f''$ independently of the degree.

It turns out that these results are closely connected with the behaviour of smooth operators, in particular the Question 0.5 (in fact, after checking Sections 3,4 of this note, the reader will realize that the validity of the estimate from [H2] is essentially equivalent to the validity of Q 0.5). We recover a sharper form of Pelczynski Theorem 0.2 from this and Theorem 1.5 (answering Q 0.5 in the positive for polynomial operators). In the subsequent section we construct a $C^{1,1}$-smooth noncompact operator which fails this description (and Q 0.5) and seems to be a half-way counterexample to Question 0.3.

**Theorem 3.1**
Let $X$ be a Banach space, $K$ be a scattered compact, $P : C(K) \to X$ be a noncompact polynomial operator (not necessarily homogeneous). Then there exists a sequence $\{v_n\}_{n=0}^{\infty} \in B_{C(K)}$, such that both $\{v_n - v_0\}_{n=1}^{\infty}$ and $\{P(v_n) - P(v_0)\}_{n=1}^{\infty}$ are equivalent to the canonical basis of $c_0$.

**Proof.** By the reduction theorem we may replace $C(K)$ by the space $c_0$. Let $k = \deg(P)$. As $P(B_{c_0})$ is not relatively compact, by [H2], Lemma 12, there exists $v_n \in B_{c_0}$, $n = 0, 1, \dots$, such that $\{v_n\}_{n=1}^{\infty}$ is equivalent to the unit basis of $c_0$ and $\|\cdot\| - \lim_{n\to\infty} P(v_0 + v_n)$ does not exist. Put $y_0 = P(v_0)$, $y_n = P(v_0 + v_n)$, $z_n = y_n - y_0$. As was shown in the proof of Lemma 2.1, by passing to a subsequence WLOG $\{z_n\}_{n=1}^{\infty}$ is a $C$-seminormalized weakly null Schauder basic sequence ($C^{-1} \le \|z_n\| \le C$ for some $C$). We claim that $\{z_n\}_{n=1}^{\infty}$ is equivalent to the canonical basis of $c_0$. To this end, it suffices to show that there exists $K \in I\!R$ such that

$$\sup_{|\alpha_n|\leq 1} \|\sum_{n=1}^{\infty} \alpha_n z_n\| \leq K.$$

which is equivalent to $\sum\limits_{n=1}^{\infty} |\phi(z_n)| \leq K$ for every $\phi \in B_{X^*}$.

Assume $\tilde{P}(x) = P(v_0 + x) - P(v_0) = \sum\limits_{l=1}^{k} P_l(x)$, where $P_l$ are homogeneous polynomials. Fixing $l$, there exists $K_l \in I\!R$, such that for every $\phi \in B_{X^*}$, $\phi \circ P_l : c_0 \to I\!R$ is a real valued $k$-homogeneous polynomial satisfying $\|(\phi \circ P_l)''(x)\| \leq K_l$ for all $x \in B_{c_0}$.
By ([H2], Lemma 15)

$$\sum_{n=1}^{\infty} |\phi \circ P_l(v_n)| \leq 16C^k K_l.$$

Consequently,

$$\sum_{n=1}^{\infty} |\phi \circ (\sum_{l=1}^{k} P_l(v_n))| = \sum_{n=1}^{\infty} |(\sum_{l=1}^{k} \phi \circ P_l(v_n))| \leq \sum_{l=1}^{k} 16C^k K_l = K < \infty.$$

This estimate is true for every $\phi \in B_{X^*}$. ■

The improvement of Theorem 3.1 over Theorem 0.2 consists of showing that $P$ actually carries a translate of the canonical basis of $c_0$ into the range space $X$, in the spirit of Theorem 0.1. The next simple example shows that Theorem 3.1 is optimal in the sense that the shifting of the $c_0$ basis by $y_0$ is necessary, so the result is necessarily of affine rather than linear nature.

**Example 3.2**
Put $P(x) : c_0 \to c_0$, $P((x_i)_{i=1}^{\infty}) = (x_1^4, x_1^2 x_2^2, x_1^2 x_3^2, \ldots)$. Then $P$ is noncompact, but $\lim\limits_{n\to\infty} P(u_n) = 0$ for every weakly null sequence in $c_0$. Choosing $v_i = e_{i+1}$, $i = 1, 2, \ldots$ gives $P(v_0) = e_1$, $P(v_0 + v_i) = e_1 + e_{i+1}$. Since the image of $P$ is contained in the positive cone of $c_0$, we also see that we cannot hope for $B_{c_0} \subset P(B_{c_0})$.

We continue with the main result of this section, a construction of a special sequence of $C^{1,1}$-smooth functions failing the good summability properties. These functions will be used later to construct a $C^{1,1}$-smooth noncompact operator which fails the statement of Theorem 3.1 (and also Q 0.5).

**Theorem 3.3**
Let $\phi_n : B_{c_0^n} \to I\!R$ be defined as

$$\phi_n(x_i) = \frac{1}{\sqrt{n}} \prod_{i=1}^{n} (1 - x_i^4).$$

Then $\exists C$ independent of $n$ such that $\phi_n'': B_{c_0^n} \to \mathcal{L}(c_0^n, \ell_1^n)$,

$$\phi_n''(x) = \Big(\frac{\partial^2 \phi_n(x)}{\partial x_i \partial x_j}\Big)_{i,j=1}^n \text{ satisfies } \|\phi_n''\|_{\mathcal{L}(c_0^n, \ell_1^n)} \le C.$$

**Proof.** First note that for $(a_{ij})_{i,j=1}^n = L \in \mathcal{L}(c_0^n, \ell_1^n)$, we have $\|L\|_{\mathcal{L}(c_0^n,\ell_1^n)} = \max_{\varepsilon_j = \pm 1} \sum_{i=1}^n |\sum_{j=1}^n a_{ij}\varepsilon_j| \le \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|$. Now we have

$$\frac{\partial^2 \phi_n(x)}{\partial x_i \partial x_j} = \begin{cases} \frac{-12}{\sqrt{n}} x_i^2 \prod\limits_{\substack{k=1 \\ k\neq i}}^n (1 - x_k^4) & \text{if } i = j, \\ \frac{16}{\sqrt{n}} x_i^3 x_j^3 \prod\limits_{\substack{k=1 \\ k\neq i,j}}^n (1 - x_k^4) & \text{if } i \neq j. \end{cases}$$

We wish to estimate for $x \in B_{c_0^n}$ the quantities

$$A = \sup_{x \in B_{c_0^n}} \sum_{i=1}^n \left| \frac{\partial^2 \phi_n(x)}{\partial x_i^2} \right|,$$

$$B = \sup_{x \in B_{c_0^n}} \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n \left| \frac{\partial^2 \phi_n(x)}{\partial x_i \partial x_j} \right|.$$

WLOG assume that $0 \le x_1 \le x_2 \dots \le x_n \le 1$, so that $(1 - x_k^4) \ge (1 - x_n^4)$ and put

$$F\big((x_i)_{i=1}^{n-1}\big) := \frac{12}{\sqrt{n}} + \frac{12}{\sqrt{n}} \Big(\sum_{i=1}^{n-1} x_i^2 \prod_{k=1}^{n-1} (1 - x_k^4)\Big) \ge$$

$$\frac{12}{\sqrt{n}} \Big(\sum_{i=1}^{n} x_i^2 \prod_{k=1}^{n-1} (1 - x_k^4)\Big) \ge \frac{12}{\sqrt{n}} \sum_{i=1}^{n} x_i^2 \prod_{\substack{k=1 \\ k \neq i}}^{n} (1 - x_k^4).$$

The reason for introducing $F$ instead of estimating directly the original term is the useful symmetry of $\nabla F$, as we will see below. There exists $z = (z_i)_{i=1}^{n-1} \in B_{c_0^{n-1}}$ such that $F(z) = \max_{x \in B_{c_0^{n-1}}} F(x) \ge A$. Clearly, either $z \in \partial B_{c_0^{n-1}}$ or else $\nabla F(z) = 0$. In the first case, $z_i = 1$ for some $i \le n-1$ and thus $F(z) = \frac{12}{\sqrt{n}}$. Suppose $z \notin \partial B_{c_0^{n-1}}$.

$$\nabla F(z) = \frac{12}{\sqrt{n}} \Big(2 z_i \prod_{k=1}^{n-1} (1 - z_k^4) - 4 z_i^3 \big(\sum_{j=1}^{n-1} z_j^2\big) \prod_{\substack{k=1 \\ k \neq i}}^{n-1} (1 - z_k^4)\Big)_{i=1}^{n-1} = 0.$$

Put $\gamma = \sum\limits_{j=1}^{n-1} z_j^2$. Unless $z_i = 0$, we have $(1 - z_i^4) - 2\gamma z_i^2 = 0$. Solving this equation for $z_i^2$ gives $z_i^2 = -\gamma \pm \sqrt{\gamma^2 + 1}$. However, since $z_i^2 > 0$, we have $z_i^2 = \sqrt{\gamma^2+1} - \gamma = z_j^2$ for every $i, j \le n-1$, for which $z_i, z_j \neq 0$. Suppose that $m = \mathrm{card}\{i : z_i \neq 0\} \le n-1$ and $|z_i| = \lambda$ whenever $z_i \neq 0$. Thus $\lambda = \frac{1}{\sqrt[4]{1+2m}}$, and

$$F(z) = \frac{12}{\sqrt{n}}\big(1 + m\lambda^2(1-\lambda^4)^m\big) = \frac{12}{\sqrt{n}}\big(1 + \frac{m}{\sqrt{1+2m}}(1 - \frac{1}{1+2m})^m\big) \le$$

$$\le \frac{12}{\sqrt{n}} + \frac{12\sqrt{m}}{\sqrt{n}}(1 - \frac{1}{1+2m})^m \le K,$$

where $K$ is a constant independent of $n$ and $m < n$. Indeed, recall that $\lim\limits_{m\to\infty}(1 - \frac{1}{1+2m})^m = \frac{1}{\sqrt{e}}$. In order to estimate $B$, suppose WLOG $0 \le x_1 \le x_2 \le \cdots \le x_n$. Then

$$\frac{16}{\sqrt{n}}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} x_i^3 x_j^3 \prod_{\substack{k=1\\ k\neq i,j}}^{n}(1 - x_k^4) \le \frac{16}{\sqrt{n}}\Big(\sum_{i=1}^{n-2}\sum_{j=1}^{n-2} x_i^3 x_j^3 \prod_{k=1}^{n-2}(1 - x_k^4) + S_0 + S_1 + 1\Big)$$

where $S_\tau = x_{n-\tau}^3 \sum\limits_{j=1}^{n-2} x_j^3 \prod\limits_{k=1}^{n-2}(1 - x_k^4)$. By comparing this expression with the formula for $F(x)$, and keeping in mind that $x_j^2 \ge x_j^3$ we get $\frac{16}{\sqrt{n}}(S_0 + S_1 + 1) \le 4K$. In order to estimate $B$, set (again for reasons of symmetry of $\nabla G$ which makes the calculations easier)

$$G(x) = \frac{16}{\sqrt{n}}\Big(\sum_{i=1}^{n-2}\sum_{\substack{j=1\\ j\neq i}}^{n-2} x_i^3 x_j^3 \prod_{k=1}^{n-2}(1 - x_k^4) + \sum_{i=1}^{n-2}\frac{1}{2}x_i^6 \prod_{k=1}^{n-2}(1 - x_k^4)\Big),$$

and note that clearly

$$\max_{x \in B_{c_0^{n-2}}} G(x) + 4K \ge \max_{x \in B_{c_0^{n-2}}} G(x) + \frac{16}{\sqrt{n}}(S_0 + S_1 + 1) \ge B.$$

Suppose $z \in B_{c_0^{n-2}}$, $G(x) = \max\limits_{x \in B_{c_0^{n-2}}} G(x)$ (and WLOG $z_i \ge 0$). In case $z \in \partial B_{c_0^{n-2}}$, we have $z_i = 1$ for some $i$ and $G(z) = 0$. Thus $z \notin \partial B_{c_0^{n-2}}$ and so $\nabla G(z) = 0$. A straitforward calculation gives

$$\frac{\partial G}{\partial x_i}(z) = \frac{16}{\sqrt{n}}\big(3z_i^2 \prod_{k=1}^{n-2}(1 - z_k^4)\alpha - 4z_i^3 \prod_{\substack{k=1\\ k\neq i}}^{n-2}(1 - z_k^4)\beta\big) = 0$$

where $\alpha = \sum_{j=1}^{n-2} z_j^3$ and $\beta = \sum_{l=1}^{n-2}\sum_{j=1}^{n-2} z_l^3 z_j^3 + \sum_{j=1}^{n-2} \frac{1}{2} z_j^6$. Therefore, whenever $z_i \neq 0$, we have $3(1-z_i^4)\alpha - 4z_i\beta = 0$. Thus $z_i, z_j \neq 0$ implies $\frac{1}{z_i} - z_i^3 = \frac{1}{z_j} - z_j^3$. As the real function $\phi(t) = \frac{1}{t} - t^3$ is decreasing on $I\!R^+$, this gives $z_i = z_j = \lambda$. Denote by $m = \operatorname{card}\{i : z_i \neq 0\}$. We have

$$G(z) = \frac{16}{\sqrt{n}}\big(m^2\lambda^6(1-\lambda^4)^m + \frac{1}{2}m\lambda^6(1-\lambda^4)^m\big) \le 2\frac{16}{\sqrt{n}}m^2\lambda^6(1-\lambda^4)^m.$$

In order to estimate the last expression, fix $m$ and define a function $\phi(\lambda) = \lambda^6(1-\lambda^4)^m$. On the interval $[0,1]$ $\phi$ has only one critical (and clearly a local maximum) point $\lambda = \sqrt[4]{\frac{3}{3+2m}}$. So $G(z) = \frac{2\cdot 16}{\sqrt{n}}m^2\big(\frac{1}{1+\frac{2}{3}m}\big)^{\frac{3}{2}}\big(1-\frac{1}{1+\frac{2}{3}m}\big)^m$. Since $m \le n-2$ and $\lim_{m\to\infty}\big(1-\frac{1}{1+\frac{2}{3}m}\big)^m = e^{-\frac{3}{2}}$, there exists a constant $L$, independent of the values $n, m < n$, for which $G(z) \le L$. Finally, setting $C = L + 5K \ge A + B$ satisfies the requirements.

■

## 4. Range of $C^{1,1}$ smooth operator.

Using the functions constructed above, we are now going to construct an $C^{1,1}$-smooth noncompact operator such that the set $T(B_{c_0})$ does not contain a translate of the canonical basis of $c_0$ (and consequently fails Question 0.5). This phenomenon cannot occur with polynomials, or real analytic operators. In fact we are able to control the "positive" span of $T(B_{c_0})$ as well. However, using negative coefficients generates the copy of $c_0$ in the range. Changing the construction somewhat, we are able to eliminate $c_0$ basis from spans containing a limited number of negative coordinates. We do not present these modifications here (as they are technical and do not suffice for a general counterexample), but they may shed some light on the delicacy of the problem.
Let $T : c_0 \to \ell_\infty$ be an operator, $T(x) = (f_n(x))_{n=1}^\infty$.

**Lemma 4.1**
Let $T : B_{c_0} \to \ell_\infty$ be a $C^1$-smooth operator. Then $T' : B_{c_0} \to \mathcal{L}(c_0, \ell_\infty)$ is uniformly continuous with modulus of continuity $\omega(t)$ iff every $f_n' : B_{c_0} \to \ell_1$ is uniformly continuous with modulus of continuity $\omega(t)$.

**Proof.** Consider an infinite matrix $(a_{ij})_{i,j=1}^\infty$, which represents $L \in \mathcal{L}(c_0, \ell_\infty)$. More precisely,

$$L(e_k) = (a_{ik})_{i=1}^\infty \in \ell_\infty.$$

Since $L$ is bounded, we have

$$\sup_{i\in I\!N} \sum_{k=1}^\infty |a_{ik}| = \|L\|_{\mathcal{L}(c_0,\ell_\infty)}.$$

Put $g_i = (a_{ik})_{k=1}^{\infty} \in \ell_1$. We can write $L = (g_i)_{i=1}^{\infty}$, $\|L\|_{\mathcal{L}(c_0,\ell_\infty)} = \sup_{i\in I\!N} \|g_i\|_{\ell_1}$. Now given $x, y \in B_{c_0}$, $\|x-y\| = t$, $\big(f_n'(x)\big)_{n=1}^{\infty} = T'(x) = L = (g_i)_{i=1}^{\infty}$, $\big(f_n'(y)\big)_{n=1}^{\infty} = T'(y) = S = (h_i)_{i=1}^{\infty}$ we have

$$\|L - S\|_{\mathcal{L}(c_0,\ell_\infty)} = \sup_{i\in I\!N} \|g_i - h_i\|_{\ell_1}.$$

Clearly, $\|L - S\|_{\mathcal{L}(c_0,\ell_\infty)} \le \omega(t)$ iff for every $i \in I\!N$ $\|g_i - h_i\|_{\ell_1} \le \omega(t)$.

■

In the rest of the note we will construct simultaneously a Banach space $X \hookrightarrow \ell_\infty$ and a $C^{1,1}$-smooth and noncompact operator $T : B_{c_0} \to X$, such that $T(B_{c_0})$ does not "contain" a canonical basis of $c_0$.

First, let $C$ be from Theorem 3.3, fix a sequence $n_i = 2^{4i}$, and put $\psi_{n_i} : I\!R^{n_i} \to I\!R$,

$$\psi_{n_i}(x) = \frac{1}{2^i C}\Big(\frac{1}{\sqrt{n_i}} - \phi_{n_i}(x)\Big).$$

Clearly, $\psi_{n_i}(0) = 0$, $\psi_{n_i}'(0) = 0$, $\psi_{n_i}$ is symmetric and $\|\psi_{n_i}''\| \le 2^{-i}$ on $B_{c_0^{n_i}}$. Since $\psi_{n_i}$ is a symmetric function, given $A \subset I\!N$, $|A| = n_i$, we may put $\psi_{n_i}^A : B_{c_0} \to I\!R$ to be

$$\psi_{n_i}^A\big((x_j)_{j=1}^{\infty}\big) = \psi_{n_i}\big((x_j)_{j\in A}\big).$$

The system of tuples of sets

$$\mathcal{S}_k = \{(A_1, A_2, \ldots, A_l) : A_i \subset I\!N, |A_1| < |A_2| < \cdots < |A_l|, |A_i| \in \{n_i\}_{i=1}^{\infty}, |A_l| = n_k\}$$

is countable, and so is $\mathcal{S} = \bigcup_{k=1}^{\infty} \mathcal{S}_k$. For $(A_1, \ldots, A_l) \in \mathcal{S}$ put $\psi^{(A_1,\ldots,A_l)} : B_{c_0} \to I\!R$, $\psi^{(A_1,\ldots,A_l)}(x) = \sum_{i=1}^{l} \psi_{|A_i|}^{A_i}(x)$. Fix a bijection $\omega : \mathcal{S} \to I\!N$. We define $\tau_n : B_{c_0} \to I\!R$ by $\tau_n(x) = \psi^{\omega^{-1}(n)}(x)$, and $T : B_{c_0} \to \ell_\infty$ by $T(x) = (\tau_1(x), \tau_2(x), \ldots)$. By Lemma 4.1, $T$ is $C^{1,1}$-smooth. We define $X = \overline{\mathrm{span}} T(B_{c_0}) \hookrightarrow \ell_\infty$.

**Theorem 4.2**
$T : B_{c_0} \to X$ is a noncompact, $C^{1,1}$-smooth operator, with the property that there is no sequence $\{y^n\}_{n=0}^{\infty}$ in $T(B_{c_0})$ such that $\{y^n - y^0\}_{n=1}^{\infty}$ is equivalent to the canonical basis of $c_0$.

**Proof.** It remains to prove the statement about $\{y^n - y^0\}_{n=1}^{\infty}$. We proceed by contradiction, assuming $y^n = T(u^n)$, where $u^n \in c_{00}$. Clearly, by passing to a subsequence of $\{u^n\}_{n=1}^{\infty}$ WLOG there exists some $m \in I\!N$, $\delta > 0$ and a sequence $m < j_1 < j_2 < \ldots$, such that $\mathrm{supp}(u^0) \subset [1, m]$, $u_{j_n}^n > \delta$. Take a set $A \subset \{j_k\}_{k=1}^{\infty}$, $|A| = n_p = 2^{4p}$. We have

$$\psi^A_{n_p}(u^n) = \frac{1}{2^pC}\frac{1}{\sqrt{n_p}}\big(1 - \prod_{i\in A}(1 - u_i^{n\,4})\big) \geq \frac{1}{2^pC\sqrt{n_p}} \cdot \delta^4 \text{ for } n \in A,$$
$$\psi^A_{n_p}(u^0) = 0.$$

Thus $\tau_{\omega((A))}(u^0) = 0$, $y^i_{\omega(A)} = \tau_{\omega((A))}(u^j) \geq \frac{1}{2^pC}\frac{1}{\sqrt{n_p}}\delta^4$. So $\|\sum_{j\in A}(y^j - y^0)\| \geq \sum_{j\in A} y^j_{\omega(A)} \geq \frac{1}{2^pC}\cdot\sqrt{n_p}\delta^4$, which is a contradiction, since the last expression can be made arbitrarily large (by the choice of $p$). ■

In fact, our construction enables us to prove a somewhat more general statement. Let $N_n \in I\!N$, $a^n_i \in I\!R$ where $1 \leq i \leq N_n$, $y^{n,i} = T(x^{n,i})$. The main conjecture on containment of $c_0$ in $X$ would be disproved if for each such system, $\{\sum_{i=1}^{N_n} a^n_i y^{n,i}\}_{n=1}^\infty$ is not equivalent to the canonical basis of $c_0$. We are able to prove this statement under assumption that $a^n_i \geq 0$. This is not sufficient to ensure that $c_0 \not\hookrightarrow X$, and in fact in our construction the sequence $\{T(\sum_{i=1}^{k+1} e_i) - T(\sum_{i=1}^{k} e_i)\}_{k=1}^\infty$ is equivalent to $\{e_k\}_{k=1}^\infty$ and thus $c_0 \hookrightarrow X$. However, further modifications of our construction may lead to the full counterexample. Since the following result is not central in this work, we present only a sketch of the argument.

**Proposition 4.3**

In the notation above, assume that $a^n_i \geq 0$. Then $\{\sum_{i=1}^{N_n} a^n_i y^{n,i}\}_{n=1}^\infty$ is not equivalent to the canonical basis of $c_0$.

**Sketch of Proof.** Assume, By contradiction, that $\{\sum_{i=1}^{N_n} a^n_i y^{n,i}\}_{n=1}^\infty$ is equivalent to the canonical basis of $c_0$. Let $\{m_n\}_{n=1}^\infty$ be a sequence form $I\!N$ such that $\exists\Delta > 0$

$$\sum_{i=1}^{N_n} a^n_i y^{n,i}_{m_n} \geq \Delta.$$

We will distinguish two cases (which involve passing to subsequences).
*Case I.*
$\exists\{m_n\}_{n=1}^\infty$ as above and such that

$$\omega^{-1}(m_n) = (A^n_1, \ldots, A^n_{l_n}) \text{ where } \lim_{n\to\infty}|A^n_1| = \infty.$$

*Case II.*
$$\lim_{\substack{n\to\infty\\k\to\infty}} \sup_{|A_1|=n_k} \sum_{i=1}^{N_n} a^n_i y^{n,i}_{\omega(A_1,\ldots,A_l)} = 0$$

In *Case I*, we may clearly assume, by passing to a subsequence, that $\tilde{n} > n$ implies $|A^n_{l_n}| < |A^{\tilde{n}}_1|$. Thus

$$\sum_{i=1}^{N_n} a_i^n y^{n,i}_{\omega(A_1^1,\dots,A_{l_1}^1,A_1^2,\dots,A_{l_2}^2,\dots,A_1^{\tilde n},\dots,A_{l_{\tilde n}}^{\tilde n})} \geq$$
$$\geq \sum_{i=1}^{N_n} a_i^n y^{n,i}_{\omega(A_1^n,\dots,A_{l_n}^n)} \geq \Delta.$$

In particular, $\| \sum_{n=1}^{p} \sum_{i=1}^{N_n} a_i^n y^{n,i} \|_\infty \geq p{\cdot}\Delta$ a contradiction. (*Note that this case can be handled without the assumption* $a_i^n \geq 0$.)
In *Case II*, we may WLOG assume that $\exists p \in I\!N$ such that $\omega^{-1}(m_n) = (A^n)$, $|A^n| = n_p$. Next, choose a set $A$, $|A| = n_r$, $A^n \subset A$ for $1 \leq n \leq \frac{n_r}{n_p}$. It is easy to observe that

$$\psi^A_{n_r}(x) \geq \frac{1}{2^r C} \frac{1}{\sqrt{n_r}} \big(1 - \prod_{i \in A^n} (1 - x_i^4)\big) = \frac{1}{2^{r-p}} \frac{\sqrt{n_p}}{\sqrt{n_r}} \psi^{A^n}_{n_p}(x).$$

Thus (due to $a_i^n \geq 0$)

$$\sum_{n=1}^{\frac{n_r}{n_p}} \sum_{i=1}^{N_n} a_i^n y^{n,i}_{\omega(A)} \geq \frac{n_r}{n_p} \frac{1}{2^{r-p}} \sqrt{\frac{n_p}{n_r}} \Delta = 2^{r-p} \Delta$$

a contradiction. ■

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