A CHARACTERIZATION OF REFLEXIVITY

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Abstract. We give a characterization of reflexivity in terms of rotundity of the norm.

Renorming characterization of various classes of Banach spaces is important and useful for applications. Some classes turn out to have very elegant descriptions, while most seem to resist the renorming point of view. The most spectacular result in this area is certainly the Enflo-Pisier characterization of super-reflexive spaces as those admitting uniformly rotund norm [E] (or even having power type modulus of uniform convexity [P]).

Restricting to separable Banach spaces allows more elegant results (not valid in the general case), such as Fréchet smooth or weakly uniformly rotund (WUR) renorming characterization of spaces with a separable dual (Asplund spaces), characterizations of subspaces of c_0 etc. A good source of references on the subject is G. Godefroy's article in [JL], or [DGZ].

In the present note we are interested in renorming characterization of reflexivity. Let us give a brief account of the known facts. Combining Theorem 5.4 of [C] with the fundamental LUR renorming of the WCG spaces [T], Troyanski obtained a characterization of reflexive spaces X as those admitting a renorming $\|\cdot\|$ with the following property (named weakly 2-rotund (W2R) by Cudia [C]):

For every sequence $\{x_n\} \subset S_X$, if there exists $0 \neq f \in X^*$ satisfying $\lim\limits_{m,n\to\infty} |f(\frac{x_n+x_m}{2})| = \|f\|$, then $\{x_n\}$ is convergent in norm. We can easily see that this happens if and only if each $f \in S_{X^*}$ strongly exposes the unit ball of X.

It is standard (using Šmulyan's criterion) and well-known that the above property of the norm $\|\cdot\|$ is equivalent to $\|\cdot\|^*$ being Fréchet smooth. In particular, every LUR renorming of a reflexive space satisfies the criterion. (Note however, that in [T] Cudia's definition of W2R is stated incorrectly, and in fact the stated condition fails to imply reflexivity as shown in [HR].)

On the other hand, Milman in [M] introduced the notions of 2-rotund (2R) and weakly 2-rotund (W2R). (See below for these definitions and note that Milman's W2R is distinct from Cudia's. In the present paper we choose to use Milman's terminology which seems more in place.). He states (without proof) that separable reflexive spaces are precisely the W2R renormable and asks whether reflexive spaces with LUR norm (this condition is redundant due to [T]) are 2R renormable. The last problem was settled positively for separable spaces by Odell and Schlumprecht [OS], but the general case remains open.

The main result of this note is a characterization of reflexive spaces as those admitting a W2R renorming. We also give examples showing that LUR renorming of a reflexive space is not necessarily W2R and vice versa, so ours is an essentially different characterization of reflexivity than Cudia-Troyanski's.

First let us fix some notation. For a finite set A we denote the number of elements of A by $|A|$. Given a vector $x = \{x(\gamma)\}_\Gamma \in c_0(\Gamma)$ and an $A \subset \Gamma$, $x{\restriction}_A$ denotes the vector defined as $x{\restriction}_A(\gamma) = x(\gamma)$ for $\gamma \in A$ and $x{\restriction}_A(\gamma) = 0$ for $\gamma \in \Gamma \setminus A$.

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Definition (Milman [M]**).** *We say that a norm $\|\cdot\|$ on a Banach space X is 2-rotund (2R) (resp. weakly 2-rotund (W2R)) if for every $\{x_n\} \subset B_X$ such that*

$$\lim_{m,n\to\infty} \|x_m + x_n\| = 2$$

there is an $x \in X$ such that $\lim_{n\to\infty} x_n = x$ in the norm (resp. weak) topology of X.

Alternative Definition. *We say that a norm $\|\cdot\|$ on a Banach space X is 2-rotund (resp. weakly 2-rotund) if for every $\{x_n\} \subset X$ such that*

$$\lim_{m,n\to\infty} 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2 = 0 \tag{1}$$

there is an $x \in X$ such that $\lim_{n\to\infty} x_n = x$ in the norm (resp. weak) topology of X.

It is a well-known fact (for a proof see e.g. [DGZ, II.1.2]) that Milman's and the alternative definitions are equivalent. The original one is somewhat more geometrically appealing, while the alternative one is more convenient for calculations, and we will use it in our paper.

Theorem 1. *Let X be a Banach space. Then X is reflexive if and only if it admits an equivalent W2R norm.*

In the proof we need the following proposition. Recall that for an arbitrary set Γ, Day's norm on $c_0(\Gamma)$ is defined by

$$\|x\| = \sup\left\{ \left(\sum_{k=1}^{n} 4^{-k} x^2(\gamma_k) \right)^{1/2} ;\ (\gamma_1, \dots, \gamma_n) \right\}$$

where the supremum is taken over all $n \in \mathbb{N}$ and all ordered n-tuples $(\gamma_1, \dots, \gamma_n)$ of distinct elements of Γ.

Proposition 2. *Let Γ be an arbitrary set and $\|\cdot\|$ be Day's norm on $c_0(\Gamma)$. Let $\{x_n\} \subset c_0(\Gamma)$ such that (1) holds, $x_{2n} \to x \in c_0(\Gamma)$ and $x_{2n+1} \to y \in c_0(\Gamma)$ in the pointwise topology. Then $x = y$.*

Proof. Let $\|\cdot\|_\infty$ denote the canonical norm on $c_0(\Gamma)$. Let $\{\alpha_k^n\}$ be the support of x_n enumerated so that $|x_n(\alpha_1^n)| \geq |x_n(\alpha_2^n)| \geq \dots$ and $\{\beta_k^{m,n}\}$ be the support of $(x_m + x_n)$ enumerated so that $|(x_m + x_n)(\beta_1^{m,n})| \geq |(x_m + x_n)(\beta_2^{m,n})| \geq \dots$. Note that $\beta_k^{m,n} = \beta_k^{n,m}$, $k \in \mathbb{N}$.

From the definition of Day's norm

$$\|x_n\|^2 = \sum_k 4^{-k} x_n^2(\alpha_k^n) \geq \sum_k 4^{-k} x_n^2(\gamma_k) \tag{2}$$

for any sequence $\{\gamma_k\} \subset \Gamma$. Hence

$$\begin{aligned} 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2 &= 2\sum 4^{-k} x_m^2(\alpha_k^m) + 2\sum 4^{-k} x_n^2(\alpha_k^n) \\ &\quad - \sum 4^{-k} (x_m + x_n)^2(\beta_k^{m,n}) \\ &\geq 2\sum 4^{-k} x_m^2(\beta_k^{m,n}) + 2\sum 4^{-k} x_n^2(\beta_k^{m,n}) \\ &\quad - \sum 4^{-k} (x_m + x_n)^2(\beta_k^{m,n}) \\ &= \sum 4^{-k} \big(x_m(\beta_k^{m,n}) - x_n(\beta_k^{m,n}) \big)^2 \geq 0 \end{aligned} \tag{3}$$

As $2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2 \geq \big(\|x_m\| - \|x_n\|\big)^2 \geq 0$, (1) implies that $\{\|x_n\|\}$ is Cauchy and hence $\{\|x_n\|_\infty\}$ is bounded. Therefore by passing to a suitable subsequence we may assume that for $i \in \{0,1\}$ there is $z_i \in \ell_\infty$ such that $|x_{2n+i}(\alpha_k^{2n+i})| \to z_i(k)$, $k \in \mathbb{N}$. Notice that $z_i(1) \geq z_i(2) \geq \dots$.

We claim that $z_0 \in c_0$. If this is not the case then there is a $C > 0$ such that $z_0(k) > C$ for $k \in \mathbb{N}$. Then there is a finite $A \subset \Gamma$ such that $\|x\restriction_{\Gamma\setminus A}\|_\infty < \frac{C}{8}$. The pointwise convergence implies that there is $n_0 \in \mathbb{N}$ such that $\|(x_{2n} - x)\restriction_A\|_\infty < \frac{C}{8}$ for $n > n_0$. By (3) and (1) there is $m_0 \in \mathbb{N}$ such that

$$\sum_k 4^{-k}\big(x_m(\beta_k^{m,n}) - x_n(\beta_k^{m,n})\big)^2 < 4^{-|A|-1}\frac{C^2}{16} \quad \text{for } m, n > m_0. \tag{4}$$

Since $\big|x_{2n}\big(\alpha^{2n}_{|A|+1}\big)\big| \to z_0(|A|+1) > C$, there is $n_1 > n_0$ such that $2n_1 > m_0$ and $\big|x_{2n_1}\big(\alpha^{2n_1}_{|A|+1}\big)\big| > C$. Thus we can choose $\gamma \in \Gamma \setminus A$ for which $|x_{2n_1}(\gamma)| > C$. Next we find a finite $B \subset \Gamma$ such that

$$\|x_{2n_1}\restriction_{\Gamma\setminus B}\|_\infty < \frac{C}{8}. \tag{5}$$

This implies that $\gamma \in B\setminus A$. Using the pointwise convergence again we choose $n_2 > n_0$ such that $2n_2 > m_0$ and $\|(x_{2n_2} - x)\restriction_B\|_\infty < \frac{C}{8}$. Therefore we have

$$\|x_{2n_2}\restriction_{B\setminus A}\|_\infty < \frac{C}{4} \tag{6}$$

and so $|x_{2n_2}(\gamma)| < \frac{C}{4}$. Further,

$$|x_{2n_1}(\gamma) + x_{2n_2}(\gamma)| > \frac{3}{4}C. \tag{7}$$

We find the smallest $k_0 \in \mathbb{N}$ for which $\beta_{k_0}^{2n_1,2n_2} \notin A$. It follows that $k_0 \le |A| + 1$ and

$$\left|(x_{2n_1} + x_{2n_2})\left(\beta_{k_0}^{2n_1,2n_2}\right)\right| \ge \big|(x_{2n_1} + x_{2n_2})(\gamma)\big|. \tag{8}$$

Now either $\beta_{k_0}^{2n_1,2n_2} \in B \setminus A$ and we can use (8), (7) and (6) to obtain

$$\begin{aligned}
&\left|x_{2n_1}(\beta_{k_0}^{2n_1,2n_2}) - x_{2n_2}(\beta_{k_0}^{2n_1,2n_2})\right| \\
&\qquad \ge \left|x_{2n_1}(\beta_{k_0}^{2n_1,2n_2}) + x_{2n_2}(\beta_{k_0}^{2n_1,2n_2})\right| - 2\left|x_{2n_2}(\beta_{k_0}^{2n_1,2n_2})\right| \\
&\qquad \ge |x_{2n_1}(\gamma) + x_{2n_2}(\gamma)| - 2\left|x_{2n_2}(\beta_{k_0}^{2n_1,2n_2})\right| \ge \frac{3}{4}C - \frac{1}{2}C \ge \frac{C}{4},
\end{aligned}$$

or $\beta_{k_0}^{2n_1,2n_2} \in \Gamma \setminus (B \cup A)$ and we use (8), (7) and (5) instead to get the same conclusion. Finally

$$\sum_k 4^{-k}\big(x_{2n_1}(\beta_k^{2n_1,2n_2}) - x_{2n_2}(\beta_k^{2n_1,2n_2})\big)^2 \ge 4^{-k_0}\big(x_{2n_1}(\beta_{k_0}^{2n_1,2n_2}) - x_{2n_2}(\beta_{k_0}^{2n_1,2n_2})\big)^2 \ge 4^{-|A|-1}\frac{C^2}{16}$$

which contradicts (4). We can show in the same way that also $z_1 \in c_0$.

Moreover, $z_0 = z_1$. Indeed, assume that there is a $k_1 \in \mathbb{N}$ such that (without loss of generality) $z_0(k_1) < z_1(k_1)$. Put $\varepsilon = \frac{1}{3}(z_1(k_1) - z_0(k_1))$. There is $n_0 \in \mathbb{N}$ such that for $n > n_0$ we have $|x_{2n}(\alpha_k^{2n})| < z_0(k_1) + \varepsilon$ for $k \ge k_1$ and $|x_{2n+1}(\alpha_k^{2n+1})| > z_1(k_1) - \varepsilon$ for $k \le k_1$. This implies the existence of $\gamma_n \in \Gamma$ such that for $n > n_0$

$$|(x_{2n} - x_{2n+1})(\gamma_n)| > z_1(k_1) - \varepsilon - z_0(k_1) - \varepsilon = \varepsilon \tag{9}$$

$$|(x_{2n} + x_{2n+1})(\gamma_n)| > z_1(k_1) - \varepsilon - z_0(k_1) - \varepsilon = \varepsilon. \tag{10}$$

Since $z_0 \in c_0$ and $z_1 \in c_0$, we can find $k_2 \in \mathbb{N}$ such that $z_0(k_2) < \frac{\varepsilon}{4}$ and $z_1(k_2) < \frac{\varepsilon}{4}$. Hence we can choose $n_1 > n_0$ such that $|x_{2n}(\alpha_k^{2n})| < \frac{\varepsilon}{2}$ and $|x_{2n+1}(\alpha_k^{2n+1})| < \frac{\varepsilon}{2}$ for $k > k_2$ and $n > n_1$. Fix any $n > n_1$ and let $A = \big\{\gamma;\ |(x_{2n} + x_{2n+1})(\gamma)| > \varepsilon\big\}$. It's easy to see that $|A| \le 2k_2$ and by (10), $\gamma_n \in A$. This together with (9) gives

$$\sum_k 4^{-k}\big(x_{2n}(\beta_k^{2n+1,2n}) - x_{2n+1}(\beta_k^{2n+1,2n})\big)^2 \ge 4^{-2k_2}(x_{2n} - x_{2n+1})^2(\gamma_n) > 4^{-2k_2}\varepsilon^2$$

which contradicts (3) and (1).

Now we stabilize the supports of the vectors x_n. By (3),

$$\begin{aligned}0 &\leq 2\sum 4^{-k}x_m^2(\alpha_k^m) + 2\sum 4^{-k}x_n^2(\alpha_k^n) - \sum 4^{-k}(x_m+x_n)^2(\beta_k^{m,n}) \\ &\quad - \Big(2\sum 4^{-k}x_m^2(\beta_k^{m,n}) + 2\sum 4^{-k}x_n^2(\beta_k^{m,n}) - \sum 4^{-k}(x_m+x_n)^2(\beta_k^{m,n})\Big) \\ &\leq 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m+x_n\|^2,\end{aligned}$$

which together with (2) and (1) gives

$$\lim_{m,n\to\infty}\Big(\sum 4^{-k}x_n^2(\alpha_k^n) - \sum 4^{-k}x_n^2(\beta_k^{m,n})\Big) = 0. \tag{11}$$

But, for every $j \in \mathbb{N}$

$$\begin{aligned}\sum_{k=1}^{\infty} 4^{-k}x_n^2(\alpha_k^n) - \sum_{k=1}^{\infty} 4^{-k}x_n^2(\beta_k^{m,n}) &= \sum_{k=1}^{\infty}\Big(4^{-k} - 4^{-(k+1)}\Big)\left(\sum_{i=1}^{k} x_n^2(\alpha_i^n) - \sum_{i=1}^{k} x_n^2(\beta_i^{m,n})\right) \\ &\geq \Big(4^{-j} - 4^{-(j+1)}\Big)\left(x_n^2(\alpha_j^n) - x_n^2(\alpha_{j+1}^n)\right),\end{aligned} \tag{12}$$

unless $\{\alpha_i^n;\ 1 \leq i \leq j\} = \{\beta_i^{m,n};\ 1 \leq i \leq j\}$.

Indeed, if $\{\alpha_i^n;\ 1 \leq i \leq j\} \neq \{\beta_i^{m,n};\ 1 \leq i \leq j\}$, then $x_n^2(\alpha_1^n) + x_n^2(\alpha_2^n) + \cdots + x_n^2(\alpha_{j-1}^n) + x_n^2(\alpha_{j+1}^n) \geq \sum_{i=1}^{j} x_n^2(\beta_i^{m,n})$.

Now assume that there is $\gamma \in \Gamma$ for which $x(\gamma) \neq y(\gamma)$ and without loss of generality $x(\gamma) \neq 0$. Let $A = \big\{\widetilde{\gamma} \in \Gamma;\ |x(\widetilde{\gamma})| \geq |x(\gamma)|\big\}$.

As $z_0(|A|) \neq 0$ and $z_0 \in c_0$, we can find $k_1 \geq |A|$ such that $z_0(k_1) > z_0(k_1+1)$. Put $\varepsilon = \frac{1}{3}\big(z_0(k_1) - z_0(k_1+1)\big)$. There is $n_1 \in \mathbb{N}$ such that $\big||x_{2n}(\alpha_k^{2n})| - z_0(k)\big| < \varepsilon$ for $n > n_1$ and $k \leq k_1+1$ and thus $|x_{2n}(\alpha_{k_1}^{2n})| - |x_{2n}(\alpha_{k_1+1}^{2n})| > \varepsilon$ for $n > n_1$. By putting this fact together with (12) and (11) we obtain $m_1 \in \mathbb{N}$ such that $\{\alpha_k^{2n};\ 1 \leq k \leq k_1\} = \{\beta_k^{m,2n};\ 1 \leq k \leq k_1\}$ for $m > m_1$ and $2n > m_1$. As $\{\alpha_k^{2m}\} = \{\beta_k^{2n,2m}\} = \{\beta_k^{2m,2n}\} = \{\alpha_k^{2n}\}$, by passing to a subsequence we may assume that $\alpha_k^{2n} = \alpha_k$ for $1 \leq k \leq k_1$ and $2n > m_1$. Moreover, as $\{\alpha_k^{2n}\} = \{\beta_k^{2n+1,2n}\}$, by passing to a further subsequence we may assume that $\beta_k^{2n+1,2n} = \alpha_k$ for $1 \leq k \leq k_1$ and $2n > m_1$.

Since $z_0 = z_1$, similarly as in previous paragraph we can conclude that $\alpha_k^{2n+1} = \beta_k^{2n,2n+1}$ for $1 \leq k \leq k_1$ and $2n > m_2$ for some $m_2 \in \mathbb{N}$. Moreover, $\alpha_k^{2n+1} = \beta_k^{2n,2n+1} = \beta_k^{2n+1,2n} = \alpha_k^{2n} = \alpha_k$ for $1 \leq k \leq k_1$ and $n > \max\{m_1, m_2\}/2$. The inequality (3) together with (1), the stabilization above and the pointwise convergence implies for $1 \leq k \leq k_1$

$$0 = \lim_{n\to\infty} x_{2n}(\beta_k^{2n,2n+1}) - x_{2n+1}(\beta_k^{2n,2n+1}) = \lim_{n\to\infty} x_{2n}(\alpha_k) - x_{2n+1}(\alpha_k) = x(\alpha_k) - y(\alpha_k).$$

We have $|x(\alpha_1)| \geq |x(\alpha_2)| \geq \cdots \geq |x(\alpha_{k_1})| \geq |x(\widetilde{\gamma})|$ for $\widetilde{\gamma} \in \Gamma \setminus \{\alpha_k;\ 1 \leq k \leq k_1\}$, because $x(\alpha_k) = \lim_{n\to\infty} x_{2n}(\alpha_k^{2n})$.

Finally we claim that $\gamma \in B = \{\alpha_k;\ 1 \leq k \leq k_1\}$. Otherwise $|x(\alpha_{k_1})| \geq |x(\gamma)|$, consequently $B \subset A$, but as $|B| = k_1 \geq |A|$, we see that $\gamma \in A = B$.

Therefore $x(\gamma) = y(\gamma)$ which contradicts the choice of γ.

$\square$

Proof of Theorem 1. The "if" part relies heavily on James' theorem: Let $\|\cdot\|$ be a W2R norm on X. Fix any $f \in X^* \setminus \{0\}$. Choose x_n in B_X such that $f(x_n) \to \|f\|$. Then $0 \leq 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m+x_n\|^2 \leq 4 - \|f\|^{-2}f(x_m+x_n)^2 \to 0$. Thus there is $x \in X$ such that $x_n \to x$ weakly, hence $f(x) = \lim f(x_n) = \|f\|$ and by James' theorem X is reflexive.

The "only if" part:

Observe that since X is reflexive, to show that $\|\cdot\|$ is W2R it only suffices to show that for any sequence $\{x_n\}$ satisfying (1) and such that $x_{2n} \to x \in X$ weakly and $x_{2n+1} \to y \in X$ weakly, we have $x = y$. Indeed, as $2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m+x_n\|^2 \geq \big(\|x_m\| - \|x_n\|\big)^2 \geq 0$, any sequence $\{x_n\}$ satisfying

(1) is bounded and hence relatively weakly compact and so we only need to show that it has only one weak cluster point. Obviously any subsequence of $\{x_n\}$ also satisfies (1).

It is very easy to construct an equivalent W2R norm on a separable reflexive X. Let $|\cdot|$ be the original norm on X, $\{f_k\}$ be a countable subset of B_{X^*} that distinguishes points of X. Define a new norm by

$$\|x\|^2 = |x|^2 + \sum_{k=1}^{\infty} 2^{-k} f_k(x)^2.$$

Clearly it is an equivalent norm on X. Let $\{x_n\}$ satisfy (1), $x_{2n} \to x \in X$ weakly and $x_{2n+1} \to y \in X$ weakly. As

$$\begin{aligned} 2\|x_{2n+1}\|^2 + 2\|x_{2n}\|^2 - \|x_{2n+1} + x_{2n}\|^2 &= 2|x_{2n+1}|^2 + 2|x_{2n}|^2 - |x_{2n+1} + x_{2n}|^2 \\ &\quad + \sum 2^{-k} \left(2f_k(x_{2n+1})^2 + 2f_k(x_{2n})^2 - f_k(x_{2n+1} + x_{2n})^2\right) \\ &\geq \big(|x_{2n+1}| - |x_{2n}|\big)^2 + \sum 2^{-k} \big(f_k(x_{2n+1}) - f_k(x_{2n})\big)^2 \geq 0 \end{aligned}$$

and all the summands in the last term are nonnegative, (1) implies $0 = \lim_{n\to\infty} f_k(x_{2n+1}) - f_k(x_{2n}) = f_k(y) - f_k(x)$ for any $k \in \mathbb{N}$. Therefore $x = y$.

In a general case of X nonseparable, let $T\colon X \to c_0(\Gamma)$ be a one-to-one bounded linear operator for some suitable Γ. (The existence of such an operator is well-known, see e.g. [DGZ, VI.5].) Define a norm on X by $\|x\|^2 = |x|^2 + \|Tx\|_D^2$, where $\|\cdot\|_D$ is Day's norm on $c_0(\Gamma)$. Clearly it is an equivalent norm on X. Let $\{x_n\}$ satisfy (1), $x_{2n} \to x \in X$ weakly and $x_{2n+1} \to y \in X$ weakly. Similarly as above, (1) implies that $\lim_{m,n\to\infty} 2\|Tx_m\|_D^2 + 2\|Tx_n\|_D^2 - \|Tx_m + Tx_n\|_D^2 = 0$ and so $\{Tx_n\}$ satisfies (1) in the norm $\|\cdot\|_D$. Since T is w–w-continuous, $Tx_{2n} \to Tx$ weakly and $Tx_{2n+1} \to Ty$ weakly (and thus also in the pointwise topology of $c_0(\Gamma)$) and we can apply Proposition 2 to the sequence $\{Tx_n\}$ to obtain $Tx = Ty$. Finally, as T is one-to-one, $x = y$.

□

Remark. The method of stabilization of the support of the sequence in $c_0(\Gamma)$ in the proof of Proposition 2 was inspired by Rainwater's proof of the fact that Day's norm is locally uniformly rotund.

The following two examples show that Troyanski's construction of the LUR norm on a reflexive space is neither sufficient for nor overcome by W2R renorming.

Example 3. *There is an equivalent norm $\|\cdot\|$ on ℓ_2 such that it is W2R but there is a point in $S_{(\ell_2,\|\cdot\|)}$ which is not strongly exposed (and thus neither $\|\cdot\|$ is LUR nor $\|\cdot\|^*$ is Fréchet differentiable).*

Proof. Let $\|\cdot\|_2$ be the canonical norm on ℓ_2 and let us define the new norm by

$$\|x\|^2 = \big(\max\{\|x\|_2, 2|x_1|\}\big)^2 + \sum_{i=2}^{\infty} 2^{-i} x_i^2.$$

This is clearly an equivalent norm.

Let us denote the i-th coordinate of a vector $x_n \in \ell_2$ by $x_n(i)$. In view of the construction of the W2R norm on a separable reflexive space in the proof of Theorem 1 if remains to show that if $\{x_n\}$ satisfies (1), $x_{2n} \to x \in \ell_2$ weakly and $x_{2n+1} \to y \in \ell_2$ weakly then $x(1) = y(1)$. By passing to a subsequence we may assume that either always $\|x_{2n} + x_{2n+1}\|_2 \geq 2|x_{2n}(1) + x_{2n+1}(1)|$ or always $2|x_{2n}(1) + x_{2n+1}(1)| \geq \|x_{2n} + x_{2n+1}\|_2$. In the first case

$$\begin{aligned} &2\|x_{2n+1}\|^2 + 2\|x_{2n}\|^2 - \|x_{2n+1} + x_{2n}\|^2 \\ &\quad= 2\big(\max\{\|x_{2n+1}\|_2, 2|x_{2n+1}(1)|\}\big)^2 + 2\big(\max\{\|x_{2n}\|_2, 2|x_{2n}(1)|\}\big)^2 - \|x_{2n+1} + x_{2n}\|_2^2 \\ &\qquad + 2\sum_{i=2}^{\infty} 2^{-i} x_{2n+1}(i)^2 + 2\sum_{i=2}^{\infty} 2^{-i} x_{2n}(i)^2 - \sum_{i=2}^{\infty} 2^{-i} \big(x_{2n+1}(i) + x_{2n}(i)\big)^2 \\ &\quad\geq 2\|x_{2n+1}\|_2^2 + 2\|x_{2n}\|_2^2 - \|x_{2n+1} + x_{2n}\|_2^2 \geq 0 \end{aligned}$$

and the uniform rotundity of $\|\cdot\|_2$ implies $x=y$. In the second case similarly

$$\begin{aligned}&2\|x_{2n+1}\|^2+2\|x_{2n}\|^2-\|x_{2n+1}+x_{2n}\|^2\\&\qquad\geq 2\cdot 4|x_{2n+1}(1)|^2+2\cdot 4|x_{2n}(1)|^2-4|x_{2n+1}(1)+x_{2n}(1)|^2=4\big(x_{2n+1}(1)-x_{2n}(1)\big)^2\geq 0.\end{aligned}$$

Thus obviously $x(1)=y(1)$ and we can conclude that $\|\cdot\|$ is W2R.

The point $\frac{e_1}{2}\in S_{(\ell_2,\|\cdot\|)}$ is not a strongly exposed point of $B_{(\ell_2,\|\cdot\|)}$. Indeed, choose any $f\in S_{(\ell_2,\|\cdot\|^*)}$ such that $f(\frac{e_1}{2})=1$ and put $x_n=\frac{1}{2}(e_1+e_n)$ and $y_n=\frac{1}{2}(e_1-e_n)$. Then, $\|x_n\|=\|y_n\|=(1+2^{-n}\frac{1}{4})^{1/2}\to 1$ and $1\geq f(x_n/\|x_n\|)\geq f\big((x_n+y_n)/\|x_n\|\big)-1=f\big(e_1/\|x_n\|\big)-1\to 1$, but $\|x_n-\frac{e_1}{2}\|=\|\frac{e_n}{2}\|=(\frac{1}{4}+2^{-n}\frac{1}{4})^{1/2}>\frac{1}{2}$.

□

Example 4. *There is an equivalent norm $\|\cdot\|$ on ℓ_2 such that it is LUR but not W2R.*

Proof. Let us define the norm on ℓ_2 by

$$\begin{aligned}|x|_{i,j}^2&=\big(|x_1|+|x_i|+|x_j|\big)^2+\frac{1}{i+j}\left(x_1^2+x_i^2+x_j^2\right)+\sum_{\substack{k=2\\k\neq i,j}}^{\infty}x_k^2\\\|x\|^2&=\sup_{1<i<j}\left\{|x|_{i,j}^2\right\}.\end{aligned}$$

This is clearly an equivalent norm.

We claim that locally (away from the origin) the supremum can be taken over a finite set. To see this fix any $x\in\ell_2\setminus\{0\}$. We have to distinguish two cases.

If we can choose $k>1$ such that $x_k\neq 0$ then there is i_0 such that $|x_i|<\frac{|x_k|}{3}$ for $i>i_0$ and if we denote by $m(y)\in\mathbb{N}$ the largest index for which $|y_{m(y)}|=\max\{|y_i|,\,i>1\}$ then clearly $m(y)\leq i_0$ for any $y\in\ell_2$ such that $\|x-y\|_2<\frac{|x_k|}{3}$. Let $\varepsilon=\frac{1}{3i_0}\frac{x_k^2}{4}\frac{1}{16\|x\|_2}$ and find j_0 such that $8\|x\|_2\big(|x_j|+\varepsilon\big)+\frac{4\|x\|_2^2}{j}<\frac{1}{3i_0}\frac{x_k^2}{4}$ for $j>j_0$. Then for any $y\in\ell_2$ such that $\|x-y\|_2<\min\left\{\frac{|x_k|}{3},\varepsilon\right\}$ we have

$$\begin{aligned}|y|_{i,j}^2&=\|y\|_2^2+2\big(|y_1y_i|+|y_1y_j|+|y_iy_j|\big)+\frac{1}{i+j}\big(y_1^2+y_i^2+y_j^2\big)\\&\leq\|y\|_2^2+2|y_1||y_i|+4\|y\|_2|y_j|+\frac{1}{i+j}\|y\|_2^2\leq\|y\|_2^2+2|y_1||y_{m(y)}|+8\|x\|_2\big(|x_j|+\varepsilon\big)+\frac{4}{i+j}\|x\|_2^2\\&<\|y\|_2^2+2|y_1||y_{m(y)}|+\frac{1}{3i_0}\frac{x_k^2}{4}\qquad\text{for } j>j_0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|y\|^2&\geq\|y\|_2^2+2|y_1||y_{m(y)}|+\frac{1}{m(y)+2m(y)}y_{m(y)}^2\\&\geq\|y\|_2^2+2|y_1||y_{m(y)}|+\frac{1}{3i_0}y_k^2>\|y\|_2^2+2|y_1||y_{m(y)}|+\frac{1}{3i_0}\frac{x_k^2}{4}.\end{aligned}$$

In the second case we have $x_i=0$ for $i>1$ and $x_1\neq 0$. Let $\varepsilon=\frac{1}{5}\frac{x_1^2}{4}\frac{1}{24\|x\|_2}$ and find j_0 such that $12\|x\|_2\varepsilon+\frac{4\|x\|_2^2}{j}<\frac{1}{5}\frac{x_1^2}{4}$ for $j>j_0$. Then for any $y\in\ell_2$ such that $\|x-y\|_2<\min\left\{\frac{|x_1|}{2},\varepsilon\right\}$ we have

$$|y|_{i,j}^2\leq\|y\|_2^2+2\|y\|_2 3\varepsilon+\frac{1}{i+j}\|y\|_2^2\leq\|y\|_2^2+12\|x\|_2\varepsilon+\frac{4}{i+j}\|x\|_2^2<\|y\|_2^2+\frac{1}{5}\frac{x_1^2}{4}\qquad\text{for } j>j_0.$$

On the other hand, $\|y\|^2\geq\|y\|_2^2+\frac{1}{5}y_1^2>\|y\|_2^2+\frac{1}{5}\frac{x_1^2}{4}$.

Because

$$|x|_{i,j}^2=\frac{1}{i+j}\|x\|_2^2+\big(|x_1|+|x_i|+|x_j|\big)^2+\left(1-\frac{1}{i+j}\right)\sum_{\substack{k=2\\k\neq i,j}}^{\infty}x_k^2,$$

it is clearly a LUR norm for each i, j and thus $\|\cdot\|$ is also LUR as it is locally a maximum of a finitely many LUR norms.

Now put $x_{2n} = \frac{1}{2}(e_1 + e_{2n})$ and $x_{2n+1} = e_{2n+1}$. Then we can easily compute $\|x_{2n}\|^2 = 1 + \frac{1}{2(2n+2)} \to 1$, $\|x_{2n+1}\|^2 = 1 + \frac{1}{2n+3} \to 1$ and

$$\|x_m + x_n\|^2 = \begin{cases} \|e_1 + \frac{e_m+e_n}{2}\|^2 = 4 + \frac{3}{2(m+n)} \to 4, \\ \|e_m + e_n\|^2 = 4 + \frac{2}{m+n} \to 4, \\ \|\frac{e_1+e_m}{2} + e_n\|^2 = 4 + \frac{3}{2(m+n)} \to 4. \end{cases}$$

But since $x_{2n} \to \frac{e_1}{2}$ weakly and $x_{2n+1} \to 0$ weakly, the norm $\|\cdot\|$ is not W2R.

□

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