# LECTURES: Infinitary combinatorics with applications in mathematical analysis
# PART 1


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## Contents



## 1 Lectures 1 + 2

We present the proof of classical Ramsey theorem, using products of ultrafilters. First we recall basic properties of ultrafilters, next we present the 2-dimensional Ramsey theorem and finally we prove the multidimensional non-symmetric version.

### 1.1 Ultrafilters

We start with some basic definitions, which will be needed later.

**Definition 1.1.** An *algebra of sets* in $X$ is a family $\mathscr{A} \subseteq \mathscr{P}(X)$ such that

(A1) $\emptyset, X \in \mathscr{A}$,

(A2) $A, B \in \mathscr{A} \implies A \cap B \in \mathscr{A}$,

(A3) $A \in \mathscr{A} \implies X \setminus A \in \mathscr{A}$.

A family of sets $\mathscr{B}$ is *centered* if $B_0 \cap \dots B_n \neq \emptyset$ whenever $B_0, \dots, B_n \in \mathscr{B}$, $n \in \omega$.

Let $\mathscr{A}$ be an algebra of sets in $X$. A family $\mathscr{F} \subseteq \mathscr{A}$ is a *filter* if it satisfies

(F1) $\emptyset \notin \mathscr{F}$, $X \in \mathscr{F}$.

(F2) $A, B \in \mathscr{F} \implies A \cap B \in \mathscr{F}$.

(F3) $A \in \mathscr{F} \implies \{B \in \mathscr{A} \colon A \subseteq B\} \subseteq \mathscr{F}$.

A filter that is maximal with respect to inclusion (i.e. not properly contained in any other filter) is called an *ultrafilter*.

Below are standard and well-known properties of ultrafilters.

**Proposition 1.2.** *Let $\mathscr{A}$ be an algebra of subsets of a nonempty set $X$.*

*(i) Every centered subfamily of $\mathscr{A}$ extends to an ultrafilter in $\mathscr{A}$.*

*(ii) A filter $\mathscr{F} \subseteq \mathscr{A}$ is an ultrafilter if and only if either $A \in \mathscr{F}$ or $X \setminus A \in \mathscr{F}$, for every $A \in \mathscr{A}$.*

*(iii) If $\mathscr{F}$ is an ultrafilter in $\mathscr{A}$ and $A_0, \dots, A_n \in \mathscr{A}$ are such that $A_0 \cup \dots \cup A_n \in \mathscr{F}$ then there is $j \leqslant n$ such that $A_j \in \mathscr{F}$.*

We shall be interested in ultrafilters in $\mathscr{P}(X)$. Namely, we say that $p$ is an *ultrafilter on* $X$ if it is an ultrafilter in the algebra $\mathscr{P}(X)$. An ultrafilter $p$ on $X$ is *principal* if it is of the form

$$p = \{A \subseteq X \colon x \in A\}$$

for some $x \in X$. Otherwise, $p$ is called *non-principal*. It is an easy exercise to check that every ultrafilter on a finite set is principal. On the other hand:

**Proposition 1.3.** *Let $X$ be an infinite set. Then there exists a non-principal ultrafilter on $X$.*

*Proof.* The family $\mathscr{F} = \{A \subseteq X \colon |X \setminus A| < \aleph_0\}$ is a filter. Every ultrafilter extending $\mathscr{F}$ is non-principal. □

Given a set $X$, we shall denote by $\beta X$ the family of all ultrafilters on $X$ and by $\beta^* X$ the family of all non-principal ultrafilters[1] on $X$.

[1] Many authors use the notation $X^*$ instead of $\beta^* X$ which, in our opinion, may be confusing.

## 1.2 Classical Ramsey theorem for pairs

We show how to use ultrafilters for proving infinitary version of Ramsey's theorem.

**Definition 1.4.** Let $X$, $Y$ be two nonempty sets and let $p \in \beta X$, $q \in \beta Y$. Given a set $S \subseteq X \times Y$ we denote by $(C)^x$ its *vertical section* at $x$, namely

$$(C)^x = \{y \in Y \colon \langle x, y \rangle \in C\}.$$

Horizontal sections $(C)_y$ are defined analogously. We define

$$p \otimes q = \{A \subseteq X \times Y \colon \{x \in X \colon (A)^x \in q\} \in p\}.$$

It is an easy exercise to show that $p \otimes q$ is an ultrafilter on $X \times Y$. We shall call $p \otimes q$ the *product* of $p$ and $q$.

It turns out that products of ultrafilters are good enough for proving Ramsey's theorem.

**Theorem 1.5** (Ramsey)**.** *Assume $X$ is an infinite set and*

$$[X]^2 = C_0 \cup \dots \cup C_{\ell-1}.$$

*Then there exist $j < \ell$ and an infinite set $Y \subseteq X$ such that $[Y]^2 \subseteq C_j$.*

*Proof.* We identify subsets of $[X]^2$ with symmetric subsets of $X \times X$. Fix a non-principal ultrafilter $p$ on $X$.

There is $j < \ell$ such that $C_j \in p \otimes p$. We define inductively a decreasing chain $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ in $p$ and a one-to-one sequence $\{x_n\}_{n\in\omega}$ such that

(0) $A_0 = \{x \in X \colon (C_j)^x \in p\}$,

(1) $x_n \in A_n$,

(2) $A_{n+1} = A_n \cap (C_j)^{x_n}$.

Actually, starting with any $x_0$ such that $(C_j)^{x_0} \in p$, conditions (1) and (2) tell us how to proceed, knowing that $p$ contains only infinite sets.

Finally, if $m < n$ then $x_n \in A_{m+1} \subseteq (C_j)^{x_m}$, which means that $\{x_m, x_n\} \in C_j$. This shows that $[Y]^2 \subseteq C_j$, where $Y = \{x_n\}_{n\in\omega}$. □

We now show how to derive the finite version of Ramsey's theorem from the infinitary one.

**Theorem 1.6.** *Given $k, \ell \in \mathbb{N}$, there exists $r \in \mathbb{N}$ such that whenever $[X]^2 = C_0 \cup \dots \cup C_{\ell-1}$ and $|X| \geqslant r$, then there exists $Y \in [X]^k$ such that $[Y]^2 \subseteq C_j$ for some $j < \ell$.*

*Proof.* Suppose the theorem fails for fixed $k, \ell \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ there is a set $X_n$ of cardinality $\geqslant n$ and a function $c_n \colon X_n \to \ell$ such that for every $Y \in [X_n]^k$ the restriction $c_n \restriction [Y]^2$ is not constant. Fix $p \in \beta^*\mathbb{N}$ and let $X = \prod_{n\in\mathbb{N}} X_n/p$ be the ultraproduct of $\{X_n\}_{n>2}$. That is, $X$ is the quotient of $\prod_{n\in\mathbb{N}} X_n$ with respect to the relation $\sim_p$ defined by

$$x \sim_p y \iff \{n \colon x(n) = y(n)\} \in p.$$

Define

$$c_\infty(x, y) = i \iff \{n \colon c_n(x(n), y(n)) = i\} \in p.$$

By Ramsey's theorem, there is an infinite set $Y \subseteq X$ such that $c_\infty[Y]^2 = \{j\}$ for some $j < \ell$. There is $A \in p$ such that $|\{y(n) \colon y \in Y\}| \geqslant k$ and $c_\infty(x, y) = c_n(x(n), y(n)) = j$ for every $x, y \in Y$ and for every $n \in A$. This is a contradiction. $\square$

## 1.3 Multidimensional Ramsey theorem

We shall present a non-symmetric version of Ramsey's theorem, using products of ultrafilters.

**Lemma 1.7.** *Let $X, Y, Z$ be sets, let $p \in \beta X$, $q \in \beta Y$, $r \in \beta Z$. Then*

$$(p \otimes q) \otimes r = p \otimes (q \otimes r),$$

*agreeing that $(X \times Y) \times Z = X \times (Y \times Z)$.*

*Proof.* We have the following sequence of equivalences:

$$\begin{aligned}
A \in (p \otimes q) \otimes r &\iff \{s \in X \times Y \colon (A)^s \in r\} \in p \otimes q \\
&\iff \Big\{x \in X \colon \Big(\{s \in X \times Y \colon (A)^s \in r\}\Big)^x \in q\Big\} \in p \\
&\iff \Big\{x \in X \colon \{y \in Y \colon ((A)^x)^y \in r\} \in q\Big\} \in p \\
&\iff \{x \in X \colon (A)^x \in q \otimes r\} \in p \\
&\iff A \in p \otimes (q \otimes r).
\end{aligned}$$

We have used the fact that $(\{s \in X \times Y \colon (A)^s \in r\})^x = \{y \in Y \colon ((A)^x)^y \in r\}$. $\square$

The lemma above says that the operation $\otimes$ is associative. Having this in mind, we shall use the abbreviation $p^{\otimes n}$ instead of $p \otimes \cdots \otimes p$ (where $p$ occurs $n$ times).

**Definition 1.8.** Given sets $A, B$, we shall denote by $B^{\hookleftarrow A}$ the set of all one-to-one functions from $A$ to $B$. If $A$ and $B$ are linearly ordered, we denote by $B^{\nwarrow A}$ the set of all strictly increasing functions from $A$ to $B$.

The next statement easily implies Ramsey's theorem.

**Lemma 1.9.** *Let $X$ be an infinite set, $p \in \beta^* X$ and assume $C \in p^{\otimes N}$, where $N \geqslant 1$ is a natural number. Then there exists $f \in X^{\hookleftarrow \omega}$ such that*

$$\{f \circ s \colon s \in \omega^{\nwarrow N}\} \subseteq C.$$

*Proof.* Given $0 < k < N$ and $s \in X^{\hookleftarrow (k-1)}$, define

$$A_s = \{x \in X \colon (C)^{s \frown x} \in p^{\otimes (N-k)}\},$$

where $s \frown x$ means the concatenation of $s$ and the one-element sequence with value $x$. Note that $A_s \in p$ whenever $(C)^s \in p^{\otimes (N-k+1)}$. In particular, $A_\emptyset \in p$. Now define inductively $f \colon \to^{\hookleftarrow X} \omega$ such that for every $n \in \omega$ the following condition

$$(\forall\, k < N)(\forall\, r \in n^{\nwarrow k}) \;\; (C)^{f \circ r} \in p^{\otimes (N-k)}.$$

We start by choosing any $f(0) \in A_\emptyset$. Once $f \restriction n$ has been defined, we choose $f(n) \in X \setminus f[n]$ so that

$$f(n) \in \bigcap_{r \in n^{\nwarrow (N-1)}} (C)^{f \circ r} \qquad \text{and} \qquad f(n) \in \bigcap_{t \in n^{\nwarrow k}} A_{f \circ t} \quad \text{for every } k < N-1.$$

This is possible, because all the sets above are in $p$. Finally, $f$ is as required. □

A symmetric variant of the lemma above could be proved slightly simpler, by using induction on $N$. We leave the details to the readers.

**Theorem 1.10** (Ramsey)**.** *Let $X$ be an infinite set, let $N > 0$ be a natural number and assume $[X]^N = C_0 \subseteq C_{k-1}$ for some $k \in \omega$. Then there exist $j < k$ and an infinite set $Y \subseteq X$ such that $[Y]^N \subseteq C_j$.*

*Proof.* Identify each $C_i$ with a symmetric subset of $X^N$ and use Lemma 1.9, noting that $C_0 \cup \dots \cup C_{k-1} \in p^{\otimes N}$ whenever $p \in \beta^* X$. □

A finite version of Ramsey's theorem can be obtained by the same argument as in the proof of Theorem 1.6.

# 2 Lectures 3 + 4

We discuss topological counterparts of Ramsey's theorem. Ramsey theorem says, roughly speaking, that if $N$-tuples of a 'big' set are partitioned into finitely many pieces then there is a 'big' subset whose all $N$-tuples are in one piece of the partition. In the classical setting, 'big' means 'infinite'. Once we deal with topology, a natural meaning of 'big' is 'perfect'. Recall that a set $P$ is *perfect* if it is nonempty, completely metrizable and dense-in-itself (i.e. has no isolated points). We are going to work with separable metric spaces, where a typical example of a perfect set is the Cantor set. In fact, proving the existence of a perfect set with certain properties is almost always reduced to constructing a suitable Cantor set.

## 2.1 Scattered spaces, Cantor sets and trees

We now recall some basic definitions from topology. We shall work mostly with second countable metric spaces, however some of the definitions below make sense for arbitrary topological spaces.

**Definition 2.1.** A topological space $X$ is *scattered* if every nonempty subset of $X$ has an isolated point.

**Definition 2.2.** Fix a topological space $X$. The *Cantor-Bendixson derivative* of $X$, denoted by $X'$, is the subset of $X$ consisting of all non-isolated points of $X$. Specifically,

$$X' = X \setminus \{x \in X \colon \text{there is a neighborhood } U \text{ of } x \text{ such that } U \cap X = \{x\}\}.$$

The Cantor-Bendixson derivative can be iterated, using ordinals. Namely, the *$\alpha$th Cantor-Bendixson derivative* $X^{(\alpha)}$ of $X$ is defined by induction as follows:

$$\begin{aligned} X^{(0)} &= X, \\ X^{(\alpha+1)} &= (X^{(\alpha)})', \\ X^{(\delta)} &= \bigcap_{\xi<\delta} X^{(\xi)} \text{ for a limit ordinal } \delta. \end{aligned}$$

Note that each $X^{(\alpha)}$ is a closed subset of $X$. The *Cantor-Bendixson rank* of $X$ is the minimal $\alpha$ such that $X^{(\alpha)} = \emptyset$. If such $\alpha$ does not exist, we say that the Cantor-Bendixson rank of $X$ is $\infty$.

**Proposition 2.3.** *Let $X$ be a topological space. Then:*

*(1) There exists an ordinal $\alpha$ such that $X^{(\alpha)} = X^{(\beta)}$ for every $\beta > \alpha$.*

*(2) $X$ is scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal $\alpha$.*

*Proof.* Property (1) is obvious, because $\{X^{(\alpha)}\}_\alpha$ is a decreasing chain of subsets of $X$, therefore it has to stabilize.

Suppose $X^{(\alpha)} = X^{(\alpha+1)} \neq \emptyset$ for some $\alpha$. Then $X^{(\alpha)}$ witnesses that $X$ is not scattered. Suppose now that $X^{(\alpha)} = \emptyset$ for some $\alpha$ and fix $A \subseteq X$. Let $\xi < \alpha$ be the minimal ordinal such that $A \not\subseteq X^{(\alpha)}$. Then $\alpha = \xi + 1$ and $A \subseteq X^{(\xi)}$. Any element of $A \setminus X^{(\xi+1)}$ is an isolated point of $A$. This shows that $X$ is scattered. $\square$

Recall that a topological space is *second-countable* if its topology is generated by countably many open sets (equivalently: it has a countable base for open sets).

**Proposition 2.4.** *Let $X$ be a second-countable topological space. If $X$ is scattered then its Cantor-Bendixson rank is a countable ordinal and $X$ is countable.*

*Proof.* In a second-countable space, every strictly decreasing chain of closed sets is countable. Thus, there exists a countable ordinal $\alpha$ such that $X^{(\alpha)} = X^{(\alpha+1)}$. If $X^{(\alpha)} \neq \emptyset$, then $X$ is not scattered. Otherwise, $X$ is scattered, its Cantor-Bendixson rank is $\leqslant \alpha$. Finally, $X$ is countable, because every second-countable space can have only countably many isolated points and therefore $X^{(\xi)} \setminus X^{(\xi+1)}$ is countable for every $\xi < \alpha$. $\square$

We have already seen the definition of a perfect set (a nonempty, dense-in-itself, completely metrizable space). It is quite obvious that being perfect is totally opposite to being scattered. In fact, being perfect is close to 'being a Cantor set'. Namely:

**Proposition 2.5.** *Every perfect set contains a subset homeomorphic to the Cantor set.*

This fact is well-known. Instead of proving it directly, we shall discuss briefly the notion of a Cantor tree. Let $\langle X, \varrho \rangle$ be a metric space. A *Cantor tree* of open subsets of $X$ is a family $\{U_s\}_{s \in 2^{<\omega}}$ consisting of nonempty open subsets of $X$ with the following properties:

1. $\operatorname{cl} U_{s^\frown i} \subseteq U_s$ for every $s \in 2^{<\omega}$,
2. $U_s \cap U_t = \emptyset$ whenever $s, t \in 2^k$ and $s \neq t$,
3. $\operatorname{diam}(U_s) \leqslant 2^{-|s|}$ for every $s \in 2^{<\omega}$.

Here, $s^\frown i$ denotes the end-extension of the sequence $s$ by adding the value $i$. It is clear that the last condition can be weakened, we only to take care that the diameters of $U_s$ tend to 0 with respect to the length of $s$. Cantor trees provide a canonical way of constructing Cantor sets with various properties. This will be demonstrated later. For the moment, let us recall the following folklore fact.

**Proposition 2.6.** *Let $\{U_s\}_{s \in 2^{<\omega}}$ be a Cantor tree of open subsets of a Polish space $X$. Then the set*

$$P = \bigcap_{n \in \omega} \bigcup_{s \in 2^n} U_s$$

*is homeomorphic to the Cantor set $2^\omega$.*

We shall call $P$ the *Cantor set induced by* $\{U_s\}_{s\in 2^{<\omega}}$.

*Proof.* Note that, given $x \in P$, there is a unique $\sigma \in 2^\omega$ such that $x \in \bigcap_{n\in\omega} U_{\sigma\restriction n}$. This defines a map $h\colon P \to 2^\omega$. Clearly, $h$ is one-to-one. Cantor's theorem says that $h$ is onto. Finally, it is straightforward to check that $h$ is a homeomorphism. $\square$

## 2.2 Partitions of planes and topology

Ramsey's theorem suggests the following question:

(∗) Assume $C \subseteq [X]^2$ is such that for every $n \in \mathbb{N}$ there is $Y_n \subseteq X$ satisfying $[Y_n]^2 \subseteq C$. Is it true that there is an infinite set $Y_\infty \subseteq X$ satisfying $[Y_\infty]^2 \subseteq C$?

The answer is easily seen to be negative:

**Example 2.7.** Let $X = \omega \times \omega$ and define $C \subseteq [X]^2$ to consist of all pairs $\{x, y\}$, where $x = \langle k, l\rangle$, $y = \langle m, n\rangle$ so that $k < m$ and $l > n$. It is obvious that for each $n \in \mathbb{N}$ there is $Y_n$ such that $[Y_n]^2 \subseteq C$ ($Y_n$ could be the graph of the function $x \mapsto -x + n$ restricted to $\omega$). On the other hand, there is no infinite $Y$ satisfying $[Y]^2 \subseteq C$, because such a set would violate the fact that $\omega$ is well-ordered.

Now suppose that the set $X$ and $C \subseteq [X]^2$ have some reasonable topological structure and we look for a criterion giving a "large" set $P$ satisfying $[P]^2 \subseteq C$. One of the obvious meanings of "large", in the realm of separable metric spaces, is the notion of being perfect. In fact, every perfect set contains a homeomorphic copy of the Cantor set.

Recall that a metrizable space is *analytic* if it is a continuous image of some Polish (i.e. separable complete metrizable) space.

Given $C \subseteq [X]^2$, we shall say that $P$ is $C$*-homogeneous* if $[P]^2 \subseteq C$. The next result offers a criterion for the existence of a homogeneous perfect set. It is attributed to Todorčević, although it was probably noticed by several people independently. The book of Todorčević & Farah [9] is probably the first text where this result is presented in full detail.

**Theorem 2.8.** *Let $X$ be an analytic space and assume $C \subseteq [X]^2$ is open. Then either there exists a perfect $C$-homogeneous set (i.e. a set $P \subseteq X$ such that $[P]^2 \subseteq C$) or else*

$$X = \bigcup_{n\in\omega} X_n$$

*where $[X_n]^2 \cap C = \emptyset$ for every $n \in \omega$.*

*Proof.* **Step 1:** Let $f\colon Y \to X$ be a continuous surjection, where $Y$ is a Polish space. Define $C' = \{s \in [Y]^2\colon f[s] \in C\}$. Then $C'$ is open, by the continuity of $f$. If $P$ is a $C'$-homogeneous perfect set then $P$ contains a homeomorphic copy of the Cantor set $K$. Furthermore, $f \restriction K$ is one-to-one and therefore $f[K]$ is homeomorphic to the

Cantor set and $[K]^2 \subseteq C$. If $Z \subseteq Y$ is such that $[Z]^2 \cap C' = \emptyset$ then $[f[Z]]^2 \cap C = \emptyset$. This shows that it suffices to prove our statement for Polish spaces. In other words, without loss of generality we may assume that $X$ is Polish.

**Step 2:** Let $W$ be the set of all $x \in X$ for which there is a neighborhood $U_x$ satisfying $U_x = \bigcup_{n\in\omega} U_x^n$, where $[U_x^n]^2 \cap C = \emptyset$ for every $n \in \omega$. Obviously, $W$ is an open set. If $W = X$ then we are done (we get the second part of the dichotomy), because $X$ has a countable base. Suppose $W \neq X$ and let $A = X \setminus W$. Since now we are aiming at the first part of the dichotomy, we may assume that $A = X$.

**Step 3:** We shall construct a suitable copy of the Cantor set in $A$. The key fact is that $[U]^2 \cap C \neq \emptyset$ whenever $U \subseteq X$ is a nonempty open set. Thus it is straightforward to construct a Cantor tree of open sets $\{U_s\}_{s\in 2^{<\omega}}$ satisfying

(1) $\{x, y\} \in C$ whenever $x \in U_s$, $y \in U_t$ and $s, t \in 2^{<\omega}$ are incomparable.

Having defined $U_s$, we choose $\{x, y\} \in [U_s]^2 \cap C$ and, using the fact that $C$ is open, we enlarge the points $x$, $y$ to open sets $U_{s^\frown 0}$, $U_{s^\frown 1}$ (of diameter $< 2^{-n}$, where $n = |s|+1$) so that $\{x', y'\} \in C$ whenever $x' \in U_{s^\frown 0}$, $y' \in U_{s^\frown 1}$. The tree $\{U_s\}_{s\in 2^{<\omega}}$ induces a $C$-homogeneous Cantor set. □

As an immediate corollary, we obtain the following classical fact:

**Corollary 2.9.** *Every analytic set is either countable or contains a perfect set.*

*Proof.* Let $X$ be an analytic set and let $C = [X]^2$. If there is no $C$-homogeneous perfect set then, by Theorem 2.8, we have that $X = \bigcup_{n\in\omega} X_n$, where each $X_n$ is at most a singleton; in other words, $X$ is countable. □

As we have mentioned, Theorem 2.8 gives a criterion for the existence of a homogeneous set:

**Corollary 2.10.** *Let $X$ be an analytic space and let $C \subseteq [X]^2$ be open. If there exists an uncountable $C$-homogeneous set then there exists a perfect one as well.*

*Proof.* Clearly, the second part of the dichotomy cannot hold if $X$ has an uncountable $C$-homogeneous set. □

We now present a Ramsey-type version of Theorem 2.8. Recall that a set $A$ in a topological space $X$ has the *Baire property* if $A = (U \setminus F) \cup G$, where $U$ is open and $F$, $G$ are of first category (i.e. meager sets). It is well-known that every Borel set has the Baire property (just because the family of all sets with the Baire property forms a $\sigma$-algebra of sets).

**Proposition 2.11** (Mycielski [7])**.** *Let $X$ be an uncountable Polish space and assume $C \subseteq [X]^2$ has the Baire property. Then there exists a perfect set $P \subseteq X$ such that $C \cap [P]^2$ is open in $[P]^2$.*

*Proof.* Passing to a closed subspace, we may assume that $X$ is dense-in-itself. By assumption, $C = (U \setminus F) \cup G$, where $F$ and $G$ are meager (it may happen that $U = \emptyset$). Let $F \cup G = \bigcup_{n\in\omega} H_n$, where $H_n$ is nowhere dense. It suffices to find a perfect set $P \subseteq X$ such that $[P]^2 \cap H_n = \emptyset$ for every $n \in \omega$. We may assume that $H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots$.

For this aim, we construct a Cantor tree $\{V_s\}_{s\in 2^{<\omega}}$ of open subsets of $X$ so that

(x) $\{x, y\} \notin H_n$ whenever $x \in V_s$, $y \in V_t$ and $s, t \in 2^n$ are such that $s \neq t$.

Once we have defined $V_s$, using the fact that $[V_s]^2 \nsubseteq \operatorname{cl} H_{n+1}$, we find $x_0 \neq y_0$ in $V_s$ such that $\{x_0, y_0\} \notin \operatorname{cl} H_{n+1}$. Enlarge $x_0$ and $y_0$ to disjoint open sets $V_{s^\frown 0}$, $V_{s^\frown 1}$ (of diameter less than $1/n$, where $n$ is the length of $s$) so that (x) holds. Finally, the Cantor set $P = \bigcap_{n\in\omega} \bigcup_{s\in 2^n} V_s$ has the property that $[P]^2 \cap (F \cup G) = \emptyset$. □

The result above is actually valid when 2 is replaced by a bigger natural number, although we shall not need it here.

**Theorem 2.12** (Galvin, Mycielski)**.** *Let $X$ be an uncountable Polish space and let*

$$[X]^2 = C_0 \cup \dots \cup C_{k-1}$$

*where each $C_i$ has the Baire property. Then there exist $j < k$ and a perfect set $P \subseteq X$ such that $[P]^2 \subseteq C_j$.*

*Proof.* Applying Proposition 2.11 $k$ times, we find a perfect set $Q \subseteq X$ such that $C_i \cap [Q]^2$ is open in $[Q]^2$ for each $i < k$. Now we apply Theorem 2.8 until we reach the required perfect set. Namely, if there is no perfect $P_0$ for which $[P_0]^2 \subseteq C_0$, then $Q$ is a countable union of sets whose symmetric squares omit $C_0$. Since $Q$ is compact, we can shrink it to a smaller perfect set $Q_1$ such that $[Q_1]^2 \cap C_0 = \emptyset$. Continue this way until we reach $j < k$ such that the first part of the dichotomy in Theorem 2.8 occurs. □

*Remark* 2.13. (a) Theorem 2.8 fails when the word "open" is replaced by "closed". A counterexample is not trivial, namely, there exists a partition $[\mathbb{R}]^2 = K_0 \cup K_1$ such that $K_0$ is open, $K_1$ is closed, there are no uncountable $K_1$-homogeneous sets and $\mathbb{R}$ cannot be covered by a countable union of $K_0$-homogeneous sets. The details are contained in the book of Todorčević & Farah [9, p. 85].

(b) Theorem 2.8 fails for co-analytic sets: there exist co-analytic sets of cardinality $\aleph_1$, without perfect subsets.

(c) Theorem 2.8 does not generalize to colorings of triples. The simplest example, perhaps first noticed by Blass [2] is as follows. Let $X = 2^\omega$ be the Cantor set with the lexicographic ordering. Let $C \subseteq [X]^3$ consist of all triples $\{x, y, z\}$ such that, assuming $x < y < z$ with respect to the lexicographic ordering, it holds that $\Delta(x, y) > \Delta(x, z) = \Delta(y, z)$, where $\Delta(s, t)$ is the maximal $n$ such that $s \restriction n = t \restriction n$. Then $C$ is open and closed at the same time and every $Y \subseteq X$ such that $[Y]^2 \subseteq C$

or $[Y]^2 \cap C = \emptyset$ has Cantor-Bendixson rank at most 1. In particular, every set that is either $C$-homogeneous or $([X]^2 \setminus C)$-homogeneous is countable.

(d) Actually, Blass [2] proved the following interesting partition result: whenever $X$ is an uncountable Polish space, $N > 0$ is a natural number and $[X]^N$ is partitioned into finitely many of open (or just having the Baire property) pieces, then there exists a perfect set $P \subseteq X$ such that $[P]^N$ intersects at most $(N-1)!$ pieces.

(e) One has to mention a set-theoretic axiom, called Open Coloring Axiom, which reads as follows:

(OCA) Given a separable metric space $X$, given an open set $C \subseteq [X]^2$, either there exists an uncountable set $Y \subseteq X$ such that $[Y]^2 \subseteq C$ or else $X = \bigcup_{n\in\omega} X_n$, where $[X_n]^2 \cap C = \emptyset$ for each $n \in \omega$.

This axiom, introduced by Abraham, Rubin and Shelah [1] is known to be independent of the usual axioms of set theory.

## 2.3 An application to real functions

Let us mention the following application of Theorem 2.8 to real functions, part of it due to Filipczak [3]. In order to avoid confusion, we assume that a function is *increasing* if it preserves the $\leqslant$ ordering, in particular a constant function is increasing. A one-to-one increasing function is called *strictly increasing*. The same remarks apply to decreasing functions.

**Theorem 2.14.** *Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function whose graph is an analytic set. Then $f$ has one of the following properties:*

(+) $\mathbb{R} = \bigcup_{n\in\omega} A_n$ *such that $f \restriction A_n$ is increasing for each $n \in \omega$.*

(−) $\mathbb{R} = \bigcup_{n\in\omega} A_n$ *such that $f \restriction A_n$ is decreasing for each $n \in \omega$.*

(¶) *There exist perfect sets $P, Q \subseteq \mathbb{R}$ such that $f \restriction P$ is continuous and strictly increasing and $f \restriction Q$ is continuous and strictly decreasing.*

*Furthermore, if $f[\mathbb{R}]$ is uncountable then there exists a perfect set $D$ such that $f \restriction D$ is strictly monotone (i.e. either strictly increasing or strictly decreasing).*

*Proof.* Let $X = \{\langle t, f(t)\rangle \colon t \in \mathbb{R}\}$ be the graph of $f$. Suppose (¶) fails, e.g., there is no perfect $P$ such that $f \restriction P$ is strictly increasing and continuous. Since an increasing function has at most countably many points of discontinuity, there is also no perfect set $P$ such that $f \restriction P$ is strictly increasing. In other words, we may forget about the continuity condition.

Define $C \subseteq [X]^2$ to consist of all $\{x, y\}$ such that $x = \langle s, f(s)\rangle$, $y = \langle t, f(t)\rangle$ and, $f \restriction \{s, t\}$ is strictly increasing. Then $C$ is open, as a symmetric subset of $X \times X$. Using the assumption and Theorem 2.8, we find that $X = \bigcup_{n\in\omega} X_n$ such that $[X_n]^2 \cap C = \emptyset$ for each $n \in \omega$. Let $A_n$ be the projection of $X_n$ onto the first coordinate. Then $f \upharpoonright A_n$ is decreasing, which shows that $(-)$ holds.

Concerning the "furthermore" part, supposing it is false, Theorem 2.8 would say that $f$ satisfies both $(+)$ and $(-)$, which is possible only if $f[\mathbb{R}]$ is countable. $\square$

## 2.4 Bernstein sets

We now show that the Ramsey-type Theorem 2.12 fails for arbitrary partitions. This is a straightforward consequence of the existence of Bernstein sets.

**Definition 2.15.** A subset $B$ of a topological space $X$ is called a *Bernstein set* if for every perfect set $P \subseteq X$ it holds that $P \cap B \neq \emptyset \neq P \setminus B$.

Below is the well-known fact concerning Bernstein sets.

**Theorem 2.16.** *Every uncountable Polish space contains a Bernstein set.*

*Proof.* Let $X$ be an uncountable Polish space and let $\{P_\alpha\}_{\alpha<\mathfrak{c}}$ be an enumeration of all perfect subsets of $X$. Construct inductively one-to-one sequences $\{a_\alpha\}_{\alpha<\mathfrak{c}}$, $\{b_\alpha\}_{\alpha<\mathfrak{c}}$ so that the following conditions are satisfied:

(i) $a_\xi \neq b_\eta$ for every $\xi, \eta < \mathfrak{c}$,

(ii) $a_\xi, b_\xi \in P_\xi$ for every $\xi < \mathfrak{c}$.

It is clear that the construction can be carried out, because every perfect subset of $X$ has cardinality $\mathfrak{c}$. By conditions (i) and (ii), it is obvious that $B = \{b_\alpha\}_{\alpha<\mathfrak{c}}$ is a Bernstein set. $\square$

**Corollary 2.17.** *Let $X$ be an uncountable Polish space and let $B$ be a Bernstein set in $X$. Then for every perfect sets $P, Q \subseteq X$ it holds that $(P \times Q) \cap (B \times B) \neq \emptyset$ and $(P \times Q) \setminus (B \times B) \neq \emptyset$.*

## 2.5 A criterion for perfect homogeneous sets

As we have seen in Remark 2.13(c) above, there is no hope for a Ramsey-type theorem involving perfect sets and open colorings of triples. Below we present a criterion for the existence of perfect homogeneous sets with respect to $G_\delta$ colorings.

**Theorem 2.18** ([4])**.** *Let $X$ be a Polish space, let $N > 0$ be a natural number and let $C \subseteq [X]^N$ be $G_\delta$. Then either*

*(s) there exists a countable ordinal $\delta$ such that $S^{(\delta)} = \emptyset$ whenever $[S]^N \subseteq C$, or else*

*(P) there exists a perfect set $P$ such that $[P]^N \subseteq C$.*

*Proof.* Suppose (s) fails. Let us call a family $\mathscr{F}$ of subsets of $X$ *admissible* if for every countable ordinal $\gamma$ there exists a $C$-homogeneous set $S$ such that $S^{(\gamma)} \cap A \neq \emptyset$ for every $A \in \mathscr{F}$. Note that $\{X\}$ is an admissible family. Fix a complete metric on $X$ such that $\operatorname{diam} X \leqslant 1$. Let $C = \bigcap_{n\in\omega} C_n$, where each $C_n$ is open. We shall construct a Cantor tree $\{U_s\}_{s\in 2^{<\omega}}$ in $X$ such that the following conditions are satisfied for each $k < \omega$:

(1) Given pairwise distinct $s_0, \dots, s_{N-1} \in 2^k$, it holds that $\{x_0, \dots, x_{N-1}\} \in C_k$ whenever $x_i \in U_{s_i}$ for $i < N$.

(2) The family $\{U_s\}_{s\in 2^k}$ is admissible.

Condition (1) ensures us that the Cantor set induced by this tree is $C_k$-homogeneous for every $k \in \omega$, therefore it is $C$-homogeneous. Condition (2) will allow us to continue the inductive construction.

We start with $U_\emptyset = X$ (here we use the fact that $\operatorname{diam}(X)$ is small enough). Fix $k > 0$ and suppose $U_s$ have already been constructed for all $s \in 2^{<k}$. Fix a countable base $\mathscr{B}$ for the open sets in $X$. Fix $\gamma < \omega_1$. Let $S$ be such that $S^{(\gamma+1)} \cap U_s \neq \emptyset$ for every $s \in 2^{k-1}$. Then $S^{(\gamma)} \cap U_s$ is infinite for every $s \in 2^{k-1}$. Choose a set $W = \{x_t\}_{t\in 2^k}$ such that $x_{s^\frown i} \in S \cap U_s$ for $s \in 2^{k-1}$, $i \in 2$ and $x_t \neq x_r$ whenever $t \neq r$. Note that $[W]^N \subseteq C$, because $W \subseteq S$. As $C_k$ is open and $W$ is finite, we can enlarge each element $x_t$ of $W$ to an open set $V_t \in \mathscr{B}$ so that $\operatorname{diam}(V_t) \leqslant 2^{-k}$, $\operatorname{cl} V_{s^\frown i} \subseteq U_s$ for every $s \in 2^{k-1}$, and

$$(*) \qquad \{x_0, \dots, x_{N-1}\} \in C_k \text{ for every } \langle x_0, \dots, x_{N-1}\rangle \in V_{t_0} \times \dots \times V_{t_{N-1}},$$

whenever $t_0, \dots, t_{N-1} \in 2^k$ are pairwise distinct. In particular, the family $\mathscr{V}_\gamma = \{V_t\}_{t\in 2^k}$ is pairwise disjoint. The problem is to show that $\mathscr{V}_\gamma$ is admissible for some $\gamma$. Notice, however, that there are only countably many possibilities for the indexed family $\mathscr{V}_\gamma$, therefore there exists an uncountable set $F \subseteq \omega_1$ such that $\mathscr{V} = \{V_t\}_{t\in 2^k}$ does not depend on $\gamma$, whenever $\gamma \in F$. We claim that the family $\mathscr{V}$ is admissible.

Fix $\alpha < \omega_1$ and choose $\gamma \in F$ such that $\gamma > \alpha$. Let $S$ be chosen above for $\gamma$. Then $S^{(\alpha)} \cap V_t \subseteq S^{(\gamma)} \cap V_t \neq \emptyset$ for every $t \in 2^k$, by the choice of the set $W$ above. This shows that $\mathscr{V}$ is admissible, completing the proof. □

A sample application to real functions yields the following:

**Corollary 2.19.** *Let $f\colon K \to L$ be a continuous map of compact metric spaces. Then either there exists $q \in L$ such that $f^{-1}(q)$ contains a Cantor set, or else there exists a countable ordinal $\gamma$ such that for every $y \in L$ the set $f^{-1}(y)$ is scattered and its Cantor-Bendixson rank is $< \gamma$.*

## 2.6 Coverings by functions

We are going to present an example showing that Theorem 2.18 fails for $\sigma$-compact colorings. There is actually a longer story here, contained in Shelah [8] and Kubiś &

Shelah [5], where $\sigma$-compact colorings with 'large' but not perfect squares are constructed. For our purposes, it suffices to present a simple example of a set consisting of countably many graphs of continuous functions.

**Definition 2.20.** Let $X$ be a set and let $\mathscr{F}$ be a family of partial functions from subsets of $X$ into $X$. We say that $A \subseteq X \times X$ is *covered by* $\mathscr{F}$ if for every $\langle a, b\rangle \in A$ there exists $f \in \mathscr{F}$ such that either $b = f(a)$ or $a = f(b)$.

**Proposition 2.21** (Sierpiński)**.** *There exists a countable family $\mathscr{F}$ of functions from $\omega_1$ to $\omega_1$ such that $\omega_1 \times \omega_1$ is covered by $\mathscr{F}$.*

*Proof.* Given $0 < \alpha < \omega_1$, choose a surjection $\varphi_\alpha \colon \omega \to \alpha$ and define $f_n(\xi) = \varphi_\xi(n)$. Then $\mathscr{F} = \{\mathrm{id}_{\omega_1}\} \cup \{f_n\}_{n\in\omega}$ is as required. $\square$

**Proposition 2.22.** *Suppose $\mathscr{F}$ is a countable family of partial functions and $S \times T$ is covered by $\mathscr{F}$. If $S$ is uncountable then $|T| \leqslant \aleph_1$.*

*Proof.* Suppose $|T| > \aleph_1$ and assume that $|S| = \aleph_1$. Given $s \in S$, let

$$F_s = \{f(s) \colon f \in \mathscr{F}\}.$$

Then $F_s \subseteq T$ is countable. Let $F = \bigcup_{s\in S} F_s$. Then $|F| \leqslant \aleph_1$, therefore $F \neq T$. Choose $t \in T \setminus F$. Then for every $s \in S$ there exists $f_s \in \mathscr{F}$ such that $f_s(t) = s$. This is impossible, because $S$ is uncountable and $\mathscr{F}$ is countable. $\square$

The two statements above provide good motivations for asking whether uncountable squares can be covered by countable families of continuous functions in the plane. This is answered below.

**Theorem 2.23** ([6])**.** *There exists a countable family $\mathscr{F}$ of continuous functions from the Cantor set $2^\omega$ into itself such that every maximal set $S$ such that $S \times S$ is covered by $\mathscr{F}$ is uncountable.*

*Furthermore, there are no perfect sets $P, Q$ such that $P \times Q$ is covered by $\mathscr{F}$.*

Note that, by the Zorn's Lemma, every set whose square is covered by $\mathscr{F}$ is contained in a maximal one.

*Proof.* Let $\omega = B \cup \bigcup_{n\in\omega} A_n$, where the family $\{B\} \cup \{A_n\}_{n\in\omega}$ is pairwise disjoint and consists of infinite sets. Given $n \in \omega$, let $\varphi_n \colon \omega \to A_n$ be a bijection and let $f_n \colon 2^\omega \to 2^\omega$ be defined by

$$f_n(x)(k) = x(\varphi_n(k)).$$

Suppose $S = \{s_n\}_{n\in\omega}$ is such that $S \times S$ is covered by $\mathscr{F} = \{f_n\}_{n\in\omega}$. Let $x \in 2^\omega$ be such that $x \restriction B_n = s_n \circ \varphi^{-1}$. Then $f_n(x) = s_n$ for every $n \in \omega$, therefore $(S \cup \{x\}) \times (S \cup \{x\})$ is covered by $\mathscr{F}$. Finally, notice that we had a freedom to define $x \restriction B$, and there are continuum many possibilities. Thus, we can ensure that $x \notin S$. This shows the first part.

Concerning the 'furthermore' part, fix two Cantor sets $P$ and $Q$. We construct a sequence $\{U_n \times V_n\}_{n\in\omega}$ of open subsets of $P \times Q$ respectively such that $\operatorname{cl} U_{n+1} \times \operatorname{cl} V_{n+1} \subseteq U_n \times V_n$ and $U_n \times V_n$ is disjoint from $f_n \cup f_n^{-1}$ for every $n \in \omega$. At each step we use the fact that $U_n$ and $V_n$ are dense-in-itself and therefore the graph of every continuous function is nowhere dense in $U_n \times V_n$. In particular, $U_n \times V_n$ cannot be covered by finitely many functions. Finally, by compactness, we find $\langle x, y\rangle \in \bigcap_{n\in\omega} U_n \times V_n$ that is not covered by $\mathscr{F}$. $\square$

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