ISOMORPHIC EMBEDDINGS AND HARMONIC BEHAVIOUR OF SMOOTH OPERATORS

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Abstract. Let Y be a Banach space, $1 < p < \infty$. We give a simple criterion for embedding $\ell_p \subset Y$, namely it suffices that the positive cone $\ell_p^+ \subset Y$. This result is applied to the study of highly smooth operators from ℓ_p into Y (p is not an even integer). The main result is that every such operator has a harmonic behaviour unless $\ell_{\frac{p}{K}} \subset Y$ for some $K \in \mathbb{N}$.

In this note we establish a natural criterion for embedding of ℓ_p or c_0 into a given Banach space and apply it to smooth operators with harmonic behaviour from ℓ_p spaces.

Recall that the well-known summing basis $\{e_i\}$ of c_0 has the property that $\|\sum a_i e_i\| = \sum a_i$ provided that $a_i \geq 0$, which means (in our notation) that $\ell_1^+ \subset c_0$. In fact, and more surprisingly, $\ell_1^+ \subset Y$ for any Banach space Y. Moreover, if Y is separable, then there exists a minimal and fundamental system in Y whose positive cone is isomorphic to ℓ_1^+ ([S1], [S2], [DJ]). In our paper we prove a result going in the opposite direction, that $Z^+ \subset Y$ already implies $Z \subset Y$ for $Z = \ell_p$, $1 < p < \infty$, or c_0.

This simple and somewhat unexpected criterion allows us to completely characterize Banach spaces Y, for which there exist separating polynomial (or smooth enough) operators from ℓ_p into Y, as those for which $\ell_{\frac{p}{k}} \subset Y$ for some integer k.

1. Embedding of the Positive Cone

Let Y be a Banach space, Z be a Banach space with a Schauder basis $\{e_i\}$. Let us denote the positive cone of Z by $Z^+ = \{\, z \in Z;\ z = \sum a_i e_i,\ a_i \geq 0 \,\}$. We say that $Z^+ \subset Y$ if there is a basic sequence $\{y_i\}$ in Y such that $\|\sum a_i e_i\|_Z \leq \|\sum a_i y_i\|_Y \leq C \|\sum a_i e_i\|_Z$ for any $\sum a_i e_i \in Z^+$. We say that C is an *isomorphism constant*.

For $a \in \mathbb{R}$, let $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$.

Theorem 1. *Let Y be a Banach space. If $c_0^+ \subset Y$ then $c_0 \subset Y$. Moreover, $\{y_i\}$ is equivalent to the canonical basis of c_0.*

Proof. Let $\sum a_i y_i \in Y$. Then by assumption

$$\Bigl\|\sum a_i y_i\Bigr\| = \Bigl\|\sum a_i^+ y_i - \sum a_i^- y_i\Bigr\| \leq \Bigl\|\sum a_i^+ y_i\Bigr\| + \Bigl\|\sum a_i^- y_i\Bigr\| \\ \leq C\max\{a_i^+\} + C\max\{a_i^-\} \leq 2C\max\bigl\{|a_i|\bigr\}.$$

But, as $\{y_i\}$ is a basic sequence,

$$\Bigl\|\sum a_i y_i\Bigr\| \geq \frac{1}{2K}\max\bigl\{\|a_i y_i\|\bigr\} \geq \frac{1}{2K}\max\bigl\{|a_i|\bigr\},$$

where K is a basis constant of $\{y_i\}$.

□

Theorem 2. *Let Y be a Banach space, $1 < p < \infty$. If $\ell_p^+ \subset Y$ then $\ell_p \subset Y$.*

First notice the following lemma:

Lemma 3. *Let Z be a Banach space with an unconditional Schauder basis $\{e_i\}$, Y be a Banach space and $Z^+ \subset Y$ such that $\{y_i\}$ is an unconditional basic sequence. Then $Z \subset Y$ (in fact $\{y_i\}$ is equivalent to $\{e_i\}$).*

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Proof. There is a $K_1 \geq 1$ such that $K_1^{-1}\left\|\sum |a_i|\, y_i\right\|_Y \leq \left\|\sum a_i y_i\right\|_Y \leq K_1\left\|\sum |a_i|\, y_i\right\|_Y$ for any $\sum a_i y_i \in Y$ and a $K_2 \geq 1$ such that $K_2^{-1}\left\|\sum |a_i|\, e_i\right\|_Z \leq \left\|\sum a_i e_i\right\|_Z \leq K_2\left\|\sum |a_i|\, e_i\right\|_Z$ for any $\sum a_i e_i \in Z$. Thus $K_1^{-1}K_2^{-1}\left\|\sum a_i e_i\right\|_Z \leq \left\|\sum a_i y_i\right\|_Y \leq K_1 C K_2\left\|\sum a_i e_i\right\|_Z$ for any $\sum a_i e_i \in Z$.

□

Proof of Theorem 2. We claim that there is an unconditional normalized block basic sequence of $\{y_i\}$ such that all its vectors have nonnegative coordinates with respect to $\{y_i\}$. Then it is easily seen by Lemma 3 that this block basic sequence is equivalent to the canonical basis of ℓ_p.

For $x = \sum a_i y_i \in Y$ we denote $x^+ = \sum a_i^+ y_i$, $x^- = \sum a_i^- y_i$ and $\widehat{x} = \sum a_i e_i \in \ell_p$.

Suppose that $\{y_i\}$ is not unconditional and $\ell_p^+ \subset Y$ with isomorphism constant C. Then for any $\varepsilon > 0$ there is $y \in \overline{\operatorname{span}}\{y_i\}$ with finite support such that $\|y^+\| = 1$ and $\|y\| < \varepsilon$. If this was not true for some $\varepsilon > 0$, then for any $x \in \overline{\operatorname{span}}\{y_i\}$

$$\|x\| \geq \varepsilon \max\Big\{\|x^+\|, \|x^-\|\Big\} \geq \frac{\varepsilon}{2}\Big(\|x^+\| + \|x^-\|\Big) \geq \frac{\varepsilon}{2}\left\|x^+ + x^-\right\|.$$

On the other hand

$$\|x\| = \left\|x^+ - x^-\right\| \leq \left\|x^+\right\| + \left\|x^-\right\| \leq C\left(\left\|\widehat{x^+}\right\|_p + \left\|\widehat{x^-}\right\|_p\right) \leq C 2^{1-\frac{1}{p}}\left\|\widehat{x^+} + \widehat{x^-}\right\|_p \leq C 2^{1-\frac{1}{p}}\left\|x^+ + x^-\right\|,$$

which means that $\{y_i\}$ would be unconditional.

Thus we can construct a block basic sequence $\{v_i\}$ of $\{y_i\}$ such that $\|v_i\| < \frac{1}{2}\frac{1}{2^i}$ and $\left\|\widehat{v_i^+}\right\|_p = 1$. Let $\{a_j\}_{j=1}^n$ be a finite sequence of nonnegative real numbers. Then

$$\left\|\sum_{j=1}^n a_j v_j\right\| \leq \sum_{j=1}^n a_j\|v_j\| \leq \max\{a_j\}\sum_{j=1}^n \|v_j\| \leq \frac{1}{2}\left(\sum_{j=1}^n a_j^p\right)^{\frac{1}{p}} \quad \text{and} \tag{1}$$

$$\begin{aligned}\left\|\sum_{j=1}^n a_j v_j\right\| &= \left\|\sum_{j=1}^n a_j v_j^+ - \sum_{j=1}^n a_j v_j^-\right\| \geq \left\|\sum_{j=1}^n a_j v_j^-\right\| - \left\|\sum_{j=1}^n a_j v_j^+\right\| \\ &\geq \left\|\sum_{j=1}^n a_j \widehat{v_j^-}\right\|_p - C\left\|\sum_{j=1}^n a_j \widehat{v_j^+}\right\|_p = \left\|\sum_{j=1}^n a_j \widehat{v_j^-}\right\|_p - C\left(\sum_{j=1}^n a_j^p\right)^{\frac{1}{p}},\end{aligned}$$

which implies

$$\left\|\sum_{j=1}^n a_j \widehat{v_j^-}\right\|_p \leq \left(C + \frac{1}{2}\right)\left(\sum_{j=1}^n a_j^p\right)^{\frac{1}{p}}. \tag{2}$$

As $\left\|\widehat{v_j^+}\right\|_p = 1$, we can easily see that

$$\left\|\sum_{j=1}^n a_j v_j^+\right\| \geq \left\|\sum_{j=1}^n a_j \widehat{v_j^+}\right\|_p = \left(\sum_{j=1}^n a_j^p\right)^{\frac{1}{p}}. \tag{3}$$

On the other hand, take $f \in S_{(\overline{\operatorname{span}}\{y_i\})^*}$ such that $f\left(\sum a_j v_j^+\right) = \left\|\sum a_j v_j^+\right\|$. Let $b_i = f(y_i)$, $i \in \mathbb{N}$ and $M = \bigcup_{j=1}^n \operatorname{supp} v_j$ (notice that this is a finite set). Define $g = \sum_{k\in M} b_k y_k^*$, $g^+ = \sum_{k\in M} b_k^+ y_k^*$, $\widehat{g} = \sum_{k\in M} b_k e_k^*$ and $\widehat{g^+} = \sum_{k\in M} b_k^+ e_k^*$, where y_k^* and e_k^* are the biorthogonal functionals to y_k and e_k respectively. Let $\frac{1}{p} + \frac{1}{q} = 1$ and put $y = \sum_{k\in M}(b_k^+)^{q-1} y_k$. Then

$$\left\|\widehat{g^+}\right\|_q = \left(\sum_{k\in M}(b_k^+)^q\right)^{\frac{1}{q}} = \frac{\sum_{k\in M}(b_k^+)^q}{\left(\sum_{k\in M}(b_k^+)^q\right)^{\frac{1}{p}}} = \frac{g(y)}{\|\widehat{y}\|_p} \leq C\frac{g(y)}{\|y\|} = C\frac{f(y)}{\|y\|} \leq C\|f\| = C. \tag{4}$$

Using (3) we have

$$\left\|\sum_{j=1}^n a_j v_j\right\| \ge f\left(\sum_{j=1}^n a_j v_j\right) = f\left(\sum_{j=1}^n a_j v_j^+\right) - f\left(\sum_{j=1}^n a_j v_j^-\right) = \left\|\sum_{j=1}^n a_j v_j^+\right\| - g\left(\sum_{j=1}^n a_j v_j^-\right)$$
$$\ge \left\|\sum_{j=1}^n a_j v_j^+\right\| - g^+\left(\sum_{j=1}^n a_j v_j^-\right) \ge \left(\sum_{j=1}^n a_j^p\right)^{\frac{1}{p}} - g^+\left(\sum_{j=1}^n a_j v_j^-\right).$$

Let us denote $M^+ = \bigcup_{j=1}^n \operatorname{supp} v_j^+$, $M^- = \bigcup_{j=1}^n \operatorname{supp} v_j^-$, $\widehat{g^+}\restriction_{M^+} = \sum_{k\in M^+} b_k^+ e_k^*$ and $\widehat{g^+}\restriction_{M^-}$ similarly. The last inequality together with (1), the Hölder inequality and (2) gives

$$\frac{1}{2}\left(\sum_{j=1}^n a_j^p\right)^{\frac{1}{p}} \le g^+\left(\sum_{j=1}^n a_j v_j^-\right) \le \left\|\widehat{g^+}\restriction_{M^-}\right\|_q \left\|\sum_{j=1}^n a_j \widehat{v_j^-}\right\|_p$$
$$\le \left\|\widehat{g^+}\restriction_{M^-}\right\|_q \left(C+\frac{1}{2}\right)\left(\sum_{j=1}^n a_j^p\right)^{\frac{1}{p}},$$

which means that

$$\left\|\widehat{g^+}\restriction_{M^-}\right\|_q \ge \frac{1}{2C+1}.$$

If we combine this inequality with (4), we obtain

$$\left\|\widehat{g^+}\restriction_{M^+}\right\|_q \le \left(\left\|\widehat{g^+}\right\|_q^q - \left\|\widehat{g^+}\restriction_{M^-}\right\|_q^q\right)^{\frac{1}{q}} \le \left(C^q - \frac{1}{(2C+1)^q}\right)^{\frac{1}{q}}. \tag{5}$$

This finally allows us to compute

$$\left\|\sum_{j=1}^n a_j v_j^+\right\| = f\left(\sum_{j=1}^n a_j v_j^+\right) = g\left(\sum_{j=1}^n a_j v_j^+\right) \le g^+\left(\sum_{j=1}^n a_j v_j^+\right) \le \left\|\widehat{g^+}\restriction_{M^+}\right\|_q \left\|\sum_{j=1}^n a_j \widehat{v_j^+}\right\|_p$$
$$= \left\|\widehat{g^+}\restriction_{M^+}\right\|_q \left(\sum_{j=1}^n a_j^p\right)^{\frac{1}{p}} \le C\left(1-\frac{1}{C^q(2C+1)^q}\right)^{\frac{1}{q}}\left(\sum_{j=1}^n a_j^p\right)^{\frac{1}{p}}.$$

The last inequality and (3) shows that we have found a semi-normalized block basis $\{v_i^+\}$ such that ℓ_p^+ embeds into $\overline{\operatorname{span}}\{v_i^+\}$ with an isomorphism constant strictly less than C. Now either $\{v_i^+\}$ is an unconditional basic sequence and we are done, or we can iterate the process to find another block basis. (Notice that in every iteration the constructed block basis is a block basis of $\{y_i\}$ such that all of its vectors have nonnegative coordinates with respect to the previous basis and hence with respect to $\{y_i\}$.) In every iteration, the isomorphism constant drops at least by the factor of $\left(1-\frac{1}{C^q(2C+1)^q}\right)^{1/q} < 1$, where C is the initial isomorphism constant corresponding to $\{y_i\}$. Therefore after finitely many steps we obtain an unconditional block basic sequence as we claimed, otherwise the isomorphism constant would eventually drop below 1, which is impossible.

□

Remarks. We have actually proven that ℓ_p is isomorphic to a subspace spanned by a normalized block basis with all the blocks having their coordinates with respect to $\{y_i\}$ nonnegative. If the embedding $\ell_p^+ \subset Y$ is almost an isometry, we can actually show that $\overline{\operatorname{span}}\{y_i\}$ is already isomorphic to ℓ_p. This is in fact contained in the previous proof, but direct proof (see Proposition 4) gives optimal constant, as we will see in Example 5.

Notice further, that in the case $Z = \ell_p$ or c_0, $1 < p < \infty$, we do not need $\{y_i\}$ to be a basic sequence in the definition of $Z^+ \subset Y$, just any sequence suffices. This can be seen as follows: Let $f \in (\overline{\operatorname{span}}\{y_i\})^*$. Similarly as in (4) we can show that $\sum\big(f(y_i)^+\big)^q < \infty$ and $\sum\big(f(y_i)^-\big)^q < \infty$. This means that $y_i \to 0$ weakly, and thus some subsequence of $\{y_i\}$ is a basic sequence (see e.g. [LT, Remark after 1.a.5]).

By closer examination of the proof, we can see that it works more generally for spaces Z with the following property: Z has an unconditional basis $\{e_i\}$ with unconditional basis constant K, and there is a nonincreasing function $G\colon (0,+\infty) \to (0,\frac{1}{K})$ such that for any two nonzero disjointly supported f,

$g \in \operatorname{span}\{e_i^*\}$, $\|f+g\|^* = 1$ we have $\|f\|^* \le G(\|g\|^*)$. (This fact will replace the inequality (5).) Then the proof gives a block basis of $\{e_i\}$ that is equivalent to the block basis of $\{y_i\}$ generated by the same (nonnegative) coefficients. (For example for ℓ_p with the canonical basis and with the canonical norm we can take $G(x) = (1-x^q)^{1/q}$. As the canonical basis in ℓ_p is equivalent to any of its block bases, we obtain the conclusion of Theorem 2. More generally, such a function exists for example for super-reflexive spaces, but also clearly for c_0.)

Proposition 4. *Let Y be a Banach space, $1 < p < \infty$. If $\ell_p^+ \subset Y$ with isomorphism constant $C < 2^{1-\frac{1}{p}}$, then $\{y_i\}$ is equivalent to the canonical basis of ℓ_p.*

Proof. By assumption there is a basic sequence $\{y_i\}$ in Y such that $\|\widehat{x^+}\|_p \le \|x^+\| \le C\|\widehat{x^+}\|_p$ for any $x \in \overline{\operatorname{span}}\{y_i\}$.

Let $x \in \overline{\operatorname{span}}\{y_i\}$. Then

$$\|x\| = \|x^+ - x^-\| \le \|x^+\| + \|x^-\| \le C\|\widehat{x^+}\|_p + C\|\widehat{x^-}\|_p \le 2^{1-\frac{1}{p}} C\, \|\widehat{x}\|_p .$$

On the other hand, choose $f \in S_{(\overline{\operatorname{span}}\{y_i\})^*}$ such that $f(x^+) = \|x^+\|$. Let $b_i = f(y_i)$, $i \in \mathbb{N}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Without loss of generality we may assume that $\|\widehat{x^+}\|_p \ge \|\widehat{x^-}\|_p$. Similarly as in (4) we can show that $\left(\sum (b_i^+)^q\right)^{\frac{1}{q}} \le C$. Further,

$$\Big(\sum_{i \in \operatorname{supp} x^+} (b_i^+)^q\Big)^{\frac{1}{q}} \ge \frac{\sum a_i^+ b_i^+}{\|\widehat{x^+}\|_p} \ge \frac{f(x^+)}{\|\widehat{x^+}\|_p} = \frac{\|x^+\|}{\|\widehat{x^+}\|_p} \ge 1.$$

Using these two estimates we obtain

$$\begin{aligned}
\|x\| &\ge f(x) = f(x^+) - f(x^-) = f(x^+) - \sum a_i^- b_i \ge f(x^+) - \sum a_i^- b_i^+ \\
&\ge \|\widehat{x^+}\|_p - \Big(\sum_{i \in \operatorname{supp} x^-} (b_i^+)^q\Big)^{\frac{1}{q}} \|\widehat{x^-}\|_p \ge \|\widehat{x^+}\|_p \bigg(1 - \Big(\sum_{i \in \operatorname{supp} x^-} (b_i^+)^q\Big)^{\frac{1}{q}}\bigg) \\
&\ge \|\widehat{x^+}\|_p \bigg(1 - \Big(\sum (b_i^+)^q - \sum_{i \in \operatorname{supp} x^+} (b_i^+)^q\Big)^{\frac{1}{q}}\bigg) \ge \|\widehat{x^+}\|_p \Big(1 - (C^q - 1)^{\frac{1}{q}}\Big),
\end{aligned}$$

and hence

$$\|x\| \ge \Big(1 - (C^q-1)^{\frac{1}{q}}\Big) \max\Big\{\|\widehat{x^+}\|_p,\ \|\widehat{x^-}\|_p\Big\} \ge 2^{-\frac{1}{p}} \Big(1 - (C^q-1)^{\frac{1}{q}}\Big) \|\widehat{x}\|_p .$$

As $C < 2^{\frac{1}{q}}$, we have $\Big(1 - (C^q-1)^{\frac{1}{q}}\Big) > 0$ and so $\{y_i\}$ is equivalent to the canonical basis of ℓ_p.

□

Example 5. *For any $1 < p < \infty$ there is a space X isomorphic to $c_0 \oplus \ell_p$ with a Schauder basis $\{y_i\}$, such that ℓ_p^+ embeds into X onto a positive cone generated by $\{y_i\}$ with isomorphism constant $2^{1-\frac{1}{p}}$.*

By Theorem 2, there is a block basis of $\{y_i\}$ equivalent to the canonical basis of ℓ_p, but as X is isomorphic to $c_0 \oplus \ell_p$, $\{y_i\}$ is not equivalent to a basis of ℓ_p. This example shows that the constant in Proposition 4 is optimal.

Proof. Let X be the completion of the space c_{00} equipped with the norm

$$\|(a_i)\| = \max\bigg\{\max\{a_i\},\ \Big(\sum |a_{2i} + a_{2i+1}|^p\Big)^{\frac{1}{p}}\bigg\}.$$

This space has a natural basis $\{y_i\}$ consisting of the vectors that has the ith coordinate equal to 1 and all the others equal to 0. For any vector $x \in X$, the decomposition

$$x = \sum a_i y_i = \Big(\sum \frac{a_{2i} - a_{2i+1}}{2}(y_{2_i} - y_{2i+1})\Big) + \Big(\sum \frac{a_{2i} + a_{2i+1}}{2}(y_{2_i} + y_{2i+1})\Big)$$

implies that X is isomorphic to $c_0 \oplus \ell_p$.

For any $x = \sum a_i y_i \in X$, where $a_i \ge 0$ for all $i \in \mathbb{N}$, we have

$$\|x\| \le \Big\|\sum a_{2i} y_{2i}\Big\| + \Big\|\sum a_{2i+1} y_{2i+1}\Big\| = \Big(\sum a_{2i}^p\Big)^{\frac{1}{p}} + \Big(\sum a_{2i+1}^p\Big)^{\frac{1}{p}} \le 2^{1-\frac{1}{p}} \Big(\sum a_i^p\Big)^{\frac{1}{p}} .$$

On the other hand
$$\|x\| \geq \left(\sum (a_{2i}+a_{2i+1})^p\right)^{\frac{1}{p}} \geq \left(\sum a_i^p\right)^{\frac{1}{p}},$$
and therefore $\ell_p^+ \subset X$ with isomorphism constant $2^{1-\frac{1}{p}}$.

□

Remark. If the space Y is complex, Theorem 1 holds by a trivial modification of the proof. Theorem 2 is also valid in the complex case, but the given proof implies only that (the real) $\ell_p \subset Y_{\mathbb{R}}$ (i.e. the space Y considered as a real vector space). The complex embedding requires some additional work, which we briefly sketch:

Suppose that we already have the real embedding, i.e. for any real sequence $\{b_j\}$ we have $C_1 \left\|\sum b_j e_j\right\|_p \leq \left\|\sum b_j y_j\right\| \leq C_2 \left\|\sum b_j e_j\right\|_p$. Suppose further that $\{y_j\}$ is not equivalent to $\{e_j\}$. Then (as the upper estimate always holds, just consider the real and imaginary parts), we can construct a block basis $\{w_j\}$ of $\{y_j\}$ such that $\|w_j\| < \frac{\varepsilon}{2^j}$ and $\|\operatorname{Re} w_j\| = 1$, where $\varepsilon < \frac{C_1}{C_2}(1+\frac{C_1}{C_2})$. Then for any complex sequence $\{a_j\}$
$$\begin{aligned}
\left\|\sum a_j \operatorname{Re} w_j\right\| &= \left\|\sum \operatorname{Re} a_j \operatorname{Re} w_j + \sum \operatorname{Im} a_j i \operatorname{Re} w_j\right\| \\
&\geq \left\|\sum \operatorname{Re} a_j \operatorname{Re} w_j + \sum \operatorname{Im} a_j \operatorname{Im} w_j\right\| - \left\|\sum \operatorname{Im} a_j (i \operatorname{Re} w_j - \operatorname{Im} w_j)\right\| \\
&= \left\|\sum \operatorname{Re} a_j \operatorname{Re} w_j + \sum \operatorname{Im} a_j \operatorname{Im} w_j\right\| - \left\|\sum i \operatorname{Im} a_j w_j\right\| \\
&\geq C_1 \left\|\sum \operatorname{Re} a_j \operatorname{Re} \widehat{w}_j + \sum \operatorname{Im} a_j \operatorname{Im} \widehat{w}_j\right\|_p - \varepsilon \left(\sum |a_j|^p\right)^{\frac{1}{p}} \\
&= C_1 \left(\sum \|\operatorname{Re} a_j \operatorname{Re} \widehat{w}_j + \operatorname{Im} a_j \operatorname{Im} \widehat{w}_j\|_p^p\right)^{\frac{1}{p}} - \varepsilon \left(\sum |a_j|^p\right)^{\frac{1}{p}} \\
&\geq \frac{C_1}{C_2} \left(\sum \|\operatorname{Re} a_j \operatorname{Re} w_j + \operatorname{Im} a_j \operatorname{Im} w_j\|^p\right)^{\frac{1}{p}} - \varepsilon \left(\sum |a_j|^p\right)^{\frac{1}{p}} \\
&\geq \frac{C_1}{C_2} \left(\sum \Big| \|\operatorname{Re} a_j \operatorname{Re} w_j + i \operatorname{Im} a_j \operatorname{Re} w_j\| - \|\operatorname{Im} a_j (\operatorname{Im} w_j - i \operatorname{Re} w_j)\| \Big|^p\right)^{\frac{1}{p}} - \varepsilon \left(\sum |a_j|^p\right)^{\frac{1}{p}} \\
&\geq \frac{C_1}{C_2} \left(\sum \Big| \|a_j \operatorname{Re} w_j\| - \frac{\varepsilon}{2^j} |a_j| \Big|^p\right)^{\frac{1}{p}} - \varepsilon \left(\sum |a_j|^p\right)^{\frac{1}{p}} \\
&= \frac{C_1}{C_2} \left(\sum |a_j|^p \left|1 - \frac{\varepsilon}{2^j}\right|^p\right)^{\frac{1}{p}} - \varepsilon \left(\sum |a_j|^p\right)^{\frac{1}{p}} \geq \left(\sum |a_j|^p\right)^{\frac{1}{p}} \left(\frac{C_1}{C_2}(1-\varepsilon) - \varepsilon\right).
\end{aligned}$$

2. Harmonic Behaviour of Smooth Operators

First let us fix some notation. By $C^n(B_X, Y)$, $1 \leq n < \infty$ we denote the space of all n-times continuously Fréchet differentiable operators from some neighbourhood of B_X into Y. We say that $T \in C^{n,+}(B_X, Y) \subset C^n(B_X, Y)$ if $T^{(n)}(x)$ is uniformly continuous and $T \in C^{n,\alpha}(B_X, Y) \subset C^{n,+}(B_X, Y)$ if $T^{(n)}(x)$ is α-Hölder.

Definition. *Let X, Y be Banach spaces. We say that an operator $T\colon B_X \to Y$ has a* harmonic behaviour *if $T(B_X) \subset \overline{T(S_X)}$. We say that T is* separating *if* $\inf_{x \in S_X} \|T(x) - T(0)\| > 0$.

The close relation of these two notions is exposed in Lemma 8. In some sense, a very smooth separating operator is an analogue of linear embedding. (This claim is justified by Theorem 9.)

Bonic and Frampton in [BF] showed that if Y admits a $C^{k,\alpha}$-smooth bump but X does not, then every $C^{k,\alpha}$-smooth operator $T\colon B_X \to Y$ has a harmonic behaviour. Some variants of this result were also presented in [DGZ, chapter III] and [BL, ch. 10], as they are related to smooth uniform homeomorphisms between Banach spaces.

Recently, Deville and Matheron in [DM] showed that if Y has a nontrivial cotype but X has not, then every $C^{1,+}$-smooth operator $T\colon B_X \to Y$ has a harmonic behaviour. It is clear that if X admits a $C^{k,\alpha}$-smooth bump then there exists for every Banach space Y a $C^{k,\alpha}$-smooth operator $T\colon B_X \to Y$ that has not a harmonic behaviour (as $\mathbb{R} \subset Y$). In our note we investigate for a given $X = \ell_p$ and $1 \geq \alpha > p - [p]$ the structural conditions on Y which imply that every $T \in C^{[p],\alpha}(B_X, Y)$ has a harmonic behaviour. (Recall that ℓ_p has a $C^{[p],p-[p]}$-smooth bump, see [DGZ].) In particular we show that every such operator has a harmonic behaviour unless $\ell_{\frac{p}{K}} \subset Y$ for some integer $K \leq [p]$. It should be noted in this connection that by [B] and [H] (see also [BL]), for every ℓ_p and Y there exists an abundance of even polynomial operators from B_{ℓ_p} into Y such that for example $T(B_{\ell_p}) = B_Y$.

The techniques used in this section have their origin in the classical work of Kurzweil ([K]), Bonic and Frampton ([BF]) and Deville ([D]), and are presented also in the book [DGZ].

Taylor's theorem provides a connection between smooth operators with a harmonic behaviour and separating polynomials on ℓ_p (as we will see in Lemma 8), so in the next we investigate the behaviour of separating polynomials.

Recall that k-homogeneous polynomials $P\colon X \to Y$ (X and Y are Banach spaces) are defined as $P(x) = M(x,\dots,x)$, where $M\colon X \to Y$ is a continuous symmetric k-linear operator. We denote the set of all k-homogeneous polynomials from X into Y by $\mathcal{P}_k(X,Y)$. Recall that a homogeneous polynomial P is separating if $\inf\limits_{x\in S_X} \|P(x)\| > 0$.

Lemma 6. *Let X be a Banach space with a normalized Schauder basis $\{e_i\}$ which is equivalent to any of its normalized block bases (i.e. X is isomorphic to c_0 or ℓ_p, $1 \leq p < \infty$). Let Y be a Banach space and $K \in \mathbb{N}$. Suppose that there is no separating polynomial in $\mathcal{P}_k(X,Y)$ for any $1 \leq k < K$. Let $P \in \mathcal{P}_K(X,Y)$ and $\varepsilon > 0$. Then we can find a normalized block basis $\{z_i\}$ of $\{e_i\}$ such that if $\|\sum a_i z_i\| \leq 1$, then*

$$\left\| P\Big(\sum_{i=m}^{\infty} a_i z_i\Big) - \sum_{i=m}^{\infty} a_i^K P(z_i) \right\| < \frac{\varepsilon}{2^m}.$$

If moreover each polynomial in $\mathcal{P}_K(X,Y)$ is non-separating then we can find a normalized block basis $\{u_i\}$ of $\{z_i\}$ such that $\sup\big\{\|P(x)\|\,;\ x \in B_{\overline{\operatorname{span}}\{u_i\}}\big\} < \varepsilon$.

Proof. Let A be the basis constant of $\{e_i\}$. We prove the lemma by induction on K.

In the case $K = 1$ pick some bounded linear operator $P\colon X \to Y$ and $\varepsilon > 0$. The "diagonalization" is trivial (we put $z_i = e_i$). Assume there is no separating bounded linear operator $\tilde{P}\colon X \to Y$. Then P is not separating and we can choose a finitely supported vector $u_1 \in S_X$ for which $\|P(u_1)\| < \frac{1}{2}\frac{\varepsilon}{2A}$. As $\overline{\operatorname{span}}\{e_i\}_{i>n}$ is isomorphic to X and so $P\restriction_{\overline{\operatorname{span}}\{e_i\}_{i>n}}$ is not separating, we can inductively construct a normalized block basis $\{u_i\}$ of $\{e_i\}$ such that $\|P(u_i)\| < \frac{1}{2^i}\frac{\varepsilon}{2A}$. If $\|\sum a_i u_i\| \leq 1$, then

$$\left\| P\Big(\sum_{i=1}^{\infty} a_i u_i\Big) \right\| \leq \sum_{i=1}^{\infty} |a_i|\, \|P(u_i)\| \leq 2A \sum_{i=1}^{\infty} \|P(u_i)\| < \varepsilon.$$

Now suppose that the assertion holds for $K-1$ and let $\varepsilon > 0$ and M be a symmetric K-linear operator such that $P(x) = M(x,\dots,x)$. Put $D = K!(2A)^{2K}$ and $z_1 = e_1$.

$M(z_1,\dots,z_1,x)$ is (by assumption) a non-separating linear operator (in x) on $\overline{\operatorname{span}}\{e_i\}$, so by the induction hypothesis we can find a normalized block basis $\{v_i^1\}$ of $\{e_i\}$ for which $\sup\big\{\|M(z_1,\dots,z_1,x)\|\,;\ x \in B_{\overline{\operatorname{span}}\{v_i^1\}}\big\} < \frac{1}{2^4}\frac{\varepsilon}{D}\binom{2+K-2}{K-1}^{-1}$. $M(z_1,\dots,z_1,x,x)$ is (by assumption) a non-separating 2-homogeneous polynomial on $\overline{\operatorname{span}}\{v_i^1\}$, so by the induction hypothesis we can find a normalized block basis $\{v_i^2\}$ of $\{v_i^1\}$ for which $\sup\big\{\|M(z_1,\dots,z_1,x,x)\|\,;\ x \in B_{\overline{\operatorname{span}}\{v_i^2\}}\big\} < \frac{1}{2^4}\frac{\varepsilon}{D}\binom{2+K-2}{K-1}^{-1}$ and so on until we find a normalized block basis $\{v_i^{K-1}\}$ of $\{v_i^{K-2}\}$ for which $\sup\big\{\|M(z_1,x,\dots,x)\|\,;\ x \in B_{\overline{\operatorname{span}}\{v_i^{K-1}\}}\big\} < \frac{1}{2^4}\frac{\varepsilon}{D}\binom{2+K-2}{K-1}^{-1}$. Put $z_2 = v_2^{K-1}$.

$M(z_1,\dots,z_1,x)$ is a non-separating linear operator on $\overline{\operatorname{span}}\{v_i^{K-1}\}$, so again by the induction hypothesis we can find a normalized block basis $\{w_i^{1,1}\}$ of $\{v_i^{K-1}\}$ for which $\sup\big\{\|M(z_1,\dots,z_1,x)\|\,;\ x \in B_{\overline{\operatorname{span}}\{w_i^{1,1}\}}\big\} < \frac{1}{2^5}\frac{\varepsilon}{D}\binom{3+K-2}{K-1}^{-1}$. $M(z_1,\dots,z_1,z_2,x)$ is a non-separating linear operator on $\overline{\operatorname{span}}\{w_i^{1,1}\}$, so we can find a normalized block basis $\{w_i^{1,2}\}$ of $\{w_i^{1,1}\}$ for which $\sup\big\{\|M(z_1,\dots,z_1,z_2,x)\|\,;\ x \in B_{\overline{\operatorname{span}}\{w_i^{1,2}\}}\big\} < \frac{1}{2^5}\frac{\varepsilon}{D}\binom{3+K-2}{K-1}^{-1}$. Further we find a normalized block basis $\{w_i^{1,3}\}$ of $\{w_i^{1,2}\}$ for which $\sup\big\{\|M(z_1,\dots,z_1,z_2,z_2,x)\|\,;\ x \in B_{\overline{\operatorname{span}}\{w_i^{1,3}\}}\big\} < \frac{1}{2^5}\frac{\varepsilon}{D}\binom{3+K-2}{K-1}^{-1}$ and so on until we can choose a normalized block basis $\{w_i^{1,K}\}$ of $\{w_i^{1,K-1}\}$ for which $\sup\big\{\|M(z_2,\dots,z_2,x)\|\,;\ x \in B_{\overline{\operatorname{span}}\{w_i^{1,K}\}}\big\} < \frac{1}{2^5}\frac{\varepsilon}{D}\binom{3+K-2}{K-1}^{-1}$.

$M(z_1,\dots,z_1,x,x)$ is a non-separating 2-homogeneous polynomial on $\overline{\operatorname{span}}\{w_i^{1,K}\}$, so we can find a normalized block basis $\{w_i^{2,1}\}$ of $\{w_i^{1,K}\}$ for which $\sup\big\{\|M(z_1,\dots,z_1,x,x)\|\,;\ x\in B_{\overline{\operatorname{span}}\{w_i^{2,1}\}}\big\}<\frac{1}{2^5}\frac{\varepsilon}{D}\binom{3+K-2}{K-1}^{-1}$. $M(z_1,\dots,z_1,z_2,x,x)$ is a non-separating 2-homogeneous polynomial on $\overline{\operatorname{span}}\{w^{2,1}\}$, so we can find a normalized block basis $\{w_i^{2,2}\}$ of $\{w_i^{2,1}\}$ for which $\sup\big\{\|M(z_1,\dots,z_1,z_2,x,x)\|\,;\ x\in B_{\overline{\operatorname{span}}\{w_i^{2,2}\}}\big\}<\frac{1}{2^5}\frac{\varepsilon}{D}\binom{3+K-2}{K-1}^{-1}$. Further we find a normalized block basis $\{w_i^{2,3}\}$ of $\{w_i^{2,2}\}$ for which $\sup\big\{\|M(z_1,\dots,z_1,z_2,z_2,x,x)\|\,;\ x\in B_{\overline{\operatorname{span}}\{w_i^{2,3}\}}\big\}<\frac{1}{2^5}\frac{\varepsilon}{D}\binom{3+K-2}{K-1}^{-1}$ and so on until we can choose a normalized block basis $\{w_i^{2,K-1}\}$ of $\{w_i^{2,K-2}\}$ for which $\sup\big\{\|M(z_2,\dots,z_2,x,x)\|\,;\ x\in B_{\overline{\operatorname{span}}\{w_i^{2,K-1}\}}\big\}<\frac{1}{2^5}\frac{\varepsilon}{D}\binom{3+K-2}{K-1}^{-1}$.

We end with a normalized block basis $\{w_i^{K-1,2}\}$ of $\{w_i^{K-1,1}\}$ for which $\sup\big\{\|M(z_2,x,\dots,x)\|\,;\ x\in B_{\overline{\operatorname{span}}\{w_i^{K-1,2}\}}\big\}<\frac{1}{2^5}\frac{\varepsilon}{D}\binom{3+K-2}{K-1}^{-1}$. Put $z_3=w_3^{K-1,2}$.

We continue inductively in the same spirit. In the nth step, in order to define z_n, we consider all the $\binom{n+K-2}{K-1}-1$ operators $M(z_{j_1},\dots,z_{j_{K-l}},\underbrace{x,\dots,x}_{l})$, $j_1\le\dots\le j_{K-l}\le n-1$, $1\le l<K$, so that $\sup\big\{\big\|M(z_{j_1},\dots,z_{j_{K-l}},x,\dots,x)\big\|\,;\ x\in B_{\overline{\operatorname{span}}\{w_i\}}\big\}<\frac{1}{2^{n+2}}\frac{\varepsilon}{D}\binom{n+K-2}{K-1}^{-1}$ for a corresponding block basis $\{w_i\}$.

Clearly, $\{z_i\}$ is a normalized block basis of $\{e_i\}$ and if $\|\sum a_iz_i\|\le 1$, then

$$\begin{aligned}\Big\|P\Big(\sum_{i=m}^{\infty}a_iz_i\Big)-\sum_{i=m}^{\infty}a_i^KP(z_i)\Big\| &\le K!\sum_{\substack{m\le j_1\le\dots\le j_K\\ j_1<j_K}}|a_{j_1}\cdots a_{j_K}|\,\|M(z_{j_1},\dots,z_{j_K})\|\\ &\le K!(2A)^K\sum_{\substack{m\le j_1\le\dots\le j_K\\ j_1<j_K}}\|M(z_{j_1},\dots,z_{j_K})\|\\ &< K!(2A)^K\sum_{n=m+1}^{\infty}\sum_{\substack{m\le j_1\le\dots\le j_K=n\\ j_1<j_K}}\frac{1}{2^{n+2}}\frac{\varepsilon}{D}\binom{n+K-2}{K-1}^{-1}\\ &\le\frac{\varepsilon}{(2A)^K}\sum_{n=m+1}^{\infty}\frac{1}{2^{n+2}}\sum_{1\le j_1\le\dots\le j_K=n}\binom{n+K-2}{K-1}^{-1}\\ &=\frac{1}{2^{m+2}}\frac{\varepsilon}{(2A)^K}<\frac{\varepsilon}{2^m}.\end{aligned}$$

In the case that all K-homogeneous polynomials are non-separating, we can (similarly as for $K=1$) find a normalized block basis $\{u_i\}$ of $\{z_i\}$ such that $\|P(u_i)\|<\frac{1}{2^{i+1}}\frac{\varepsilon}{(2A)^K}$. Let $u_i=\sum_{j=\alpha_i}^{\beta_i}b_jz_j$. Then (as u_i is normalized) $\Big\|P(u_i)-\sum\limits_{j=\alpha_i}^{\beta_i}b_j^KP(z_j)\Big\|<\frac{1}{2^{\alpha_i+2}}\frac{\varepsilon}{(2A)^K}\le\frac{1}{2^{i+2}}\frac{\varepsilon}{(2A)^K}$. Thus, if $\|\sum a_iu_i\|\le 1$, we have

$$\begin{aligned}&\Big\|P\Big(\sum_{i=1}^{\infty}a_iu_i\Big)-\sum_{i=1}^{\infty}a_i^KP(u_i)\Big\|\\ &\quad\le\Big\|P\Big(\sum_{i=1}^{\infty}a_i\sum_{j=\alpha_i}^{\beta_i}b_jz_j\Big)-\sum_{i=1}^{\infty}a_i^K\sum_{j=\alpha_i}^{\beta_i}b_j^KP(z_j)\Big\|+\Big\|\sum_{i=1}^{\infty}a_i^KP(u_i)-\sum_{i=1}^{\infty}a_i^K\sum_{j=\alpha_i}^{\beta_i}b_j^KP(z_j)\Big\|\\ &\quad<\frac{\varepsilon}{4}+\sum_{i=1}^{\infty}|a_i|^K\Big\|P(u_i)-\sum_{j=\alpha_i}^{\beta_i}b_j^KP(z_j)\Big\|<\frac{\varepsilon}{2},\end{aligned}$$

and so

$$\Big\|P\Big(\sum_{i=1}^{\infty}a_iu_i\Big)\Big\|\le\Big\|P\Big(\sum_{i=1}^{\infty}a_iu_i\Big)-\sum_{i=1}^{\infty}a_i^KP(u_i)\Big\|+\sum_{i=1}^{\infty}|a_i|^K\,\|P(u_i)\|<\varepsilon.$$

□

Theorem 7. *Let Y be a Banach space, $1\le p<\infty$, $K\in\mathbb{N}$.*

Suppose that all polynomials in $\mathcal{P}_k(\ell_p, Y)$ are non-separating for all $1 \leq k < K$. If K is odd and $K \leq p$, or if K is even and $K < p$, then there is a separating $P \in \mathcal{P}_K(\ell_p, Y)$ if and only if $\ell_{\frac{p}{K}} \subset Y$.

There is a separating homogeneous polynomial $P\colon c_0 \to Y$ if and only if $c_0 \subset Y$ if and only if there is a separating homogeneous polynomial $P \in \mathcal{P}_K(c_0, Y)$ for any $K \in \mathbb{N}$.

Proof. First we prove the ℓ_p case.

The "if" part: Clearly, $P\colon \ell_p \to \ell_{\frac{p}{K}}$ defined as $P(\sum a_i e_i) = \sum a_i^K e_i$ is a separating K-homogeneous polynomial. Hence if T is an isomorphism of $\ell_{\frac{p}{K}}$ into Y, then $T \circ P$ is a corresponding separating K-homogeneous polynomial.

The "only if" part: Put $\varepsilon = \inf_{S_{\ell_p}} \|P(x)\| > 0$. By Lemma 6 we can construct an appropriate "ε-diagonal" normalized block basis $\{z_i\}$. Put $y_i = P(z_i)$. If K is odd then for any sequence $\{a_i\}$ satisfying $\sum_i |a_i|^{\frac{p}{K}} = 1$ we have

$$\left\|\sum_{i=1}^{\infty} a_i y_i\right\| = \left\|\sum_{i=1}^{\infty} \left(a_i^{\frac{1}{K}}\right)^K P(z_i)\right\| < \left\|P\left(\sum_{i=1}^{\infty} a_i^{\frac{1}{K}} z_i\right)\right\| + \frac{\varepsilon}{2} \leq \|P\| \left\|\sum_{i=1}^{\infty} a_i^{\frac{1}{K}} z_i\right\|^K + \frac{\varepsilon}{2} = \|P\| + \frac{\varepsilon}{2}.$$

On the other hand,

$$\left\|\sum_{i=1}^{\infty} a_i y_i\right\| = \left\|\sum_{i=1}^{\infty} \left(a_i^{\frac{1}{K}}\right)^K P(z_i)\right\| > \left\|P\left(\sum_{i=1}^{\infty} a_i^{\frac{1}{K}} z_i\right)\right\| - \frac{\varepsilon}{2} \geq \varepsilon \left\|\sum_{i=1}^{\infty} a_i^{\frac{1}{K}} z_i\right\|^K - \frac{\varepsilon}{2} = \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

This implies that $\overline{\operatorname{span}}\{y_i\} \subset Y$ is a subspace isomorphic to $\ell_{\frac{p}{K}}$.

If K is even, $a_i = (a_i^{1/K})^K$ only if $a_i \geq 0$ and therefore we obtain merely $\ell^+_{\frac{p}{K}} \subset Y$. (In view of the second remark after Theorem 2 we do not need $\{y_i\}$ to be a basic sequence.) Now Theorem 2 finishes the proof for K even.

For c_0, we start by considering the separating polynomial of the smallest degree, and analogously as above we conclude that $c_0 \subset Y$. Then we use the fact that $P\colon c_0 \to c_0$ defined as $P(\sum a_i e_i) = \sum a_i^K e_i$ is a separating K-homogeneous polynomial.

$\square$

Theorem 7 implies the well-known fact that there is no separating $P \in \mathcal{P}_k(\ell_p, \mathbb{R})$ for $1 \leq k < p < \infty$ (otherwise $\ell_{p/k} \subset \mathbb{R}$ for some $k < p$) and there is no separating $P \in \mathcal{P}_p(\ell_p, \mathbb{R})$ for p odd integer. If p is an even integer then $P(x) = \|x\|^p$ is a separating p-homogeneous polynomial and so the statement of the Theorem 7 does not hold for $K = p$.

Notice that $C^{n,+}(B_X, Y) \subset C^{n-1,1}(B_X, Y)$ and $\mathcal{P}_k(X, Y) \subset C^{n,1}(B_X, Y)$ for any $k, n \in \mathbb{N}$.

Lemma 8. *Let Y be a Banach space, $1 \leq p < \infty$. Let $n \in \mathbb{N}$ and $\alpha \in (0, 1]$ be such that $n + \alpha > p$. All $T \in C^{n,\alpha}(B_{\ell_p}, Y)$ have a harmonic behaviour if and only if there is no separating $P \in \mathcal{P}_k(\ell_p, Y)$ for all $1 \leq k \leq n$. All $T \in C^{p,+}(B_{\ell_p}, Y)$, $p \in \mathbb{N}$, have a harmonic behaviour if and only if there is no separating $P \in \mathcal{P}_k(\ell_p, Y)$ for all $1 \leq k \leq p$.*

Proof. Clearly, a separating polynomial has not a harmonic behaviour. Let $T \in C^{n,\alpha}(B_{\ell_p}, Y)$ have not a harmonic behaviour. Pick a finitely supported $y \in B_X \setminus S_X$ such that $\varepsilon = \inf\limits_{x \in S_X} \|T(x) - T(y)\| > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{n!}(1 - \|y\|^p)^{\frac{n+\alpha}{p}} N^{1 - \frac{n+\alpha}{p}} < \frac{\varepsilon}{2}$. By Taylor's theorem, for any $x, x + h \in B_{\ell_p}$,

$$T(x + h) - T(x) = \sum_{k=1}^{n} \frac{1}{k!} T^{(k)}(x)(h) + R_n(x)(h), \qquad \text{where } \|R_n(x)(h)\| \leq \frac{\|h\|^{n+\alpha}}{n!}. \tag{6}$$

(We use an abbreviation $T^{(k)}(x)(h) = T^{(k)}(x)(h, \dots, h)$, which is a k-homogeneous polynomial in h.) Suppose that all polynomials in $\mathcal{P}_k(\ell_p, Y)$ for all $1 \leq k \leq n$ are non-separating. By Lemma 6 we can find normalized block bases $\{u_i^k\}$ of $\{e_i\}$ such that $\overline{\operatorname{span}}\{u_i^k\} \subset \overline{\operatorname{span}}\{u_i^{k-1}\}$ and $\sup\{\|T^{(k)}(y)(h)\|;\ h \in B_{\overline{\operatorname{span}}\{u_i^k\}}\} < \frac{\varepsilon}{2} \frac{k!}{nN}$ for $1 \leq k \leq n$. Thus we can pick a finitely supported $h_1 \in \ell_p$ such that $\max \operatorname{supp} y < \min \operatorname{supp} h_1$, $N \|h_1\|^p = 1 - \|y\|^p$ and $\frac{1}{k!} \|T^{(k)}(y)(h_1)\| < \frac{\varepsilon}{2} \frac{1}{nN}$ for all $1 \leq k \leq n$. Similarly for $1 < j \leq N$ we choose finitely supported $h_j \in \ell_p$ such that $\max \operatorname{supp} h_{j-1} < \min \operatorname{supp} h_j$, $N \|h_j\|^p = 1 - \|y\|^p$ and $\frac{1}{k!} \left\|T^{(k)}\left(y + \sum_{i=1}^{j-1} h_i\right)(h_j)\right\| < \frac{\varepsilon}{2} \frac{1}{nN}$ for all $1 \leq k \leq n$. Then $\left\|y + \sum_{i=1}^{N} h_i\right\|^p = \|y\|^p + \sum_{i=1}^{N} \|h_i\|^p = 1$

and (6) gives

$$\begin{aligned}\Big\|T\Big(y+\sum_{i=1}^{N}h_i\Big)-T(y)\Big\| &\le \sum_{j=1}^{N}\Big\|T\Big(y+\sum_{i=1}^{j}h_i\Big)-T\Big(y+\sum_{i=1}^{j-1}h_i\Big)\Big\|\\ &\le \sum_{j=1}^{N}\Bigg(\sum_{k=1}^{n}\frac{1}{k!}\Big\|T^{(k)}\Big(y+\sum_{i=1}^{j-1}h_i\Big)(h_j)\Big\|+\Big\|R_n\Big(y+\sum_{i=1}^{j-1}h_i\Big)(h_j)\Big\|\Bigg)\\ &< \frac{\varepsilon}{2}+\frac{N}{n!}\left(\frac{1-\|y\|^p}{N}\right)^{\frac{n+\alpha}{p}}<\varepsilon,\end{aligned}$$

which is a contradiction.

The proof for $C^{p,+}$ is analogous.

□

Let Y be any Banach space, $0 \neq y \in Y$. We put $T(x)=\|x\|_p^p\, y$, $x \in \ell_p$, which is an operator without a harmonic behaviour from B_{ℓ_p} into Y. If p is an even integer, then $T \in \mathcal{P}_p(\ell_p, Y)$. If p is not an even integer and we let n be the largest integer strictly smaller than p, then $T \in C^{n,p-n}(B_{\ell_p}, Y)$. Therefore if we want all sufficiently smooth operators to have a harmonic behaviour, we need to rule out p even integer and consider smoothness higher than $C^{[p],p-[p]}$. By putting together Lemma 8 and Theorem 7 we immediately obtain

Theorem 9. *Let Y be a Banach space, $1 \le p < \infty$, p is not an even integer. Let $\mathcal{C} = C^{[p],\alpha}(B_{\ell_p}, Y)$ for some $1 \ge \alpha > p-[p]$ if p is not an integer, or $\mathcal{C} = C^{p,+}(B_{\ell_p}, Y)$ if p is an odd integer. Then either all operators in $\mathcal{C}$ have a harmonic behaviour or $\ell_{\frac{p}{k}} \subset Y$ for some $1 \le k \le [p]$.*

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