ANALYTIC NORMS IN ORLICZ SPACES

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Abstract. It is shown that an Orlicz sequence space h_M admits an equivalent analytic renorming if and only if it is either isomorphic to ℓ_{2n} or isomorphically polyhedral. As a consequence, we show that there exists a separable Banach space admitting an equivalent C^∞-Fréchet norm, but no equivalent analytic norm.

In this note, we denote by h_M as usual the subspace of an Orlicz sequence space ℓ_M generated by the unit vector basis.

More terminology and notation concerning Orlicz spaces can be found in [LT].

Let us also point out that by C^k-smoothness (or analyticity) of a norm we always mean away from the origin (as is usual in renorming theory).

The characterization of the best order of C^k-Fréchet smoothness of some renorming, $k \in \mathbb{N} \cup \{+\infty\}$, for h_M was obtained in [M], [MT1], [MT2]. In our present note, we complete the characterization also for analytic renormings. We show that an Orlicz sequence space h_M has an analytic renorming if and only if $h_M \cong \ell_{2n}$, $n \in \mathbb{N}$ or h_M is isomorphically polyhedral. Let us recall that a separable Banach space X is isomorphically polyhedral if it has an equivalent polyhedral norm. By a theorem of Fonf [F], this is the case if and only if X admits an equivalent norm with a countable boundary. More precisely, there exists a sequence $\{f_i\}_{i\in\mathbb{N}}$ in X^* such that

$$\|x\| = \max\{|f_i(x)|, \quad i \in \mathbb{N}\}.$$

According to one of the results from [DFH], we have the following:

Theorem 1. *Every separable isomorphically polyhedral Banach space X admits an equivalent analytic form.*

We prove that the converse is also true if we impose additional conditions on the space X. In connection with our result it should be noted that by recent work of Gonzalo and Jaramillo ([GJ]) every separable Banach space with a symmetric basis and C^∞-Fréchet smooth norm is isomorphic to ℓ_{2n}, provided it does not contain a copy of c_0.

Our approach is entirely different from that in [MT1] and relies on methods from [DFH] and [H1-2-3]. As a corollary, relying on an example of Leung [L], we show that there exists

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a separable Banach space with C^∞-Fréchet smooth norm which admits no analytic norm. A search of such an example was in fact a motivation of our work, since the previously known examples of such spaces (e.g. $c_0(\Gamma), \Gamma$ uncountable, see [P] and [BF] for a result of Kuiper) were nonseparable.

Let us recall that a Banach space X with an unconditional basis is said to satisfy an *upper p-estimate*, $p \geq 1$ if for some $C > 0$:

$$\left\| \sum_{i=1}^{n} u_i \right\| \leq C \left(\sum_{i=1}^{n} \|u_i\|^p \right)^{\frac{1}{p}}$$

whenever u_i are disjointly supported in X.

An important notion in our consideration is that of weak sequential continuity.

Definition 2. Let $\mathcal{U} \subseteq X$ be an open, convex and bounded subset of a Banach space X, f be a real function on $\mathcal{U}$. We say that f is *weakly sequentially continuous* (*wsc*-for short) if it maps weakly Cauchy sequences from $\mathcal{U}$ into convergent ones. A function f defined on an open subset $\mathcal{O} \subseteq X$ is said to be *locally wsc* if there exists a covering of $\mathcal{O}$ by a family of open sets $\mathcal{U}$ as above such that f is wsc on $\mathcal{U}$ for all $\mathcal{U}$.

In order to verify wsc-property for polynomials, it is sufficient to check the convergence only for weakly convergent sequences in $\mathcal{U}$ ([AHV]).

Using this fact, the following lemma follows from results in [G].

Lemma 3. *Let X be a Banach space with an unconditional basis satisfying an upper p-estimate. Then all polynomials of degree $n < p$ on X are wsc (on B_X).*

The importance of the notion of wsc stems from the following lemma, which comes from [H3], and which was shown for polynomials in [AHV].

Lemma 4. *Let X be a Banach space $\ell_1 \not\hookrightarrow X$, f be a C^2-Fréchet differentiable real function defined on some open set $\mathcal{O} \subseteq X$. TFAE:*

(1) *f is locally wsc,*
(2) *f' is locally norm compact.*

By f' being locally norm compact we mean that there exists a covering by a family of open sets $\mathcal{U}$ of $\mathcal{O}$ such that $f'(\mathcal{U})$ is relatively norm compact in X^* for all $\mathcal{U}$.

The following is a generalization of the main result in [H1].

Theorem 5. *Let $(X, \|\cdot\|)$ be a Banach space, where $\|\cdot\|$ is analytic. If all polynomials on X are wsc, then X is separable and isomorphically polyhedral.*

Proof. By ∂ we denote the duality map corresponding to $\|\cdot\|$, i.e.

$$\partial\colon\ X\backslash\{0\} \to S_{X^*} \quad \text{and} \quad \partial x(x) = \|x\| \text{ for all } x \in X\backslash\{0\}.$$

Since $\|\cdot\|$ is differentiable, ∂x is the derivative of $\|x\|$ at $x \in X\backslash\{0\}$.

Let us first show that $\|\cdot\|$ is locally wsc on $X\backslash\{0\}$.

Fix $x \neq 0$. Since $\|\cdot\|$ is analytic at x we can find $\delta > 0$ so that if $\|h\| < \delta$ we have

$$\|x+h\| = \sum_{n=1}^{\infty} p_n(h),$$

where p_n are homogeneous polynomials of degree n, and the convergence is uniform with respect to all $\|h\| < \delta$. Since all polynomials p_n are wsc, we see that $\|\cdot\|$ is wsc on $\{y\colon \|x-y\| < \delta\}$.

We proceed by showing that X is separable. Since X is an Asplund space, from Lemma 4 we get that there exist $0 < \eta < \delta$ so that the set $\{\partial y\colon \|x-y\| < \eta\}$ is norm relatively compact. Thus the subspace Y of X^* generated by $\{\partial y\colon \|y-x\| < \eta\}$ is separable. If $Y = X^*$ the proof is finished. Otherwise, assume $Y \neq X^*$. By the Hahn-Banach theorem, there exists $x^{**} \in S_{X^{**}}$ such that

$$x^{**}(\partial y) = 0 \quad \text{whenever} \quad \|y-x\| < \eta.$$

Since $\|\cdot\|$ is analytic on $X\backslash\{0\}$ we get that ∂ is analytic as well on $X\backslash\{0\}$. Hence $f = x^{**} \circ \partial$ is a real analytic function on $X\backslash\{0\}$.

Since $f(y) = 0$ for $\|y-x\| < \eta$, clearly $f \equiv 0$ on $X\backslash\{0\}$. On the other hand, by the Bishop-Phelps theorem, ∂S_X is dense in S_{X^*}, so there exists $y \in S_X$ such that $f(y) = x^{**}(\partial y) \neq 0$, a contradiction. So X is separable.

Since X is separable and $\|\cdot\|$ is locally wsc, by Lemma 4 and the Lindelöf property, $(S_X, \|\cdot\|)$ can be covered by a countable system $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of norm open convex bounded subsets of X such that $\partial\mathcal{U}_n$ is relatively compact. Thus the boundary of $(X, \|\cdot\|)$ can be covered by a countable system of compacts, and the result follows from [H2]. □

Theorem 6. *Let M be an Orlicz function. Then h_M admits an equivalent analytic norm if and only if either $h_M \cong \ell_{2n}$, $n \in \mathbb{N}$, or h_M is isomorphically polyhedral. In particular if*

(1) $\lim_{t\to 0} \frac{M(2t)}{M(t)} = +\infty$ *then h_M has an equivalent analytic norm.*

(2) $a_M = +\infty$ *and there exists a sequence $t_i \searrow 0$ such that* $\sup_{i\in\mathbb{N}} \frac{M(at_i)}{M(t_i)} < +\infty$ *for all $a \geq 1$ then h_M does not admit an equivalent analytic norm.*

Proof. The "if" part follows from the well-known result that the canonical norm on ℓ_{2n}, $n \in \mathbb{N}$, is analytic and from Theorem 1.

The "only if" part. By classical results ([LT]), the existence of an analytic norm on X implies $\alpha_M = \beta_M \in \{2n\}_{n\in\mathbb{N}} \cup \{+\infty\}$.

The case $\alpha_M = 2n$ implies that $X \cong \ell_{2n}$ by [MT1].

If $\alpha_M = \infty$, then ([LT]) X has an upper p-estimate for every $p > 1$.

Combination of Lemma 3 and Theorem 5 finishes the proof of the "only if" part.

Leung [L] showed that if M satisfies (1) then h_M is isomorphically polyhedral and if M satisfies (2) h_M is not isomorphically polyhedral. □

Corollary 7. *There exists a c_0-saturated separable Banach space which admits an equivalent C^∞-Fréchet norm but no equivalent analytic norm.*

Proof. Leung [L] constructed an Orlicz function M satisfying (2). By a result of [MT2] the corresponding space h_M admits an equivalent C^∞-Fréchet smooth norm. On the other hand, by Theorem 6, no equivalent analytic norm exists on this space. □

Let us pass to some final remarks.

A natural question is the following: Is there a separable c_0-saturated non-polyhedral Banach space with an equivalent analytic norm?

By a careful analysis of [DFH], we obtain that on every separable polyhedral space there exists a dense set of equivalent analytic norms whose boundaries can be covered by countably many compacts. Such norms in turns immediately imply the polyhedrality of the space (using [H2]).

However, there are examples of polyhedral spaces (e.g. [S], [PS]) with analytic norms failing this property.

More precisely, the space S of Schreier has an unconditional basis $\{e_n\}$ such that the formal identity operator id from S into ℓ_2 is bounded. It is easy to show that given an equivalent analytic norm $\|\cdot\|$ on S whose boundary is covered by countably many compacts, the equivalent analytic norm $|||x||| = (\|x\|^2 + \|\text{id } x\|_2^2)^{\frac{1}{2}}$ fails the covering property.

The problem is therefore how to recognize the polyhedrality of S based on its norm $|||\cdot|||$.

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