# Analytic approximations of norms in separable Banach spaces

Robert Deville
Department of Mathematics
Université de Bordeaux
Bordeaux, France

Vladimir Fonf
Khar'kov Institute of Railroad Transportation Engineers
Khar'kov, Ukraine

Petr Hájek
Department of Mathematics
University of Alberta
Edmonton, T6G 2G1, Canada

## Introduction

It is well known that in separable Banach spaces, or more generally in WCD Banach spaces, the existence of a $C^k$-Fréchet differentiable bump function implies the possibility of uniform approximation of continuous functions by $C^k$-smooth functions. However, the more subtle question of the uniform approximation on bounded sets of arbitrary equivalent norm on a Banach space by a $C^k$-smooth norm – assuming the existence of some equivalent $C^k$-smooth norm on the space – seems to be of a different nature, and until now there has not been avaliable examples of spaces with this property.

In our paper, we give a satisfactory affirmative answer to our question for several classes of separable normed spaces.

We show that normed spaces with countable algebraic basis, polyhedral Banach spaces, in particular $c_0$, $\ell_p$ spaces for $p$ even integer, and $L_p[0,1]$ for $p$ even integer allow for approximations by analytic norms. This result should be compared with [D] where it is proved that every Banach space with an equivalent $C^\infty$-smooth norm (bump) contains an isomorphic copy of $c_0$ or $\ell_p$, $p$ even integer.

We further show that spaces with Schauder basis that admit a $C^k$-smooth equivalent norm, whose all derivatives are bounded on bounded sets, also admit approximations by $C^k$-smooth norms (in general without bounded derivatives). We will comment on the boundedness condition later on.

Thus approximations in $\ell_p$, $L_p[0,1]$ for arbitrary $1 < p < \infty$ are settled in the best possible way, since by [Ku] $\ell_p$, $p$ non-even, does not admit a $C^k$ equivalent norm where $k > p$.

Since there is a natural correspondence between closed, convex and bounded (CCB) sets in a normed space, containing $\vec{0}$ as an interior point and their Minkowski functionals, the previous statements can be reformulated in the language of convex sets.

The proof of the above statements is done in two steps.

First it is showed that an arbitrary CCB set $S_1$, $\vec{0} \in \mathrm{int} S_1$, can be arbitrary well approximated by another CCB set $S_2 = \{x \in X,\ f_i(x) \le 1,\ i \in I\!N\}$ where $\{f_i\}_{i \in I\!N}$ are $C^k$-smooth convex functions, satisfying some other technical conditions. Above

all, for every $x \in \partial S_2$ there exists an $i \in I\!N$, $f_i(x) = 1$. (In case $f_i$ are linear they form the so-called boundary of the set $S_2$.)
Then a general Theorem 1.3 is applied. This theorem can be viewed as a nonlinear generalization of Theorem 1 from [H], using ideas from [FPWZ]. Intuitively, what this theorem does is "smoothenning the corners" of the body $S_2$.
The uniform boundedness conditions of the derivatives of $\{f_i\}_{i\in I\!N}$ in Theorem 1.3 are local. Yet some global boundedness condition on the derivatives of an equivalent norm on $X$ seem to be necessary in the first step of the constuction, in order to obtain $\{f_i\}_{i\in I\!N}$ that meet the local conditions.
Related to this is an example by [NS] of an equivalent norm on $\ell_2$ not allowing for approximations by $C^2$-smooth norms whose second derivative is uniformly continuous.
Since the first step of the proof differs for different classes of norms, we decided to prove the general smoothenning Theorem 1.3 in Section 1, and then deal with different classes of spaces in the subsequent sections.
Throughout the paper we use the standard notation and terminology of Banach space theory. By saying that a homogeneous function is of some class of smoothness we always mean away from the origin. Whenever we say Minkowski functional, we always mean corresponding to a CCB set containing $\vec{0}$ as an interior point. By saying that a CCB set $S_1$, $\vec{0} \in \mathrm{int} S_1$ in $(X, \|\cdot\|)$ is arbitrarily approximable by CCB sets from some class $\mathcal{C}$ we mean that for every $\varepsilon > 0$ there exists $S_2 \in \mathcal{C}$ such that

$$(1-\epsilon)S_2 \subset S_1 \subset (1-\epsilon)S_2.$$

This is equivalent to the approximations of the corresponding Minkowski functionals on bounded sets.

## Section 1

As we have already indicated, the question of the possibility of approximation of a given Minkovski functional on a Banach space $X$ by Minkovski functionals of higher order of smoothness is equivalent to the possibility of approximations of a CCB set $A$, $\vec{0} \in \mathrm{int} A$, by higer order smooth CCB sets, i.e. precisely the sets $F^{-1}\big([-\infty, 1]\big)$ where $F$ is a convex function of the corresponding smoothness. This is in fact the content of the Implicit Function Theorem, as stated in [DGZ], or [FPWZ] in the analytic case. Therefore it is our aim to find for a given CCB set $A$, $\vec{0} \in \mathrm{int} A$ a suitable smooth and convex function $F$, such that $F^{-1}([-\infty, 1])$ approximates $A$.
Let $(X, \|\cdot\|)$ be a normed space, $A$ be a CCB set in $X$, $\vec{0} \in \mathrm{int} A$. Let $\{f_i\}_{i\in I\!N}$ be a sequence of homogeneous, continuous and convex functions on $X$ such that

$$A = \{x,\ f_i(x) \le 1,\ i \in I\!N\}.$$

**Definition 1.1** We say that $\{f_i\}_{i\in I\!N}$ as above forms a countable generalized boundary (c.g.b.) of $A$ if for every $x \in \partial A$ there exists some $i \in I\!N$ such that $f_i(x) = 1$.

The following facts on complex spaces and functions can be found in [Ku] and references therein. Given $(X, \|\cdot\|)$ a real normed (Banach) space, we can pass to

its complexification $(X^c, \|\cdot\|^c)$ which, considered as a real normed (Banach) space, is isomorphic to $X \oplus X$ with a norm $\|\!|(x,y)|\!\| = \|x\| + \|y\|$.
For $P$ a $k$-homogeneous polynomial on $X$ and $A_P(x_1, \ldots, x_k)$ the corresponding symmetric $k$-linear form, we define a complexified polynomial $P^c$ on $X^c$ by the formula:

$$P^c\big((x,y)\big) = A_P(x+iy, x+iy, \ldots, x+iy).$$

Then

$$\|\!|P^c|\!\| \le 2^k \|A_P\| \le 2^k \frac{k^k}{k!} \|P\|.$$

For the last inequality see [N, p.7].
It follows from Stirling's formula that for some $K' > 0$

$$\frac{k^k}{k!} < K' \cdot e^k \text{ for every } k \in I\!N.$$

Find $K > 0$ such that $K \cdot \|\!|\cdot|\!\| \ge \|\cdot\|^c$. Then

$$\|P^c\|^c \le K' \cdot (2Ke)^k \cdot \|P\|.$$

Thus whenever $f$ is a real analytic function at $x \in X$ with the radius of convergence $r$, we can pass to its holomorphic complexification $f^c$ at $(x, \vec{0})$ with the $\|\cdot\|^c$-radius of convergence at least $\frac{r}{2Ke}$.

**Definition 1.2** Let $k \in I\!N \cup \{+\infty\} \cup \{\omega\}$. We say that a sequence $\{f_i\}_{i \in I\!N}$ of real functions defined on an open convex set $U \subset (X, \|\cdot\|)$ satisfies the condition (k) if the following holds:

(i) $f_i$ are convex and continuous.

(ii) If $k \in I\!N$ then for every $l \le k$ and every $\vec{0} \ne x \in U$ there exists a neighbourhood $O \subset U$ such that $f_i\big|_O$ are $C^k$-Fréchet differentiable and $\|D^l f_i\|\big|_O$ are uniformly bounded.

(iii) If $k = +\infty$ then for every $l \in I\!N$ and $\vec{0} \ne x \in U$ there exists a neighbourhood $O \subset U$ such that $f_i\big|_O$ are $C^\infty$-Fréchet differentiable and $\|D^l f_i\|\big|_O$ are uniformly bounded.

(iv) If $k = \omega$ then $f_i$ are real analytic on $U \setminus \{\vec{0}\}$, and for every $\vec{0} \ne x \in U$ and $\delta > 0$ there exist an $r > 0$ and $i \in I\!N$ such that:

$$|f_j^c(z)| < 1 + \delta \text{ for } \|z - (x, \vec{0})\|^c < r \text{ and } j > i.$$

(v) The convex set $A = \{x,\ f_i(x) \le 1,\ i \in I\!N\}$ is bounded and $\vec{0} \in \mathrm{int} A$.

**Theorem 1.3** *Let $(X, \|\cdot\|)$ be a separable normed space, $D \subset X$ be a CCB set, $\vec{0} \in intD$. Suppose $\{f_i\}_{i \in I\!N}$ is a c.g.b. of $D$ satisfying the condition (k) where $k \in I\!N \cup \{+\infty\} \cup \{\omega\}$. Then the Minkowski functional of $D$ can be approximated by $C^k$-smooth Minkowski functionals.*

**Proof.** Choose $\varepsilon_i \searrow 0$. Put $\tilde{f}_i = (1 + \varepsilon_i) f_i$. It is standard to check that $\{\tilde{f}_i\}_{i \in I\!N}$ again satisfy the (k) condition. Moreover they form a c.g.b. of the set

$$\tilde{D} = \{x,\ \tilde{f}_i(x) \leq 1,\ i \in I\!N\}.$$

Also for every $x \in \partial\tilde{D}$ there exists an $i \in I\!N$ such that

$$j > i \ \text{ implies } \ \tilde{f}_j(x) \leq \frac{1+\varepsilon_{i+1}}{1+\varepsilon_i} < 1. \tag{1}$$

Letting $\varepsilon_1 \to 0$ gives us arbitrary good approximation of $D$ by $\tilde{D}$. Therefore it is enough to prove our result for $\tilde{D}$.
Denote by $\psi(x) = \frac{1}{e} \cdot e^x$, $x \in I\!R$. Put $h_i = \psi \circ \tilde{f}_i$. It is again standard to check that $\{h_i\}_{i\in I\!N}$ satisfies the condition (k), $h_i$ are non-negative, $\tilde{D} = \{x,\ h_i(x) \leq 1,\ i \in I\!N\}$ and moreover, from (1), there exists a sequence $\delta_i \searrow 0$ such that for every $x \in \tilde{D}$ there exists $i(x) \in I\!N$ and a neighbourhood $O(x) \subset X$ of $x$ with the properties:

$$j > i(x) \ \text{ implies } \ h_j(y) < 1 - \delta_{i(x)} \ \text{ for } \ y \in O(x). \tag{2}$$

In case $k = \omega$ we require in addition that for some neighbourhood $O^c \subset X^c$ of $(x, \vec{0})$ where $O^c \cap X = O$ there exist $\delta > 0$ and $i \in I\!N$ such that:

$$|h_j^c(z)| < 1 - \delta \ \text{ for } \ z \in O^c \ \text{ and } \ j > i. \tag{3}$$

Let $\{p_i\}_{i\in I\!N}$ be an increasing sequence of even integers. It follows from (2) that

$$G(x) = \sum_{i=1}^{\infty} \big(h_i(x)\big)^{p_i}$$

is a well-defined function on $\tilde{D}$, $A = G^{-1}\big([0,1]\big) \subset \mathrm{int}\tilde{D}$.
The formula for a sum of a geometric series persuades us that letting $p_1 \to +\infty$ gives us arbitrary good approximation of $\tilde{D}$ by $A$. We will prove that if the sequence $\{p_i\}_{i\in I\!N}$ grows fast enough, the function $G$ on $\mathrm{int}\tilde{D} \setminus \{\vec{0}\}$ has the same smoothness properties as functions $h_i$. This will finish the proof of Theorem 1.3.
Let us first prove the case $k \in I\!N \cup \{+\infty\}$.
Using the Lindelöf property of $(X, \|\cdot\|)$, choose sequences $\{x_j\}_{j\in I\!N} \subset \mathrm{int}\tilde{D}$ and $\{O_j\}_{j\in I\!N}$ consisting of open neighbourhoods of the points $x_j$ such that:

(i) $\mathrm{int}\tilde{D} \subset \bigcup_{j\in I\!N} O_j$

(ii) For $l \in I\!N$, $l \leq k$, $\|D^l h_i(\cdot)\|$ are unifomly bounded on $O_j$

(iii) $x, y \in O_j$ implies $i(x) = i(y)$.

Now let us form a sequence $\{[z_j, l_j]\}_{j\in I\!N}$ consisting of all pairs $[x_j, l]$ where $j \in I\!N$, $l \in I\!N$, $l \leq k$.
By induction with respect to $m \in I\!N$, we construct a system $\{p_{m,n}\}_{n\in I\!N}$ of increasing sequences of even integers such that

$$\{p_{(m+1),n}\}_{n\in I\!N} \subset \{p_{m,n}\}_{n\in I\!N}$$

and for every $m \in I\!N$ and every subsequence $\{q_n\}_{n\in I\!N}$ of $\{p_{m,n}\}_{n\in I\!N}$ the function

$$G(x) = \sum_{i=1}^{\infty} \big(h_i(x)\big)^{q_n}$$

restricted to $O_m$ is $l_m$-times continuously differentiable.
Put $\{p_{0,n}\}_{n\in I\!N} = \{2n\}_{n\in I\!N}$.
Induction step from $m$ to $m+1$.
According to the generalized chain rule, see [Fe, p.222] for the notation, we compute the $\beta$-th differential of a composition of a $\beta$-differentiable real function $f$ on $X$ with $x^p$, $p$ an even integer, at $a \in X$ as follows:

$$D^\beta\big(f(a)\big)^p = \sum_{\alpha\in S(\beta)} \frac{D^{\Sigma\alpha}\big((f(a))^p\big) \circ \Big(\big(D^1 f(a)\big)^{\alpha_1} \odot \cdots \odot \big(D^k f(a)\big)^{\alpha_k}\Big)}{\alpha!}, \tag{4}$$

where $S(\beta)$ is the set of all $\beta$-termed sequences $\alpha$ of nonnegative integers such that $\sum_{i=1}^{\beta} i\alpha_i = \beta$. Notice that (4) is a formula with a fixed number of terms on the right hand side, regardless of the value of $p$. If $|f(a)| < 1$, we obtain:

$$|D^{\Sigma\alpha}\big((f(a))^p\big)| \to 0 \ \text{ as } \ p \to +\infty \ \text{ for every } \ \alpha \in S(\beta).$$

Consequently $\|D^\beta\big(f(a)\big)^p\| \to 0$ as well.
The induction step is as follows:
We put $p_{(m+1),n} = p_{m,n}$ for $n \le i(z_{m+1})$. For $n > i(z_{m+1})$ we put $p_{(m+1),n}$ to be as large an element from $\{p_{m,n}\}_{n\in I\!N}$ that

$$\|D^\beta\big(h_n(y)\big)^{p_{m+1,n}}\| < \frac{1}{2^n}$$

for all $y \in O_q$ (where $z_{m+1} = x_q$), for all $\beta \le l_{m+1}$.
Putting $\{p_n\}_{n\in I\!N}$ to be $\{p_{n,n}\}_{n\in I\!N}$ and

$$G(x) = \sum_{i=1}^{\infty} \big(h_i(x)\big)^{p_i}$$

finishes the proof for $k \in I\!N \cup \{+\infty\}$.
The analytic case.
As $\{h_i\}_{i\in I\!N}$ satisfy the condition $(\omega)$(iv) and (3), the complex series

$$G^c(z) = \sum_{i=1}^{\infty} (h_i^c(z))^{2i+k} \ \text{ where } \ k \ \text{ is an even integer}$$

is uniformly convergent on some neighbourhood of every point $(x, \vec{0}) \in X^c$ where $x \in$int$\tilde{D}$. According to the uniform convergence theorem for holomorphic functions, $G^c(z)$ is holomorphic as well. As a result, $G(x) = \sum_{i=1}^{\infty} (h_i(z))^{2i+k}$ is real analytic on int$\tilde{D} \setminus \{\vec{0}\}$. Letting $k \to +\infty$ gives us arbitrary good approximations of $\tilde{D}$ by the sets $G^{-1}([0,1])$.

**Remark 1.4** It is easy to check that the same proof as above also works for $k$-times Gateaux-differentiable approximations.

## Section 2

In this section we prove that in normed spaces with countable algebraic basis, every CCB set can be approximated by polytopes. As a consequence every equivalent Minkowski functional can be approximated by analytic Minkowski functional. Let us start with some definitions.

**Definition 2.1** Let $P$ be a CCB subset of $\mathbb{R}^n$. Then $P$ is called a (finitely dimensional) polytope iff there exists a finite set $\{p_k\}_{k=1}^l, P = \mathrm{co}\{p_k\}_{k=1}^l$. Analogously, let $(X, \|\cdot\|)$ be a normed space, $P \subset X$ be a CCB subset. Then $P$ is called a polytope if every finite dimensional section of $P$ is a finite dimensional polytope.

**Definition 2.2** Let $(X, \|\cdot\|)$ be a normed space, $P \subset X$ be a CCB set, $\vec{0} \in \mathrm{int}P$, $P^0$ be the polar set. The set $S \subset P^0 \subset X^*$ is called a boundary of $P$ , if for every $x \in P$ there exists $f \in S$ such that $f(x) = 1$. Of course we have $w^*$-clco$S = P^0$

The following are some elementary propertis of finite dimensional polytopes:
(*) Every finite dimensional polytope containing $\vec{0}$ as an interior point has a finite boundary (namely the set of the extremal points of the dual polytope).
(**) Suppose $P$ is a convex body in $\mathbb{R}^n$, containing $\vec{0}$ as an interior point, $\varepsilon > 0$. Then there exists a polytope $Q$ s.t. $P \subset Q \subset (1+\varepsilon)P$.
(***) Suppose $P$, $Q$ are polytopes in $\mathbb{R}^n$, $\varepsilon > 0$ such that $P \cap \mathrm{span}\, Q \subset Q$, $\vec{0} \in \mathrm{int}Q$. Then $\mathrm{ext}((1+\varepsilon)Q) \subset \mathrm{ext}(\mathrm{co}(P \cup (1+\varepsilon)Q))$.

**Theorem 2.3** *Let $(X, \|\cdot\|)$ be a normed linear space with countable algebraic basis . Then every CCB set $B$, $0 \in \mathrm{int}B$, can be approximated by polytopes.*

**Proof.** Choose $\varepsilon > 0$. Choose a sequence $\varepsilon_k \searrow 0$, $\varepsilon_1 < 1/10$ so that $\prod\limits_{k=1}^{\infty} (1+\varepsilon_k) < 1+\varepsilon$. Suppose $\{x_k\}_{k\in\mathbb{N}}$ is the algebraic basis of $X$. Denote by $X_n = \mathrm{span}\{x_k\}_{k=1}^n$, $B_n = X_n \cap B$. (We have $B_{n+1} \cap X_n = B_n$). We construct by induction a sequence $\{K_n\}_{n\in\mathbb{N}}$ of polytopes in $X_n$ satisfying $B_n \subset K_n \subset \prod\limits_{k=1}^{n} (1+\varepsilon_k)B_n$, $K_{n+1} \cap X_n = (1+\varepsilon_{n+1})K_n$ as follows:
$K_1 = (1+\varepsilon_1)B_1$.
Inductive step:
Denote by $D_{n+1}$ a polytope (by (**)) such that

$$B_{n+1} \subset D_{n+1} \subset (1+\frac{\varepsilon_{n+1}}{4})B_{n+1}.$$

So we have $D_{n+1} \cap X_n \subset (1+\frac{\varepsilon_{n+1}}{4})B_n \subset (1+\frac{\varepsilon_{n+1}}{4})K_n$. Consequently

$$(1+\frac{\varepsilon_{n+1}}{4})(D_{n+1} \cap X_n) \subset (1+\frac{3\varepsilon_{n+1}}{4})K_n.$$

Put $K_{n+1} = \mathrm{co}(D_{n+1} \cup (1+\varepsilon_{n+1})K_n)$. By (***) we have $K_{n+1} \cap X_n = (1+\varepsilon_{n+1})K_n$. Define $\tilde{K_n} = \prod_{k=n}^{\infty}(1+\varepsilon_k)K_n$. Then $B_n \subset \tilde{K_n} \subset (1+\varepsilon)B_n$ and $\tilde{K}_{n+1} \cap X_n = \tilde{K_n}$.
This allows us to define a new polytope $\tilde{K}$ in $X$ such that $\tilde{K} \cap X_n = \tilde{K_n}$.
The approximation condition obviously holds.

**Corollary 2.4** *On every normed space with countable algebraic boundary every Minkowski functional (resp.equivalent norm) can be approximated by analytic Minkowski functionals (resp. equivalent analytic norms).*

**Proof.** The polytope $\tilde{K}$ has a countable boundary $\{b_i\}_{i \in I\!N}$ due to (*) and the fact that $X = \bigcup_{n \in I\!N} X_n$.
It is standard to check that $\{b_i\}_{i \in I\!N}$ satisfies the $(\omega)$ condition. The rest of the proof follows from Theorem 1.3.

**Remark 2.5** It should be noted that every separable Banach space $E$ contains an isomorphic copy of a normed space $X$ with countable algebraic basis that is dense in $E$.
Consider for example $E = \ell_1$, $X$ be the space of finitely supported vectors in $\ell_1$. Then the extension of arbitrary analytic norm on $X$ to the completion $\ell_1$ of the space $X$ looses all the differentiability properties, since $\ell_1$ admits no equivalent Fréchet differentiable norm.

## Section 3

In this section we prove results analogous to Theorem 2.3 and Corollary 2.4 for polyhedral Banach spaces. The method of the proof is however different due to the completness of the space and the fact that the dual space is separable.
We begin with some definitions. A Banach space $E$ is called polyhedral [K] if its unit ball $B_E$ is a polytope.
As proved in [F5], the algebraic and the topological interiors of a polytope $P$ coincide (we mean the topological interior in the space $[P]$, i.e. in the closure of the linear span $P$).
So it is easily seen that polytopes with non-empty algebraic interior exist in polyhedral Banach spaces only. Thus the problem of finding a polytope that approximates some body can be posed in polyhedral Banach space only.
So let $W$ be a CCB body in a polyhedral Banach space $E$ such that $\vec{0} \in \mathrm{int}W$. A polytope $P$ is called a tangential polytope to the body $W$ if $P \supset W$, and each maximal face of $P$ is tangent to the body $W$.
We will call a polytope $P$ $\varepsilon$-approximating for the body $W$ if $W \subset P \subset (1+\varepsilon)W$.
The main purpose of this section is to prove the following:

**Theorem 3.1** *Let $W$ be an arbitrary CCB body in a separable polyhedral Banach space $E$ and $\varepsilon > 0$. Then there exists a tangential $\varepsilon$-approximating polytope $P$ to the body $W$.*

Let us first list some auxiliary results. Let $W$ be a CCB body in a Banach space $E$. From now on we will assume without loss of generality that $\vec{0} \in \mathrm{int}W$.

The following theorem (see [F3,4,7]) summarizes some of the properties of polyhedral Banach spaces that we will use.

**Theorem 3.2** *Let $(E, \|\cdot\|)$ be a polyhedral Banach space of weight $\alpha$. Then there exists a boundary $B \subset S_{E^*}$ of $B_E$ of cardinality $\alpha$ such that for every $f \in B$ a face $\{x \in S_E : f(x) = 1\}$ has non-empty interior in the hyperplane $\{x \in E : f(x) = 1\}$. Conversely, if a separable Banach space $E$ has a countable boundary $\{f_i\}_{i\in \mathbb{N}}$ then this space is polyhedral in the equivalent norm*

$$|||x||| = \sup\{(1+\varepsilon_i)f(x_i), i \in \mathbb{N}\}.$$

*In the norm $|||\cdot|||$, the space $E$ also has a countable boundary $B = \{h_i\}_{i\in\mathbb{N}}$ (actually $h_i = (1+\varepsilon_i)f_i$) with the following property: every $w^*$-limit point $f$ of the set $B$ such that $|||f||| = 1$ does not attain its norm. Each functional $f \in E^*$ attaining its norm $|||\cdot|||$ belongs to the set span$\{h_i\}$.*

Proof of the following proposition is based on Theorem 3.2 and uses some ideas from [F1,2,3,4].

**Proposition 3.3** *Let $P$ be a polytope (recall that polytope is a body) in a separable Banach space $E$. Then there exists a countable boundary $B \subset P^0$ for $P$.*
*Conversely, let $A$ be CCB body with a countable boundary (for example a polytope) and $\varepsilon > 0$. Then there exists a polytope $P$ with the following properties:*

1) *$A \subset P \subset (1+\varepsilon)A$*
2) *There exists a countable boundary $\{h_i\} \subset P^0$ such that each $w^*$-limit point $h$ of the set $\{h_i\}_{i\in\mathbb{N}}$ with the property $h \in \partial P^0$ does not attain its supremum on $P$.*
3) *For every sequence $\gamma_i \searrow 0$ of positive numbers there exists a sequence $\{t_i\}_{i\in\mathbb{N}}$ of linear functionals such that:*
   *(a)$\|h_i - t_i\| < \gamma_i$*
   *(b) For every sequence $\{l_i\}_{i\in\mathbb{N}}$ possessing the property $\|l_i - t_i\| < \gamma_i/4$*

$$w^* - clco\{l_i\} \supset P^0$$

   *(c) The set $P_1 = \{x \in E : l_i(x) \le 1,\ i \in \mathbb{N}\}$ is a polytope.*

**Proof.** Let $y \in \text{int}P$. We define an affine mapping $A_y : P \to E$ by the formula:

$$A_y(x) = 2y - x, x \in P.$$

Put $P_y = A_y(P)$ and $V_y = P \cap P_y$. Of course $V_y$ is a symmetric polytope and by Theorem 3.2 there exists a countable boundary $B_y = \{f_y^j\}$ for $V_y$. It is obvious that $\partial V_y \subset \partial P \cup \partial P_y$. Let $x \in \partial V_y \cap \partial P$ be such a point that there exists $\delta > 0$ with the property:

$$(x + \delta B_E) \cap \partial P \subset \partial V_y.$$

If $f_y^j \in B_y$ is a supporting functional at a point $x$, i.e. $f_y^j(x) = 1 = \sup f_y^j(V_y)$ then it is easily verified that $f_y^j(x) = 1 = \sup f_y^j(P)$. Let $\{y_i\}_{i\in\mathbb{N}}$ be a dense subset in int$P$. Simple consideration shows that for each $x \in \partial P$ there exist $y_i$ and $\delta > 0$ such that:

$$(x + \delta B_E) \cap \partial P \subset \partial V_{y_i}.$$

Thus the countable set $B = \bigcup\limits_{i \in I\!N} \{f^j_{y_i}\}_{j=1}^{\infty}$ is a boundary for the polytope $P$.
Now let $A \subset E$ be a CCB body with a countable boundary $\{f_i\}_{i \in I\!N}$ and $\varepsilon > 0$. Without loss of generality we can assume that $\vec{0} \in \text{int}A$. Let $\varepsilon_i \searrow 0$ , $\varepsilon_1 < \varepsilon$. Put

$$h_i = \frac{1+\varepsilon_i}{1+\varepsilon} f_i,\ K = w^* - \text{clco}\{h_i\}, P = \{x \in E:\ h_i(x) \leq 1,\ i \in I\!N\}.$$

It is clear that $A \subset P \subset (1+\varepsilon)A$, $P^0 = K$, and $A^0 \supset K \supset (1+\varepsilon)^{-1}A^0$. Let $h_0 = w^*\text{-lim}\, h_{i_k}$, $h_0 \in \partial K$ and $x_0 \in \partial P$ be such that $h_0(x_0) = \max h_0(P) = 1$. Since $\varepsilon_i \searrow 0$, we have $h_0 = w^*\text{-lim}\, \frac{f_i}{1+\varepsilon}$ and therefore $h_0 \in (1+\varepsilon)^{-1}A^0$. From $x_0 \in (1+\varepsilon)A$ we have $\sup x_0 \frac{1}{1+\varepsilon}A^0) \leq 1$ and so

$$h_0(x_0) = 1 = \max x_0(1/(1+\varepsilon)A^0).$$

Since $\{\frac{1}{1+\varepsilon}f_i\}_{i \in I\!N}$ is a boundary for $\frac{1}{1+\varepsilon}A$ there exists a functional $\frac{1}{1+\varepsilon}f_j$ such that $\frac{1}{1+\varepsilon}f_j(x_0) = 1$. Thus

$$\frac{1+\varepsilon_j}{1+\varepsilon} f_j(x_0) = 1 + \varepsilon_j > 1 = \max x_0(K).$$

This contradiction shows that each $w^*$-limit point $h_0$ of the set $\{h_j\}_{j \in I\!N}$ such that $h_0 \in \partial K$ does not attain its supremum on the set $P$. Let $L \subset E$ be a finite-dimensional subspace. Then by compactness of the set $L \cap P$ it follows that there exist a positive number $\alpha$ and an integer $m$ such that:

$$\sup\{h_j(x),\ x \in L \cap P\} < 1 - \alpha \ \text{ for every }\ j > m.$$

This proves both that $P$ is a polytope and $\{h_i\}_{i \in I\!N}$ is a boundary. To prove c) it is enough to set $t_i = (1+\gamma_i)h_i$, for $i \in I\!N$ and to observe that the property:

$$w^* - \text{clco}\{l_i\} \supset w^* - \text{clco}\{h_i\}$$

is equivalent to the following one:

$$\max x(w^* - \text{clco}\{l_i\}) \geq \max x(w^* - \text{clco}\{h_i\}) \ \text{ for every }\ x \in E.$$

Since $\gamma_i \to 0$ for $i \to \infty$ it follows that each $w^*$-limit point of the set $\{t_i\}_{i \in I\!N}$ belongs to the set $w^* - cl\{h_i\}$. Thus $P_1$ is a polytope by the same argument as that $P$ is a polytope. The proof is completed.

**Remark 3.4** The proof of the first part of Proposition 3.3 shows that the structure of a topological boundary of an infinite-dimensional separable polytope $P$ (with non-empty interior) is similar to the structure of a boundary of a symmetric one (see Theorem 3.2), i.e. the boundary $\partial P$ consists of countably many maximal faces that are solid parts of hyperplanes.

The following lemma is similar to Lemma 4.1 [Zp] and to the first part of Theorem 1 [Ben]. We give a proof that is very close to the consideration in [Ben].

**Lemma 3.5** *Let $E$ be a Banach space with separable dual $E^*$ and $\{x_i\}_{i\in I\!N} \subset S_E$ be an $M$-basis of $E$ such that the linear span of biorthogonal system $\{x_i^*\}_{i\in I\!N}$ is dense in $E^*$. Let $W \subset E$ be CCB body such that $\vec{0} \in intW$ and $0 < \varepsilon < 1/2$. Then there exists a $w^*$-compact subset $F \subset W^0$ such that:*
*1) $\frac{1}{1+4\varepsilon}W^0 \subset w^* - clcoF \subset \frac{1}{1+\varepsilon}W^0$*
*2) For each integer $i$ the set $x_i(F)$ is finite.*

**Proof.** Let $d = \inf\{\|g\| : g \in \partial W^0\}$ and $T_i = \{f(x_i) : f \in W^0\}$ for $i \in I\!N$. Each set $T_i$ is bounded and thus there exists a $\frac{\varepsilon}{2^{i+2}\|x_i^*\|}$-net $C_i$ in $T_i$. Put

$$A = \left\{ \sum_{i=1}^{n} a_i x_i^* \in \frac{1}{1+\varepsilon}W^0 : \ n \in N, \ a_i \in C_i \right\}, \ F = w^* - \mathrm{cl}A.$$

It is obvious that $x_i(F) = x_i(A) \subset C_i$ where $i \in I\!N$. Thus the condition 2) is satisfied. Of course $w^* - \mathrm{clco}F \subset \frac{1}{1+\varepsilon}W^0$. In order to check that $\frac{1}{1+4\varepsilon}W^0 \subset w^* - \mathrm{clco}F$ let us take $f \in \frac{1}{1+2\varepsilon}W^0$. Since $\mathrm{span}\{x_i^*\}$ is dense in $E^*$ there exists

$$g = \sum_{i=1}^{n} b_i x_i^* \in \frac{1}{1+2\varepsilon}W^0,$$

such that $\|f - g\| < \varepsilon d/6$. We have $b_i \in T_i$ and there exists $a_i \in C_i$ such that $|b_i - a_i| < \frac{\varepsilon d}{6\cdot 2^i \|x_i^*\|}, i \in I\!N$. Hence

$$\|g - \sum_{i=1}^{n} a_i x_i^*\| < \frac{\varepsilon d}{6},$$

and a straight verification shows that $h = \sum_{1}^{n} a_i x_i^* \in \frac{1}{1+\varepsilon}W^0$. Thus by definition $h \in F$ and obviously $\|f - g\| < \frac{\varepsilon d}{3}$. From the last inequality taking into account $0 < \varepsilon < 1/2$ one can deduce that $\frac{1}{1+4\varepsilon}W^0 \subset w^* - \mathrm{clco}F$. The proof is completed.

The following Lemma 3.6 is close to some results from [Zp] too. We use the notation of Lemma 3.5. In addition put $M_n = [x_i]_1^{n\perp}, n \in I\!N$.

**Lemma 3.6** *For arbitrary $\varepsilon > 0$ there exists a sequence of points $\{g_k\}$ in the set $F$, a sequence of integers $\{n_k\}, n_k \to \infty$, and a decreasing sequence $\{F_\alpha\}$ of $w^*$-closed subsets of $F$ such that:*
*1) $\bigcup_{k\in I\!N} ((g_k + M_{n_k}) \cap F_k) = F$.*
*2) $diam((g_k + M_{n_k}) \cap F_k) < \varepsilon$.*

**Proof.** We will use the following well-known property of $w^*$-compacts in a separable dual space: for every $\varepsilon > 0$ there exist a point $g \in F$ and $w^*$-neighborhood $G$ of $g$ such that $G \cap F \neq \emptyset$ and $\mathrm{diam}(G \cap F) < \varepsilon$.

Becouse of the structure of the set $F$, the sets $(h + M_n) \cap F, h \in F, n \in N$ form a base of $w^*$- topology on $F$ and each such set is both closed and open subset of $(F, w^*)$. Moreover the family

$$\Im = \{h + M_n : h \in F, n \in N\}$$

contains countably many (different) sets and obviously each $w^*$- compact subset of $F$ has the same structure as $F$.
For each ordinal $\alpha$ we define sets $F_\alpha$ and $(h_\alpha + M_{n(\alpha)})$ by transfinite induction as follows:

$$F_0 = F, F_{\alpha+1} = F_\alpha \setminus (h_\alpha + M_{n(\alpha)}),$$

where $(h_\alpha + M_{n(\alpha)})$ is a member of the family $\Im$ such that $(h_\alpha + M_{n(\alpha)}) \cap F_\alpha \neq \emptyset$ and diam $((h_\alpha + M_{n(\alpha)}) \cap F_\alpha) < \varepsilon$. If $\alpha$ is a limit ordinal we put $F_\alpha = \cap_{\beta<\alpha} F_\beta$. Since the family $\Im$ is countable and each set $F_\alpha$ is $w^*$-compact there exists a countable ordinal $\eta$ such that $F_\eta \neq \emptyset$ and $F_{\eta+1} = \emptyset$. It is clear that:

$$\bigcup_{\alpha \leq \eta} \left((h_\alpha + M_{n(\alpha)}\right) \cap F_\alpha) = F.$$

Let us reindex the countable family $\{h_\alpha + M_{n(\alpha)}\}_{\alpha \leq \eta}$ into $\{h_k + M_{n_k}\}_{k=1}^\infty$. Since for each integer $q$ there exist only finite many members $h + M_n$ of the family $\Im$ such that $n \leq q$ it follows that $n_k \to \infty$ for $k \to \infty$. Lemma is proved.

The following lemma was proved in [H].

**Lemma 3.7** *Let $X$ be a Banach space that admits a boundary that can be covered by countable union of $\|.\|$-compacts.Then for every $\varepsilon > 0$ there exists an $\varepsilon$-isometric norm on $X$ with countable boundary.*

**Lemma 3.8** *Let $E$ be a separable polyhedral Banach space with norm possesing the properties of the second part of Theorem 3.2. Let $L \subset E$ be a finite-dimensional subspace of $E$ , $M = L^\perp$, $x \in E$ and $h \in S_M$ be such that $h(x) = \max x(S_M)$. Then there exists $h_0 \in (span\{h_i\} \cap S(M)$, ($\{h_i\}_{i \in \mathbb{N}}$ comes from Theorem 3.2), such that $h_0(x) = \max x(S_M)$.*

**Proof.** If $\max x(S_M) = 0$ then let $h_0$ be arbitrary functional from $(\text{span}\{h_i\}) \cap S_M$. If $h(x) \neq 0$, we can assume without loss of generality that $h(x_0) = 1$. Let $q : E \to E/L$ be a quotient map. Of course $\|q(x_0)\| = 1$ and since $L$ is a finite-dimensional subspace there exists an element $x_1 \in S_E \cap (x_0 + L)$. It is clear that $h(x_1) = h(x_0) = 1$. Thus funcional $h$ attains its norm and by Theorem 3.2 $h \in \text{span}\{h_i\}$,. It is enough to put $h_0 = h$ to complete the proof.

**Lemma 3.9** *Let $W$ be CCB body in a Banach space $E$ , $0 \in intW$, $\varepsilon > 0$ and $A$ be a polytope possessing the property:*

$$W \subset A \subset (1 + \varepsilon)W.$$

*Then for every $\varepsilon_1 > \varepsilon$ there exists a tangential polytope $P_1$ for body $W$ such that $W \subset P_1 \subset (1+\varepsilon_1)W$.*

**Proof.** Let $\varepsilon_2 > 0$ be such that $(1+\varepsilon)(1+\varepsilon_2) < (1+\varepsilon_1)$. By Proposition 3.3 there exists a polytope $P$ possessing the properties 1)-2)-3), with $\varepsilon_2$ instead of $\varepsilon$. From property 1) of Proposition 3.3 and the above assumption we have $W \subset P \subset (1+\varepsilon_2)(1+\varepsilon)W$. Thus

$$W^0 \supset P^0 \supset \frac{1}{(1+\varepsilon_2)(1+\varepsilon)} W^0,$$

and hence

$$W^0 \supset \frac{(1+\varepsilon)(1+\varepsilon_2)}{1+\varepsilon_1} P^0 \supset \frac{1}{1+\varepsilon_1} W^0.$$

Put $\lambda = (1+\varepsilon)(1+\varepsilon_2)/(1+\varepsilon_1) < 1$. Actually we have $\lambda P^0 \subset \lambda W^0$. Using the notation of Proposition 3.3 we can assert that:

$1^0$) $\{\lambda h_i\}_{i\in I\!N}$ is a boundary for $\frac{1}{\lambda}P$ such that each $w^*$-limit point $h$ of the set $\{\lambda h_i\}_{i\in I\!N}$ such that $h \in \partial(\lambda P^0)$ does not attain its supremum on the set $\frac{1}{\lambda}P$.

$2^0$) For every $\alpha > 0$ there exists a sequence of linear functionals $\{t_i\}_{i\in I\!N}$ such that:

(a) $\|\lambda h_i - t_i\| < \alpha/2^i$.

(b) For every sequence $\{l_i\}_{i\in I\!N}$ possessing the property $\|l_i - t_i\| < \alpha/2^{i+2}$

$$w^* - \mathrm{clco}\{l_i\} \supset \lambda P^0$$

(c) The set $P_1 = \{x \in E : l_i(x) \le 1, i \in I\!N\}$ is a polytope.

Let a number $\alpha > 0$ be small enough to have every sequence $\{l_i\}_{i\in I\!N}$ (from property (b)) inside $\mathrm{int}W^0$ and let $\{f_i\}_{i\in I\!N} \subset S_E^*$ be an arbitrary sequence $w^*$-tending to zero. Let us denote by $T_i$ the straight line that the contains functionals $l_i$ and $l_i + f_i$ (as points). Denote by $u_i^1$ and $u_i^2$ the points (i.e.functionals) of intersection of the line $T_i$ with the boundary $\partial W^0$ (recall that $l_i \in \mathrm{int}W^0$). Using Bishop-Phelps theorem on the density of the set of functionals that attain their supremum on the set $W$ it is not difficult to establish the existence of the functionals $l_i$ ($\|l_i - t_i\| < \alpha/2^{i+2}$) and $g_i$ ($\|f_i - g_i\| < 2^{-i}$) such that both functionals $u_i^1$ and $u_i^2$ attain their supremum on the set $W$. It is easily seen that:

$$w^* - \lim(u_i^1 - \lambda h_i) = w^* - \lim(u_i^2 - \lambda h_i) = 0. \tag{5}$$

Moreover the point $l_i$ lies on the segment $[u_i^1, u_i^2]$, $i \in I\!N$ and hence

$$w^* - \mathrm{clco}\{u_i^1, u_i^2\}_{i\in I\!N} \supset w^* - \mathrm{clco}\{l_i\} \supset \lambda P^0.$$

Put $P_1 = \{x \in E : u_i^1 \le 1, u_i^2 \le 1, i \in I\!N\}$, then $P_1 \subset \frac{1}{\lambda}P$. Let $h$ be $w^*$-limit point of the set $\{u_i^1, u_i^2\}$ such that $h \in \partial P_1^0$. From (5) and $P_1^0 \supset \lambda P^0$ it follows that $h \in \partial(\lambda P^0)$ and by $1^0$) functional $h$ does not attain its supremum on the set $\frac{1}{\lambda}P$. But $\sup h(P_1) = \sup h(\frac{1}{\lambda}P) = 1$ and $P_1 \subset \frac{1}{\lambda}P$, hence the functional $h$ does not attain its supremum on the set $P_1$ either. Thus $P_1$ is a polytope with the boundary $\{u_i^1, u_i^2\}$. Since $P_1^0 = w^* - \mathrm{clco}\{u_i^1, u_i^2\}_1^\infty \subset W^0$ and $P_1^0 \supset \lambda P^0 \supset \frac{1}{1+\varepsilon_1} W^0$ we have:

$$W \subset P_1 \subset (1+\varepsilon_1)W.$$

It is clear from our construction that $P_1$ is a tangential polytope for $W$. The proof is completed.

Now we are ready to give the proof of Theorem 3.1.

**Proof.** We assume that the norm on the space $E$ possesses all the properties from Theorem 3.2. In view of Lemma 3.9 it is sufficient to prove the existence of a $\delta$-approximating polytope. Let $\varepsilon > 0$ and let $w^*$-compact $F \subset W^0$, sequence of functionals $\{g_k\}_{k\in I\!N}$ and subspaces $M_{n_k}, k \in I\!N$ come from Lemmas 3.5 and 3.6. It is clear that:

$$\bigcup_{k\in I\!N}(g_k + \varepsilon B_{M_{n_k}}) \supset F, \tag{6}$$

where $B_{M_{n_k}}$ is a unit ball of the space $(M_{n_k}, \|\cdot\|)$.
By Lemmas 3.7 and 3.8 there exists an $\varepsilon$-isometric norm $|\!|\!|\cdot|\!|\!|$ on the quotient-space $X_k = E/[x_i]_1^{n_k}$ possessing a countable boundary. Let $V_{M_{n_k}}$ be a unit ball of the space $(M_{n_k}, |\!|\!|\cdot|\!|\!|)$ and $\{v_i^k\}_{i=1}^{\infty} \subset V_{M_{n_k}}$ be a countable boundary. Of course we can assume that:

$$(1-\varepsilon)V_{M_{n_k}} \subset B_{M_{n_k}} \subset V_{M_{n_k}} \text{ for } k \in I\!N.$$

From (6) we have

$$C = \bigcup_{k\in I\!N}(g_k + \varepsilon V_{M_{n_k}}) \supset F$$

Define $B = \bigcup_{k\in I\!N}\{g_k + v_i^k\}_{i=1}^{\infty}, Q = \{x \in E : f(x) \le 1, f \in B\}$ and let us show that $B$ is a boundary for the body $Q$. We show first that $C$ is $w^*$-closed set. Let $\{h_m\}_{m\in I\!N} \subset C$, $h_m \xrightarrow{w^*} h_0$. If infinitely many of $h_m$ are in one of the sets $g_k + \varepsilon V_{M_{n_k}}$ then of course $h_0 \in C$. So let us assume that $h_m \in g_{k_m} + \varepsilon V_{M_{n_{k_m}}}$ where $m \in I\!N$, $k_m \to \infty$, $m \to \infty$. Set $h_m = g_{k_m} + \varepsilon u_m$ where $u_m \in V_{M_{n_{k_m}}}$, $m \in I\!N$. Since $k_m \to \infty$, we have $n_{k_m} \to \infty$ (see Lemma 3.6) and therefore $w^* - \lim u_m = 0$. Thus $h_0 = w^* - \lim g_{k_m} \in F \subset C$, which completes the proof of $w^*$-closeness of the set $C$. Also:

$$Q^0 = w^* - \text{clco}C = w^* - \text{clco}B,$$

and hence $C$ is a boundary for $Q$. Let $x \in \partial Q$ and $h \in C$, so that $h(x) = 1 = \max x(Q^0)$. Since $h \in C$ there exists an integer $k$ such that $h \in g_k + \varepsilon V_{M_{n_k}}$. If $g_k(x) = 1$ then $x(V_{M_{n_k}}) = 0$ and for each $i \in I\!N$:

$$(g_k + v_i^k)(x) = 1. \tag{7}$$

If $g_k(x) < 1$ then

$$\sup x(V_{M_{n_k}}) = \frac{1 - g_k(x)}{\varepsilon} \neq 0.$$

Hence there exists a functional $v_i^k$ such that $v_i^k(x) = \sup x(V_{M_{n_k}})$, (the set $\{v_i^k\}$ forms a boundary). Thus it is proved that $Q$ has a countable boundary and we can apply Proposition 3.3. It is clear that for $\varepsilon$ small enough we will be able to make $\delta$-approximation. The theorem is proved.

**Corollary 3.10** *Arbitrary Minkowski functional (resp. equivalent norm) on a separable polyhedral Banach space can be approximated by analytic Minkowski functionals (resp. equivalent analytic norms).*

**Proof.** The countable boundary of the approximating polytope satisfies the condition $(\omega)$, so Theorem 1.3 applies.

Let us note that among classical Banach spaces, in particular $c_0$ and $C(K)$ where $K$ is a countable compact are polyhedral.

## Section 4

**Theorem 4.1** *Let $(X, \|\cdot\|)$ be a separable Banach space with Schauder basis $\{x_i\}_{i\in I\!N}$. Let $k \in I\!N \cup \{+\infty\}$, $\|\cdot\|$ be $C^k$-(Gateaux or Fréchet) smooth, and $D^l\|\cdot\|$ be bounded on $B_X$ for $l \in I\!N$, $l \leq k$. Then every Minkowski functional (resp.equivalent norm) on $X$ can be approximated by $C^k$-smooth Minkowski functionals (resp.equivalent norms).*

**Proof.** For $k = 1$ the above result is known-see [DGZ, p.53].
For $k > 1$ the space $X$ is superreflexive [DGZ, p.203], so the basis $\{x_i\}_{i\in I\!N}$ is shrinking.
Suppose $\{x_i^*\}_{i\in I\!N}$ is the dual basis, $K$ be the basis constant of $\{x_i\}_{i\in I\!N}$. Suppose $W$ is a CCB subset of $X$, $\vec{0} \in \mathrm{int}W$. Using the basis $\{x_i\}_{i\in I\!N}$ in Lemmas 3.5 and 3.6 from Section 3 we obtain sequences $\{g_k\}_{k\in I\!N} \subset F$ and $\{M_{n_k}\}_{k\in I\!N}$ such that

$$\bigcup_{k\in I\!N} \left(g_k + \varepsilon \cdot B_{X^*} \cap M_{n_k}\right) \supset F.$$

Let us define a sequence $\{P_k\}_{k\in I\!N}$ of linear isomorphisms on $X$ defined as:

$$P_k\Big(\sum_{i=1}^{\infty} a_i x_i\Big) = \varepsilon_k \cdot \sum_{i=1}^{n_k} a_i x_i + \sum_{i=n_k+1}^{\infty} a_i x_i,$$

where $1 > \varepsilon_1$, $\varepsilon_k \searrow 0$ are chosen so that:

$$\sup\{\mathrm{dist}\big(x^*, M_{n_k}\big),\ x^* \in P_k^*\big(B_{X^*}\big)\} \to 0 \ \text{ as } \ k \to +\infty. \tag{8}$$

As

$$\|P_k\| \leq \sup_{x\in B_X} \Big(\|\varepsilon_k \cdot \sum_{i=1}^{n_k} a_i x_i\| + \|\sum_{i=n_k+1}^{\infty} a_i x_i\|\Big),$$

we have

$$\|P_k\| \le 3K \text{ for } k \in I\!N. \tag{9}$$

Thus

$$\operatorname{diam}_{\|\cdot\|^*} P_k^*\big(B_X^*\big) \le 3K. \tag{10}$$

Put

$$S^* = \bigcup_{k \in I\!N} \big(g_k + \varepsilon \cdot P_k^*\big(B_{X^*}\big)\big).$$

Then $F \subset S^*$, and a similar argument to the one in proof of Theorem 3.1 yields that $S^*$ is $w^*$-closed.
From (10) we have that $w^*$-clco$S^*$ approximate $W^\circ$ arbitrary well. Put $S = \{x \in X,\ S^*(x) \le 1\}$. It follows that $\{f_k\}_{k \in I\!N}$, where:

$$f_k(x) = \sup\{y(x),\ y \in g_k + \varepsilon P_k^*\big(B_{X^*}\big)\} = g_k(x) + \varepsilon \cdot \|P_k(x)\| \tag{11}$$

forms a generalized boundary of $S$. It follows from (6) and the generalized chain rule that $\{f_k\}_{k \in I\!N}$ satisfies the condition (k). By Theorem 1.3 we are done.

**Corollary 4.2** *On spaces $L_p[0,1]$, $\ell_p$ where $1 < p < +\infty$, $p \notin I\!N$, every equivalent norm can be approximated by $C^{[p]}$-Fréchet smooth norms. On spaces $L_p[0,1]$, $\ell_p$ where $p$ is odd, every equivalent norm can be approximated by $C^{p-1}$-Fréchet smooth norms.*

**Proof.** It is well-known that these spaces have a Schauder basis. The explicit calculation of the derivatives of its canonical norm, carried out e.g. in [DGZ, 184], finishes the proof. It should be noted that this is the best possible result besause, as shown in [DGZ, p.222], these spaces do not admit equivalent norms of higher order of Fréchet smoothness than the ones used for the approximation.

**Theorem 4.3** *Let $(X, \|\cdot\|)$ be a separable Banach space with Schauder basis $\{x_i\}_{i \in I\!N}$. Assume that there exist an even $p \in I\!N$ and a convex homogeneous $p$-polynomial $P(\cdot)$ on $X$ such that $\|\cdot\| = P(\cdot)^{\frac{1}{p}}$. Then every Minkowski functional (resp. equivalent norm) on $X$ can be approximated by analytic Minkowski functionals (resp. equivalent norms).*

**Proof.** The construction of $\{f_k\}_{k \in I\!N}$ is exactly the same as in Theorem 4.1. In order to verify that $\{f_k\}_{k \in I\!N}$ satisfy the $(\omega)$ condition, it is enough to realize that:

$$f_k^c = g_k^c + \varepsilon \cdot \big(P^c(P_k^c)\big)^{\frac{1}{p}} \tag{12}$$

where $g_k^c, P_k^c$ are uniformly continuous on a neighbourhood of $(x, \vec{0})$ and $P^c$ is uniformly continuous on every bounded set. As for $x \in X$ we have: $\|P_k^c((x, \vec{0}))\|^c \to 0$ for $k \to 0$ and $g_k$ lie in the polar of $\tilde{D}$, we are done.

**Corollary 4.4** *On spaces $L_p[0,1]$, $\ell_p$ where $p$ is an even integer, every Minkowski functional (resp. equivalent norm) can be approximated by analytic Minkowski functionals (resp. analytic norms).*

As we mentioned in the Introduction, the existence of a $C^k$-Fréchet smooth bump function on a separable space implies the possibility of approximations on bounded sets of arbitrary continuous function by $C^k$-Fréchet smooth functions.
Similarily, the existence of a separating polynomial implies the possibility of analytic approximations, as shown in [Ku].
A refinement of these results can be obtained for convex functions on certain spaces. We will give only a sketch of proof of the following statement:

**Corollary 4.5** *Let $(X, \|\cdot\|)$ be a separable normed space that satisfies the assumptions of any of the following theorems contained in this paper: Corollary 2.4, Corollary 3.10, Theorem 4.1 and Theorem 4.3.*
*Then arbitrary convex function on $X$ can be approximated on bounded sets by convex functions of the same degree of smoothness as are the approximations of norms in the corresponding theorem.*

**Sketch of proof.** It is easy to verify that if a space $X$ satisfies the above assumptions, the space $X \times I\!R$ does as well. Now use the ideas from [Ph2, p.96]. For a given convex function $f$ on $X$, pass to its epigraph in $X \times I\!R$. Now use the above results to approximate the Minkowski functional of a suitable bounded part of the epigraph in $X \times I\!R$ by smooth Minkowski functionals. Then use the Implicit Function Theorem to go back to $X$.

## References


[Ben] Y. Benyamini, *An extension theorem for separable Banach spaces,* Israel J. Math., Vol. 29, No. 1, 1978, 24-30.

[D] R. Deville, *Geometrical implications of the existence of very smooth bump functions in Banach spaces,* Israel J. Math., 6(1989), 1-22.

[DGZ] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces,* Pitman Monographs and Surveys in Pure and Applied Mathematics, 64, 1993.

[D-P] R. Durier and P. Papini, *Polyhedral norms and related properties in infinite-dimensio-
nal Banach spaces,* Atti Sem. Fis. Univ. Modena, 40, 1992, 623-645.

[FPWZ] M. Fabian, D. Preiss, J.H.M. Whitfield and V. Zizler, *Separating Polynomials on Banach Spaces,* Quart. J. Math. Oxford (2), 40, 1989, 409–422.

[Fe] H. Federer, *Geometric theory of measure,* Springer-Verlag, 1970.

[F1] V. Fonf, *Classes of Banach spaces connected to the massiveness of the set of extreme points of the dual ball,* Author's Abstract of Candidate's Dissertation, Sverdlovsk, 1979.

[F2] V. Fonf, *On massiveness of the set of extreme points of the unit ball of Banach space and polyhedral spaces,* Funct. Anal. Appl., 12, 1978.

[F3] V. Fonf, *Some properties of polyhedral Banach spaces,* Funct. Anal. Appl., 14, 1980.

[F4] V. Fonf, *On polyhedral Banach spaces,* Math. Notes Acad. Sci. USSR, 30, 1981.

[F5] V. Fonf, *On one kind of infinite-dimensional polytopes,* Funct. Anal. Appl., 18, 1984.

[F6] V. Fonf, *Weakly extremal properties of Banach spaces,* Math. Notes Acad. Sci. USSR, 45, 1989.

[F7] V. Fonf, *Three characterization of polyhedral Banach spaces,* Ukrain. Math. J., 42, No. 9, 1990.

[H] P. Hájek, *Smooth Norms That Depend Locally on Finitely Many Coordinates,* to appear.

[K] V. Klee, *Polyhedral sections of convex bodies,* Acta Math. 103, 1960, 243-267.

[Ku] J. Kurzweil, *On approximation in real Banach spaces,* Studia Math. 14, 1954, 213–231.

[N] L. Nachbin, *Topology on Spaces of Holomorphic Mappings,* Springer-Verlag, 1969.

[NS] A.M. Nemirovskii and S.M. Semenov, *On polynomial approximation of functions on Hilbert space,* Mat. Sbornik, 21, 1973.

[Ph1] R. Phelps, *Infinite-dimensional convex polytopes,* Math. Scand., 24, 1969, 5-26.

[Ph2] R. Phelps, *Convex Functions, Monotone Operators and Differentiability,* Lecture Notes in Math. 1364, Springer-Verlag, 1989.

[Zp] M. Zippin, *The separable extension problem,* Isr. J. Math., Vol. 26, 1977, 372-387.