Smooth functions on C(K)

Petr Hájek

§1. Introduction.

S. Bates has recently investigated separable Banach spaces X satisfying the condition that for every separable Banach space Y there exists a surjective C^∞-Fréchet smooth (nonlinear) operator from X onto Y. We will denote the class of all these spaces by $\mathcal{B}$. Bates has shown that every separable superreflexive space belongs to $\mathcal{B}$ and he also characterized spaces for which his method of proof fails.

Theorem 1 (Bates)
Let $X \notin \mathcal{B}$ be an infinite dimensional Banach space. Then at least one of the following conditions hold:
(i) Every seminormalized weakly null sequence in X^* has a subsequence with a spreading model isomorphic to ℓ_1
(ii) X^* has the Schur property.

Natural examples of spaces satisfying (i) or (ii) are c_0 and the original Tsirelson space, and Bates asked whether indeed $c_0 \notin \mathcal{B}$. The question was settled in [9] (i.e. $c_0 \notin \mathcal{B}$), a paper which was conducted without any knowledge of S. Bates' work, and which was mainly concerned with the behavior of C^2-smooth real functions on c_0.
In order to reveal the connection between these matters, let us denote by $\mathcal{C}$ the class of Banach spaces X such that for any real function f defined on an open subset $\mathcal{U}$ of X, with locally uniformly continuous derivative, f' is locally compact. That is to say, for every $x \in \mathcal{U}$ there exists open neighbourhood $\mathcal{V} \subset \mathcal{U}$, $x \in \mathcal{V}$, such that $f'(\mathcal{V})$ is relatively compact in X^*.
A simple use of the Baire category principle implies that if $T : X \to Y$ is a surjective operator with locally continuous derivative (e.g. C^2-Fréchet smooth), and $X \in \mathcal{C}$, then $Y \in \mathcal{C}$. If, on the other hand, $Y \in \mathcal{B}$ then $X \in \mathcal{B}$. Since $\ell_2 \in \mathcal{B}$, $\ell_2 \notin \mathcal{C}$ we have the following implication: $X \in \mathcal{C} \implies X \notin \mathcal{B}$, and moreover whenever $Y \in \mathcal{B}$, there exists no surjective $Y : X \to Y$ with locally uniformly continuous derivative. A little more can be said under some additional assumptions.

Proposition 2.

Let $X \oplus X \in \mathcal{C}$, Y be an infinitely dimensional Banach space with nontrivial type, $T : X \to Y$ be a Fréchet differentiable operator with locally uniformly continuous derivative. Then T is locally compact.

The proof of this statement is identical with that of Corollary 11 of [9], using Lemma 5 below instead of Corollary 10 of [9]. It should be noted that some additional assumptions must be put on Y, because as follows from the Josefson-Nissenzweig theorem, every infinite dimensional Banach space admits a noncompact linear operator into c_0.
Proposition 2 is particularly useful if $X \oplus X \cong X$, as is the case when $X = C(K)$, K countable, or $X = T^*$ (the original Tsirelson space) (for these results see [4], [5]). Thus in what follows, we will be mainly interested in showing that $X \in \mathcal{C}$ for these spaces.
In section 2 we develop methods from [9] to show that the original Tsirelson space T^* belongs to $\mathcal{C}$. Also, $C(K)$, K scattered, belong to $\mathcal{C}$. On the other hand, the Schreier space B ([5,10,11]) yields an example of a polyhedral subspace of $C(\omega^\omega)$ which belongs to $\mathcal{B}$. In particular, B is an example of a subspace of $C(\omega^\omega)$ which is not a quotient of $C(K)$, K scattered.
In section 3 we prove a somewhat finer statement that there exists no surjective operator from c_0 onto T^* or from T^* onto c_0 with locally uniformly continuous derivative. This suggests that there may be many "incomparable elements" with respect to smooth surjections.
Section 4 is devoted to proving certain estimates for homogeneous polynomials on c_0^n, independent of n and the degree of the polynomial, in the spirit of [2].

We are indebted to R. Haydon who first observed that the methods of [9] apply also in case of the Tsirelson space, and who informed us about S. Bates' work.

Our paper is a natural continuation of [9], but for the convenience of the reader we will repeat some important definitions and statements.
Let X, Y be real Banach spaces. We say that an operator $T : X \to Y$ is locally compact if for every $x \in X$ there exists an open neighbourhood $x \in \mathcal{U}$, such that $T(\mathcal{U})$ is norm relatively compact in Y. We say that T is weakly (w)-sequentially continuous on $\mathcal{U} \subset X$ if it maps w-Cauchy sequences from $\mathcal{U}$ into norm convergent ones.
A *modulus of continuity* for a given uniformly continuous function f from a metric space $\big(X_1, \mathrm{d}_1\big)$ into a metric space $\big(X_2, \mathrm{d}_2\big)$ is an increasing real function $\omega(\delta)$, $\delta \geq 0$, $\lim_{\delta \to 0} \omega(\delta) = 0$, such that

$$\mathrm{d}_1(x_1, x_2) \leq \delta \quad \text{implies} \quad \mathrm{d}_2(f(x_1), f(x_2)) \leq \omega(\delta).$$

The following two statements have been proved in [9], and will be used frequently.

Lemma 3.
Let $\varepsilon > 0$, f be a real function on $B_{c_0^m}$ with uniformly continuous derivative (with modulus of continuity $\omega(\delta)$) and such that $\sup_{B_{c_0^m}} \|f'\|_1 \le \omega(2)$. Let $v \in B_{c_0^m}$ and $\{u_i\}_{i=1}^n$ be a block sequence such that $v + u_i \in B_{c_0^m}$. If n is large enough (the estimate depends only on $\omega(\delta)$), then $\min_{1\le i\le n} |f(v+u_i) - f(v)| < \varepsilon$.

Lemma 4.
Let f be a Fréchet differentiable real function with uniformly continuous derivative defined on B_{c_0}. Then f is weakly sequentially continuous on B_{c_0}.

§2. The class $\mathcal{C}$.

Before we state our next lemma, let us remark that if $\ell_1 \hookrightarrow X$, then by classical results in [7], ℓ_2 is a linear quotient of X, so $X \in \mathcal{B}$.

Lemma 5.
Let X be a Banach space, $\ell_1 \not\hookrightarrow X$. Let f be a real function with uniformly continuous derivative on B_X. TFAE:
(i) f is w-sequentially continuous
(ii) $f'(B_X)$ is relatively compact.

Proof: **(ii) $\implies$ (i).** Since $K = \overline{f'(B_X)}$ is norm compact, given a weakly Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in B_X we have:

$$\lim_{n,m\to\infty} \langle \phi, x_n - x_m \rangle = 0 \text{ uniformly in } \phi \in K.$$

By the mean value theorem, for some point x in the interval joining x_n and x_m, we have:

$$|f(x_n) - f(x_m)| = \langle f'(x), x_n - x_m \rangle| \le \sup_{\phi\in K} |\langle \phi, x_n - x_m \rangle| \to 0 \text{ as } m,n \to \infty.$$

(i) $\implies$ (ii). Denote $\omega(\delta)$ the modulus of continuity of f' on B_X. Note that f is Lipschitz on B_X. If $f'(B_X)$ is not relatively compact, there exist $\varepsilon > 0$ and (by Rosenthal's theorem) a w-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(1-\varepsilon)B_X$ such that $f_n = f'(x_n)$ satisfy

$\frac{1}{\varepsilon} > \|f_n\| > \varepsilon, \|f_n - f_m\| > \varepsilon$. If $\lim f(x_n)$ does not exist, we are done. Otherwise, by standard argument, we may in addition assume that $f(x_1) = f(x_n), n \in I\!N$. By induction, we find a subsequence n_k of $I\!N$ and a sequence $\{y_k\}_{k\in I\!N}$ in B_X such that:

$$|(f_{n_k} - f_{n_l})(y_l)| > \frac{\varepsilon}{4} \text{ for } k > l \tag{1}$$

$$|(f_{n_{k_1}} - f_{n_{k_2}})(y_l)| < \frac{\varepsilon}{100} \qquad \text{for } k_1, k_2 > l. \tag{2}$$

This is done as follows: Choose $y_1 \in B_X$ such that $(f_1 - f_2)(y_1) > \frac{\varepsilon}{2}$. There exists an increasing subsequence $\{n^1_k\}_{k\in I\!N}$ of $I\!N$ satisfying (2) for $l = 1$, and satisfying (1) for either $n_1 = 1$ or $n_1 = 2$. Fix the choice of n_1 and assume $n_1 < n^1_1$. Find $y_2 \in B_X$ such that $(f_{n^1_1} - f_{n^1_2})(y_2) > \frac{\varepsilon}{2}$. There exists an increasing subsequence $\{n^2_k\}_{k\in I\!N}$ of $\{n^1_k\}_{k\in I\!N}$ satisfying (2) for $l = 2$ and (1) for either $n_2 = n^1_1$ or $n_2 = n^1_2$. We continue in an obvious manner.

We may assume that $\{y_k\}_{k\in I\!N}$ is w-Cauchy. Conditions (1) and (2) imply that for every $k > 3$ we have either $|f_{n_k}(y_{k-2} - y_{k-1})| > \frac{\varepsilon}{8}$ or $|f_{n_{k-1}}(y_{k-2} - y_{k-1})| > \frac{\varepsilon}{8}$. Passing to a suitable subsequence of $\{y_{k-2} - y_{k-1}\}_{k\in I\!N}$ and $\{f_{n_k}\}_{k\in I\!N}$, we obtain a w-null sequence $\{z_l\}_{l\in I\!N}$ such that $f_{n_l}(z_l) > \frac{\varepsilon}{8}$. For $\alpha > 0$ small enough, we have $x_{n_l} + \alpha z_l \in B_X$ and $f(x_{n_l} + \alpha z_l) > f(x_{n_l}) + \frac{1}{2}\alpha\frac{\varepsilon}{8}$. This is a contradiction, since $x_1, x_{n_1} + \alpha z_1, x_2, x_{n_2} + \alpha z_2, \ldots$ is w-Cauchy.

♢

Proposition 6.
Let T^* be the original Tsirelson space, then $T^* \in \mathcal{C}$.

Proof: Let f be a real function on B_{T^*} with uniformly continuous Fréchet derivative. Since T^* is reflexive, using Lemma 5 it is enough to show that $f(x_n)$ converges to $f(0)$ for every w-null sequence in B_{T^*}. Assume the contrary, i.e. for some $\{x_n\}_{n\in I\!N}$, which may be choosen to be a l-normalized block sequence, and some $\varepsilon > 0$, $|f(x_n) - f(0)| > \varepsilon$. By properties of T^* [5], for every $N \in I\!N$, $x_N, x_{N+1}, \ldots, x_{2N}$ is $2l$-equivalent to the canonical basis of c_0^N. This is a contradiction with Lemma 3.

♢

Clearly, the same proof also works for T^*_θ, $0 < \theta < 1$ (see [5]), so we have a continuum of mutually totally incomparable reflexive spaces from $\mathcal{C}$.

Theorem 7.
Let K be a scattered compact. Then $C(K) \in \mathcal{C}$.

Proof: Since every separable subspace of $C(K)$, K scattered, is contained in a separable subspace of $C(K)$ isomorphic to $C(K_1)$, where K_1 is countable (e.g. [6]), we may assume that K is countable.
Bessaga and Pelczynski in [4] provided an isomorphic classification of $C(K)$ spaces, K countable, as those isomorphic to $C[0,\alpha]$, where α is a countable ordinal. We will prove by transfinite induction on $\alpha \in [\omega_0, \omega_1)$ that a function $f : B_{C[0,\alpha]} \to I\!R$ with a uniformly continuous derivative is w-sequentially continuous on $B_{C[0,\alpha]}$.
Case $\alpha = \omega_0$ was proved in [9].
Inductive step.
Assume our claim is true for all $\alpha \in [\omega_0, \beta), \beta \in [\omega_0, \omega_1)$. We may clearly assume that $C[0,\alpha] \not\cong C[0,\beta]$ for $\alpha < \beta$ and β is a limit ordinal. Choose an increasing sequence $\alpha_k \nearrow \beta$ such that $[\alpha_k + 1, \alpha_{k+1}]$ are clopen. We will work in $C_0[0,\beta]$, continuous functions on $[0,\beta]$ which vanish at β, as $C_0[0,\beta] \cong C[0,\beta]$. Define for $l < m$, $P^{l,m} : C_0[0,\beta] \to C_0[0,\beta]$ as

$$P^{l,m}(\phi)(\alpha) = \begin{cases} 0 & \text{if } \alpha \in [\alpha_l + 1, \alpha_m] \\ \phi(\alpha) & \text{otherwise.} \end{cases}$$

Let $f : B_{c_0} \to I\!R$ have a uniformly continuous derivative on B_{c_0}, $f(0) = 0$, $f'(0) = 0$.
Let us assume, by contradiction, that there is a w-Cauchy sequence $\{\phi\}_{n \in I\!N} \in B_{C_0[0,\beta]}$, $f(\phi_{2n}) < 0$, $f(\phi_{2n+1}) > 1$ and each ϕ_n is supported by $[0, \alpha_i]$ for some $i \in I\!N$. Similarly to Claim 7 of [9], and with the same proof, we obtain that there is $k \in I\!N$ and some infinite sets M_1 of odd integers and M_2 of even integers satisfying, whenever $k \le l < m$:

$$f\big(P^{l,m}(\phi_n)\big) < \frac{1}{4} \quad \text{for all but finitely many } \ n \in M_2,$$

$$f\big(P^{l,m}(\phi_n)\big) > \frac{3}{4} \quad \text{for all but finitely many } \ n \in M_1.$$

We may assume $k = 1$ and using the above claim pass to another subsequence $\{\phi_{p_i}\}_{i \in I\!N}$, $p_i \in M_2$ for i even, $p_i \in M_1$ for i odd, such that

$$f(\psi_{2k}) < \frac{1}{4}, \;\; f(\psi_{2k+1}) > \frac{3}{4}, \;\; \text{where } \; \psi_i = P^{1,i}(\phi_{p_i}).$$

In addition, we may also assume $\mathrm{supp}(\psi_i) \subset [0,\alpha_1] \cup [\alpha_i + 1, \alpha_{i+1}]$. By construction, $\{\psi_i\}_{i\in I\!N}$ is w-Cauchy. Consider the linear operator $L : C[0,\alpha_1] \cong C[0,\alpha_1] \oplus c_0 \to C_0[0,\beta]$, defined by formulas:

$$L\big((\phi,0)\big) = \phi,$$

$$L\big((0,e_i)\big) = \psi_i\Big|_{[\alpha_i+1,\alpha_{i+1}]}.$$

The real function $f \circ L$ has uniformly continuous Fréchet derivative on $B_{C[0,\alpha_1]}$, but $f \circ L\big((\psi_i\Big|_{[0,\alpha_1]}, e_i)\big)$ is not convergent, a contradiction.

$\diamondsuit$

The following suprising example, based on a construction of Schreier [11], was investigated in [10].

Example 8.
There exists a subspace B of $C(\omega^\omega)$ with unconditional shrinking basis $\{e_n\}$ and a biorthogonal basis $\{e_n^*\}$ such that $e_n^* \xrightarrow{w} 0$ and the spreading model built on $\{e_n^*\}$ is c_0.

It follows immediately from Theorem 1, that $B \in \mathcal{B}$. Using [10], one can show by standard argument that the canonical injection from B into ℓ_2 is bounded. Yet, the space B as a subspace of a polyhedral space is itself (isomorphically) polyhedral and thus saturated by copies of c_0. The space B also indicates that the structure of w-Cauchy sequences in general $C(K)$, K scattered, is more complicated than that of c_0. This is the main obstacle in trying to prove analogous statements to Proposition 11 for $C(K)$ instead of c_0. On the other hand, leaning on the results from [8], with a little bit of work one can show that every w-Cauchy sequence in the Hagler space JH contains a subsequence equivalent to either the canonical or the summing basis of c_0. By Lemma 4 and 5 $JH \in \mathcal{C}$.

§3. Operators from $\mathbf{c_0}$.

The main Proposition 11 of this section implies that a C^2-smooth operators from c_0 into a space Y with an unconditional basis is locally compact unless $c_0 \hookrightarrow Y$. Together with Proposition 9 this statement implies that there is no surjective C^2-smooth operator from c_0 onto T^* or vice versa.

Proposition 9.
Let $X \in \mathcal{C}$ be a reflexive space, $T : X \to Y$ be an onto operator with locally uniformly continuous Fréchet derivative. Then $Y \in \mathcal{C}$ is reflexive.

PROOF: The fact that $Y \in \mathcal{C}$ is valid in general without the reflexivity assumption on X. Indeed, it is an easy application of the Baire category principle which implies that for some open $\mathcal{U} \subset X$ such that $T\Big|_{\mathcal{C}}$ has uniformly continuous Fréchet derivative, $\overline{T(\mathcal{U})}$ has nonempty interior. To show that Y is reflexive, note that for every open ball $\mathcal{U} \subset X$ such that $T\Big|_{\mathcal{U}}$ has uniformly continuous derivative, by Lemma 5, T maps weakly convergent sequences from $\mathcal{U}$ into weakly convergent sequences in Y. By the Eberlein-Šmulyan theorem, $T(\mathcal{U})$ is relatively weakly compact. However, for some $\mathcal{U}$, $\overline{T(\mathcal{U})}$ must have nonempty interior and thus Y is reflexive.

$\diamondsuit$

In particular, there is no C^2 operator from T^* onto c_0.

Lemma 10.
Let $T : B_{c_0} \to Y$ be an operator with uniformly continuous Fréchet derivative on B_{c_0}. Assume that for every given $u \in B_{c_0}$ and $\{v_n\}_{n\in\mathbb{N}} \subset c_0$ equivalent to the canonical basis of c_0, such that $u + v_n \in B_{c_0}$, we have:

$$\lim_{n\to\infty} T(u + v_n) = T(u).$$

Then T is w-sequentially continuous on B_{c_0}.

PROOF: Assume, by contradiction, that T is not w-sequentially continous on B_{c_0}, i.e. there exist $\varepsilon > 0$ and a w-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \in B_{c_0}$ such that $T(x_n)$ is not convergent. If $\{T(x_n)\}_{n\in\mathbb{N}}$ is relatively compact, then there exists $y^* \in Y^*$ such that $\{y^* \circ T(x_n)\}_{n\in\mathbb{N}}$ is not convergent, a contradiction with Lemma 4. We therefore assume that $\{T(x_n)\}_{n\in\mathbb{N}}$ is not relatively compact. By passing to a subsequence, changing notation and disregarding quantities that can be made arbitrary small, we can assume that there is a w-Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \in B_{c_0}$ satisfying:

(i) $\mathrm{dist}\{\mathrm{span}\{T(x_1), \ldots, T(x_n)\}, T(x_{n+1})\} > \beta > 0$

(ii) $\{x_n\}$ are supported in an increasing sequence of finite intervals $I_n = [1, m_n]$

(iii) all the x_j, for $j > n$ are equal on I_n.

By assumption, for every x_n and every block sequence $\{y_k\}_{k\in\mathbb{N}}$ such that $x_n + y_k \in B_{c_0}$,

$$\lim_{k\to\infty} T(x_n + y_k) = T(x_n).$$

Thus for every $n \in \mathbb{N}$ there exists $l_n \in \mathbb{N}$, $l_n > m_n$ such that $\|T(x_n + u) - T(x_n)\| < \frac{\beta}{2}$ for every $u \in c_0$, $x_n + u \in B_{c_0}$, $\operatorname{supp}(u) \subset [l_n, \infty)$. Consequently, for every $N \in \mathbb{N}$ we can choose a finite sequence $x_{n_1}, \dots, x_{n_{2N}}$ satisfying $l_{n_i} < m_{n_{i+1}}, i = 1, \dots, 2N-1$. We obtain the following:

$$\|T(x_{n_i} + \chi_{[l_{n_i}, m_{n_N}]} \cdot x_{n_N}) - T(x_{n_i})\| < \frac{\beta}{2} \quad i = 1, \dots, 2N-1.$$

Put $u_{n_i} = x_{n_{2N}} - (x_{n_{2i}} + \chi_{[l_{n_{2i}}, m_{n_N}]} \cdot x_{n_N}), \quad i = 1, \dots, N-1$. Then u_{n_i} is a block 2-sequence, supported by $[m_{n_{2i-1}}, l_{n_{2i}}]$. Using (i), choose $y^* \in B_{Y^*}$ satisfying:

$$y^*(T(x_{n_i})) = 0 \quad i = 1, \dots, 2N-1,$$

$$y^*(T(x_{n_{2N}})) > \beta.$$

Thus we have $|y^* \circ T(x_{n_N} - u_{n_i})| < \frac{\beta}{2} \quad i = 1, \dots, N-1$, $|y^* \circ T(x_{n_N})| > \beta$. Because N is arbitrary large, it is a contradiction with Lemma 3.

$\diamondsuit$

Proposition 11.
Let Y be a separable Banach space with an unconditional basis. Suppose $T : c_0 \to Y$ has a locally continuous Fréchet derivative. Then either $c_0 \hookrightarrow Y$ or T is locally compact.

PROOF: We proceed by contradiction, assuming that $c_0 \not\hookrightarrow Y$ and T is not locally compact. By standard shifting and scaling arguments together with Lemma 10, we may assume that T has uniformly continuous derivative on B_{c_0}, $T(0) = 0$ and $\|T(e_k)\| \geq 2\varepsilon > 0$. Denote by $\{x_k\}_{k\in\mathbb{N}}$ the unconditional normalized basis of Y, $\{x_k^*\}_{k\in\mathbb{N}}$ its dual basis. By Lemma 4, $\{T(e_k)\}_{k\in\mathbb{N}}$ is w-null, so on passing to a subsequence we may assume that there exist a sequence $J_k = [i_k, j_k]$ of consecutive intervals of integers and $f_k \in B_{Y^*}$, $f_k \in \operatorname{span}\{x_{i_k}^*, \dots, x_{j_k}^*\}$, such that $f_k \circ T(e_k) > \frac{3\varepsilon}{2} > 0$. Put $P^k : Y \to Y$ to be a projection defined as $P^k(\sum_{i=1}^{\infty} \alpha_i x_i) = \sum_{i=i_k}^{j_k} \alpha_i x_i$. Our aim now is to pass to a subsequence $\{k_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that:

$$f_{k_l} \circ T(\sum_{i=1}^{n} e_{k_i}) \geq \varepsilon \text{ for every } 1 \leq l \leq n.$$

Before we present the construction, let us observe how this implies the statement of Proposition 11. By compactness, we may find an increasing sequence of integers $\{n_p\}_{p\in I\!N}$ such that for every $l \in I\!N$

$$\lim_{p\to\infty} P^{k_l}\big(T(\sum_{i=1}^{n_p} e_{k_i})\big) = u_l$$

exists.

By the unconditionality of $\{x_k\}_{k\in I\!N}$ and boundedness of T, $\{u_l\}_{l\in I\!N}$ forms a block basis in Y satisfying

$$C_1 \max\{|\alpha_l|\} \leq \|\sum_{l=1}^{n} \alpha_l u_l\| \leq C_2 \max\{|\alpha_l|\} \quad \text{ where } C_1, C_2 \geq \varepsilon.$$

In other words, $\{u_l\}_{l\in I\!N}$ is equivalent to the canonical basis of c_0.

The sequence $\{k_i\}_{i\in I\!N}$ is constructed by induction as follows. Given $r \in I\!N$, put n_r to be a large enough integer (Lemma 3) so that whenever $f \in B_{Y^*}$, $v \in B_{c_0}$, $\{u_i\}_{i=1}^{n_r} \in c_0$ are such that $v + u_i \in B_{c_0}$, and u_i are 1-equivalent to the canonical basis of $c_0^{n_r}$, we have

$$|f \circ T(v + u_i) - f \circ T(v)| < \big(\frac{\varepsilon}{2}\big)^{r+1}$$

for some $i \in [1, n_r]$.

Using Lemma 3 again, there exists $Q_1 \in I\!N$, $Q_1 > n_1$ such that $f_i \circ T(e_i + u) \geq (1 + \frac{1}{4})\varepsilon$ whenever $i \in [1, n_1]$, $u \in B_{c_0}, \mathrm{supp}(u) \subset [Q_1, \infty)$. On the other hand, for every $j > Q_1$ there exists some $i \in [1, n_1]$ such that $f_j \circ T(e_i + e_j) \geq (1 + \frac{1}{4})\varepsilon$. Thus there exists $k_1 \in [1, n_1]$ and an infinite increasing sequence $\{m_1^1, m_2^1, \dots\} = M_1 \subset I\!N$ such that for every $u \in B_{c_0}, \mathrm{supp}(u) \subset M_1$ and every $k \in M_1$ we have $k > k_1$ and

$$f_{k_1} \circ T(e_{k_1} + u) \geq (1 + \frac{1}{4})\varepsilon, \;\; f_k \circ T(e_{k_1} + e_k) \geq (1 + \frac{1}{4})\varepsilon.$$

Similarly, there exists $Q_2 > m_{n_2}^1$ such that $f_i \circ T(e_{k_1} + e_i + u) \geq (1 + \frac{1}{8})\varepsilon$ whenever $i \in \{m_1^1, \dots, m_{n_2}^1\}$, $u \in B_{c_0}, \mathrm{supp}(u) \subset [Q-2, \infty)$. Also, whenever $j > Q_2$, there exists $i \in \{m_1^1, \dots, m_{n_2}^1\}$ such that $f_j \circ T(e_{k_1} + e_i + e_j) \geq (1 + \frac{1}{8})\varepsilon$. Thus, there exist $k_2 \in$

$\{m_1^1, \dots, m_{n_2}^1\}$ and an infinite increasing sequence $\{m_1^2, m_2^2, \dots\} = M_2 \subset M_1$ such that for every $u \in B_{c_0}, \mathrm{supp}(u) \subset M_2$ and every $k \in M_2$ we have $k > k_2$ and

$$f_{k_2} \circ T(e_{k_1} + e_{k_2} + u) \geq (1 + \frac{1}{8})\varepsilon, \;\; f_k \circ T(e_{k_1} + e_{k_2} + e_k) \geq (1 + \frac{1}{8})\varepsilon.$$

The inductive process continues in an obvious manner, at the r-th step choosing $k_r \in \{m_1^{r-1}, \dots, m_{n_r}^{r-1}\} \subset M_{r-1}$ and a subset $M_r \subset M_{r-1}$ satisfying

$$f_{k_r} \circ T(\sum_{i=1}^{r} e_{k_i} + u) \geq (1 + \frac{1}{2^{r+1}})\varepsilon, \;\; f_k \circ T(\sum_{i=1}^{r} e_{k_i} + e_k) \geq (1 + \frac{1}{2^{r+1}})\varepsilon,$$

whenever $u \in B_{c_0}$, $\mathrm{supp}(u) \subset M_r$ and $k \in M_r$. This finishes the proof.

$\diamondsuit$

As an immediate consequence, there exists no C^2 operator form c_0 onto T^*.

§4. Analytic functions on $\mathbf{c_0}$.

In the last part of our paper, we will obtain a finer description of the behavior of real analytic functions on c_0, in the spirit of Lemma 4. A similar statement was obtained in the complex setting by Aron and Globevnik in [1]. In fact, using the standard complexification argument, their result implies our Proposition 13.

Our proof uses ideas from [2], but adds a new ingredient of estimating the second derivative, which yields certain estimates independent of the degree of the polynomial and is of independent interest.

We refer to [2] for most of our notation.

Given a real C^2-smooth function f on some domain $\mathcal{U}$ in c_0^n, we denote by $D^2 f : \mathcal{U} \to \mathcal{L}(c_0^n, \ell_1^n)$ the usual second derivative of f, which can be represented by a symmetric matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\dots,n}$. For $T \in \mathcal{L}(c_0^n, \ell_1^n)$, $\|T\|$ stands for the usual operator norm. Let us denote $\overline{\Delta} f = \sum_{i=1}^{n} |\frac{\partial^2 f}{\partial x_i^2}|$.

Lemma 12.

Let $f \in C^2$, $f : B_{c_0^n} \to I\!R$, $\|D^2 f\| \leq 1$ on $B_{c_0^n}$. Then $\overline{\Delta} f \leq 1$ on $B_{c_0^n}$.

Proof: Let $x \in B_{c_0^n}$. Put $T = D^2 f(x) = (a_{ij})_{i,j=1,\dots,n}$. Clearly,

$$\|T\| = \max\{\|T(x)\|_1, x = \sum_{i=1}^{n} \pm e_i\}.$$

For any choices of signs $\varepsilon_j = \pm 1, \delta_i = \pm 1, 1 \leq i, j \leq n$, we have

$$\|T\| \geq \sum_{i=1}^{n} |\sum_{j=1}^{n} \varepsilon_j a_{ij}| \geq \sum_{i=1}^{n} (\delta_i a_{ii} + \delta_i \sum_{j \neq i} \varepsilon_j a_{ij}).$$

Keeping δ_i fixed and averaging over all choices of ε_j we obtain $\|T\| \geq \sum_{i=1}^{n} \delta_i a_{ii}$, so $\|T\| \geq \overline{\Delta} f(x)$. ♢

Lemma 13.
Let p be a homogeneous polynomial of degree k on $B_{c_0^n}$. If $\overline{\Delta} p \leq 1$ on $B_{c_0^n}$, then $\sum_{i=1}^{n} |p(e_i)| \leq 16$.

Proof: We may assume that n is odd and p is a symmetric polynomial, and we need to prove our estimate with 8 rather than 16. Indeed, otherwise assuming $p(e_i) \geq 0$ (here is why we need a better estimate, in general we have to pass to a suitable subset of $\{e_i\}_{i=1}^{n}$, where the signs of f remain constant) we can consider $\tilde{p}$ defined on $B_{c_0^m}$, $m \geq n$, m odd, as

$$\tilde{p}(\sum_{i=1}^{m} a_i e_i) = \frac{1}{m!} \sum_{\pi \in \Pi_m} p(\sum_{i=1}^{n} a_{\pi(i)} e_i),$$

where Π_m is the group of permutations of $\{1, \ldots, m\}$. Clearly, $\tilde{p}$ is symmetric, $\overline{\Delta} \tilde{p} \leq 1$ and $\sum_{i=1}^{m} |\tilde{p}(e_i)| = \sum_{i=1}^{n} |p(e_i)|$.
Assume $p(x_1, \ldots, x_n) = \sum_{|\alpha|=k} a_\alpha x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, denote $\tilde{a}_i$ the coefficient by x_i^k. Clearly, $\sum_{i=1}^{n} |p(e_i)| = \sum_{i=1}^{n} |\tilde{a}_i|$. To estimate $\sum_{i=1}^{n} |\tilde{a}_i|$, consider the polynomial

$$q(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i \frac{\partial^2 p}{\partial x_i^2}.$$

Then q is a homogeneous polynomial of degree $k-2$, $|q| \leq 1$ on $B_{c_0^n}$ and due to the symmetry of p and n being odd, the leading coefficients of q by x_i^{k-2} are $(-1)^i k(k-1)\tilde{a}_i$. By Theorem 1.2 of [2], $k(k-1)\sum_{i=1}^{n} |\tilde{a}_i| \leq 4k^2$. Thus $\sum_{i=1}^{n} |\tilde{a}_i| \leq 8$, and the proof is completed. $\diamondsuit$

Unfortunately, uniform estimates of this type, independent of the dimension n and degree of the polynomial are not valid for nonhomogeneous polynomials (consider e.g. $\prod_{i=1}^{n}(1-x_i^4)$ on c_0^n). This is the reason for which no analogue of the following proposition is valid under the weaker assumption of C^2 smoothness rather than analyticity.

Proposition 14.
Let f be a real analytic function on some domain $\mathcal{U}$ in c_0, $0 \in \mathcal{U}$, $f(0) = 0$ and $f'(0) = 0$. Then there exists some $\varepsilon > 0$ such that $\sum_{i=1}^{\infty} |f(\varepsilon e_i)| < \infty$.

Proof: Let us assume that the Taylor series of $D^2 f$ at 0:

$$D^2 f(x) = P_0 + P_1(x) + P_2(x) + \dots,$$

where $P_k(x)$ is a k-homogeneous polynomial form c_0 into $\mathcal{L}(c_0, \ell_1)$, is uniformly convergent on εB_{c_0} and moreover satisfies

$$\sup_{x \in \varepsilon B_{c_0}} \|P_k(x)\| \leq K(1-\varepsilon)^k,$$

where K is some constant. By Lemma 12, 13 and an easy homogeneity argument we obtain $\sum_{i=1}^{\infty} |f(\varepsilon e_i)| \leq 16K\varepsilon^2 \sum_{k=0}^{\infty} (1-\varepsilon)^k = 16K\varepsilon$.

References

[1] R. Aron and J. Globevnik, *Analytic functions on c_0*, Rev. Mat. Univ. Complut. Madrid 2 (1989), 27-33.

[2] R. Aron, B. Beauzamy and P. Enflo, *Polynomials in many variables: real vs complex norms*, J. Aproximation Theory 74 (1993), 181-198.

[3] S.M. Bates, *On smooth, nonlinear surjections of Banach space*, to appear in Israel J. Math..

[4] C. Bessaga and A. Pelczynski, *Spaces of continuous functions (IV)*, Studia Math. 19 (1960), 53-62.
[5] P. Casazza and T. Shura, Tsirelson's space, LNM 1363, Berlin-Heidelberg-New York 1989.
[6] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Monograph Surveys Pure Appl. Maths. 64 (Pitman, 1993).
[7] J. Hagler, *Some more Banach spaces which contain* ℓ_1, Studia Math. 46 (1973), 35-42.
[8] J. Hagler, *A counterexample to several questions about Banach spaces*, Studia Math. 60 (1977), 289-308.
[9] P. Hájek, *Smooth functions on* c_0, to appear in Israel J. Math..
[10] A. Pelczynski and W. Szlenk, *An example of a non-shrinking basis*, Rev. Roum. Math. Pures et Appl. 10 n.7 (1965), 961-965.
[11] J. Schreier, *Ein Gegenbeispiel zur Theorie der schwachen Konvergenz*, Studia Math. 2 (1930), 58-62.

Department of Mathematics, University of Alberta, Edmonton, T6G 2G1, Canada,
and
Mathematical Institute, Czech Academy of Science, Žitná 26, Prague, Czech Rep.
E-mail address: phajek@vega.math.ualberta.ca