Independent bases of admissible rules

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Abstract

We show that *IPC*, *K*4, *GL*, and *S*4, as well as all logics inheriting their admissible rules, have independent bases of admissible rules.

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1 Introduction

The study of nonclassical logics usually revolves around provability of formulas. When we generalize the problem from formulas to inference rules, there arises an important distinction between *derivable* and *admissible* rules, introduced by Lorenzen [12]. A rule is derivable if it can be inferred from the postulated axioms and rules of the logic (such as modus ponens, or necessitation); and it is admissible if the set of theorems of the logic is closed under the rule. In classical logic, these two notions coincide, but nonclassical logics often admit rules which are not derivable. For example, all intermediate (superintuitionistic) logics admit the Kreisel–Putnam rule

$$\neg\varphi \to \psi \vee \chi \;/\; (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi),$$

whereas many of these logics (such as *IPC* itself) do not derive this rule. A set of admissible rules in a given logic is a *basis* of admissible rules, if every admissible rule is derivable from the basis and the postulated inference rules of the logic.

The research of admissible rules was stimulated by a question of H. Friedman [3], asking whether admissibility of rules in *IPC* is decidable. The problem was investigated mainly by Rybakov (see [13]), who has shown that admissibility is decidable for a large class of modal and intermediate logics, found semantic criteria for admissibility, proved nonexistence of finite bases of admissible rules for many logics (including *IPC* and *K*4), and obtained other results

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on various aspects of admissibility. Ghilardi [5, 6] discovered the connection of admissibility to projective formulas and unification, which provided another criteria for admissibility in certain modal and intermediate logics. Based on this result, Iemhoff [7] constructed an elegant explicit basis for rules admissible in *IPC*, generalized to some other intermediate logics in [8, 9]. Similar bases for admissible rules of some modal logics were constructed by Jeřábek [10]. A basis for admissible rules of $S4$ was also constructed earlier by Rybakov [14].

In many contexts (such as linear algebra), the notion of a "basis" involves *independence*: a basis is a generating set which has no proper generating subset. Bases of admissible rules are not required to satisfy this property, and a natural question is when *independent bases* of admissible rules exist. The question is nontrivial even for axiomatization of logics by formulas: there are modal logics without an independent axiomatization by Chagrov and Zakharyaschev [1]. (In contrast, notice that every countable classical first-order theory has an independent axiomatization.) The problem for rules was investigated in Rybakov [13], who constructed a tabular logic without an independent basis of admissible rules. Rybakov et al. [16] have shown that all pretabular extensions of $S4$ or *IPC* have an independent basis of admissible rules, and posed the problem whether the basic transitive logics ($K4$, $S4$, *IPC*) posses independent bases. The known bases of rules admissible in these logics from [7, 14, 10] are not independent, as they consist of increasing (with respect to logical consequence) chains of rules.

We use a modification of the bases from [7, 10] to solve the problem affirmatively: *IPC*, $K4$, GL, and $S4$ do have independent bases of admissible rules, and the same is true for every logic which inherits the admissible rules of any of these four systems. In fact, the same basis works for all logics of unbounded width which inherit admissible rules of *IPC*, whereas logics of bounded width (which actually implies width at most 2) have finite bases. A similar dichotomy holds in the modal cases.

2 Preliminaries

We will use various tools from the theory of modal and intermediate logics, such as the general frame semantics; we briefly review the relevant definitions below. More background can be found in Chagrov and Zakharyaschev [2].

We work with modal logics in a language which contains a single unary connective $\Box$, besides (any complete set of) connectives of the propositional classical logic. A *normal modal logic* is a set of formulas L which contains all classical tautologies, the axiom

$$(K) \qquad \Box(p \to q) \to (\Box p \to \Box q),$$

and which is closed under substitution, modus ponens (MP), and necessitation (Nec):

$$\text{(MP)} \qquad \varphi, \varphi \to \psi \;/\; \psi,$$

$$\text{(Nec)} \qquad \varphi \;/\; \Box\varphi.$$

The smallest normal modal logic is called K, and $L \oplus X$ denotes the normal closure of a logic L, and a set of formulas X. Some normal modal logics which we need to refer by name

logic	axiomatization
$K4$	$K \oplus \Box p \to \Box\Box p$
$S4$	$K4 \oplus \Box p \to p$
GL	$K4 \oplus \Box(\Box p \to p) \to \Box p$
	$= K \oplus \Box(\Box p \to p) \to \Box p$
$GL.3$	$GL \oplus \Box(\Box p \to q) \vee \Box(\boxdot q \to p)$
$K4Grz$	$K4 \oplus \Box(\Box(p \to \Box p) \to p) \to \Box p$
$S4Grz$	$K4Grz \oplus S4$
	$= K \oplus \Box(\Box(p \to \Box p) \to p) \to p$
$S4.1$	$S4 \oplus \Box\Diamond p \to \Diamond\Box p$

Table 1: some normal modal logics

are listed in table 1. The symbols $\Diamond\varphi$, $\boxdot\varphi$, $\diamonddot\varphi$, and $\Box^n\varphi$, are respectively abbreviations for $\neg\Box\neg\varphi$, $\varphi \wedge \Box\varphi$, $\varphi \vee \Diamond\varphi$, and $\underbrace{\Box \cdots \Box}_{n}\varphi$.

The language of the intuitionistic logic contains the connectives $\to$, $\wedge$, $\vee$, and $\bot$. Negation is defined as an abbreviation $\neg\varphi = (\varphi \to \bot)$. An *intermediate* (or *superintuitionistic*) *logic* is a set L of intuitionistic formulas which is closed under substitution and MP, and contains all tautologies of the intuitionistic propositional calculus (*IPC*, see e.g. [2] for an axiomatization). Normal modal logics extending $K4$, and intermediate logics are also called *transitive logics.*

A (modal) *Kripke frame* is a pair $\langle F, < \rangle$, where $<$ is a binary relation on a set F. As all modal logics we encounter are extensions of $K4$, we will require all frames to be transitive. We will usually denote accessibility relations by the ordering symbol $<$, in which case $\leq$ is the reflexive closure of $<$. (The notation $<$ does not imply that the relation is irreflexive. In particular, if the accessibility relation is already reflexive, then $< \; = \; \leq$.) A *valuation* (or *truth assignment*) in $\langle F, < \rangle$ is a binary relation $\Vdash$ between elements of F and formulas, which locally respects Boolean connectives, and satisfies

$$x \Vdash \Box\varphi \quad \text{iff} \quad \forall y \in F\,(x < y \Rightarrow y \Vdash \varphi).$$

The triple $\langle F, <, \Vdash \rangle$ is then called a *Kripke model.* A *general frame* is a triple $\langle F, <, V \rangle$, where $\langle F, < \rangle$ is a Kripke frame, and $V \subseteq \mathcal{P}(F)$ is closed under Boolean operations, and under the operation

$$X{\downarrow} = \{y \in F;\ \exists x \in X\ y < x\}.$$

A subset $X \subseteq F$ is *admissible* in $\langle F, <, V \rangle$, if $X \in V$. A valuation $\Vdash$ is *admissible*, if the set

$$\Vdash(\varphi) = \{x \in F;\ x \Vdash \varphi\}$$

is admissible for every formula φ (or equivalently, for every propositional variable). We will identify a Kripke frame $\langle F, < \rangle$ with the general frame $\langle F, <, \mathcal{P}(F) \rangle$. A Kripke model $\langle F, <, \Vdash \rangle$ *induces* the general frame $\langle F, <, V \rangle$, where

$$V = \{\Vdash(\varphi);\ \varphi \text{ is a formula}\}.$$

We will often denote admissible sets or valuations as *definable* in induced frames (and, par abus de langage, in other general frames).

A formula φ is *valid* or *satisfied* in a model $\langle F,<,\Vdash\rangle$ if $x\Vdash\varphi$ for every $x\in F$, otherwise it is *refuted*. A formula is valid in a general frame $\mathcal{F}=\langle F,<,V\rangle$ if it is valid under all admissible valuations. A logic L is valid in $\mathcal{F}$ if all axioms (equivalently: all theorems) of L are valid in $\mathcal{F}$; in such a case we call $\mathcal{F}$ an *L-frame*. The set of all formulas valid in $\mathcal{F}$ is called the *logic of the frame* $\mathcal{F}$, and is denoted by $L(\mathcal{F})$. A logic L is *complete* with respect to a class $\mathcal{C}$ of general frames, if $L=\bigcap\{L(F);\ F\in\mathcal{C}\}$. A logic has the *finite model property* if it is complete with respect to a class of finite frames.

A general frame $\langle F,<,V\rangle$ is *refined* if it satisfies

$$\forall X\in V\,(x\in X\Leftrightarrow y\in X)\Rightarrow x=y,$$
$$\forall X\in V\,(x\in\Box X\Rightarrow y\in X)\Rightarrow x<y,$$

for any $x,y\in F$. Recall that a family of sets has the *finite intersection property* (*fip*) if every its finite subfamily has a nonempty intersection. A refined frame $\langle F,<,V\rangle$ is called *descriptive*, if every subset of V with fip has a nonempty intersection. All Kripke frames are refined, and all finite refined frames are Kripke frames. A Kripke frame is descriptive iff it is finite. For any logic L, *canonical frames* are particular descriptive L-frames $\langle C,<,V\rangle$ constructed as follows. We fix a set P of propositional variables, and let C consist of maximal (with respect to inclusion) L-consistent sets of formulas over P, where a set of formulas is called L-consistent if $\nvdash_L\neg\bigwedge X$ for every its finite subset X. We define an accessibility relation $<$ on C, and a valuation $\Vdash_c$, by

$$X<Y\quad\text{iff}\quad\forall\varphi\,(\Box\varphi\in X\Rightarrow\varphi\in Y),$$
$$X\Vdash_c\varphi\quad\text{iff}\quad\varphi\in X.$$

Equivalently,

$$X<Y\quad\text{iff}\quad\Diamond Y\subseteq X,$$

where $\Diamond Y=\{\Diamond\varphi;\ \varphi\in Y\}$. We let $\langle C,<,V\rangle$ be the general frame induced by the model $\langle C,<,\Vdash_c\rangle$. An important corollary of Zorn's lemma states that every L-consistent set of formulas is included in a maximal L-consistent set (in other words, it is satisfied in a point of $\langle C,<,\Vdash_c\rangle$).

Frame semantics for intuitionistic logic is introduced similarly to modal logic, we will only indicate the differences. An *intuitionistic Kripke frame* is a partially ordered set $\langle F,\le\rangle$. Valuations $\Vdash$ in intuitionistic Kripke models are required to make $\Vdash(\varphi)$ an upper subset of F for every formula φ (or equivalently, for every propositional variable); the monotone connectives $\wedge$, $\vee$, $\bot$ are evaluated locally as in classical logic, and for $\to$ we have

$$x\Vdash\varphi\to\psi\quad\text{iff}\quad\forall y\in F\,(x\le y\wedge y\Vdash\varphi\Rightarrow y\Vdash\psi).$$

An *intuitionistic general frame* is $\mathcal{F}=\langle F,\le,V\rangle$, where V is a set of upper subsets of F, closed under monotone Boolean operations and under the operation

$$X\to Y=F\smallsetminus(X\smallsetminus Y)\!\downarrow=\{x\in F;\ \forall y\ge x\,(y\in X\Rightarrow y\in Y)\}.$$

The frame $\mathcal{F}$ is refined if

$$\forall X \in V\,(x \in X \Rightarrow y \in X) \Rightarrow x \leq y,$$

and it is descriptive if in addition every subset of $V \cup \{F \smallsetminus X;\ X \in V\}$ with fip has a nonempty intersection. In the intuitionistic case, a canonical L-frame consists of L-consistent deductively closed sets X with the disjunction property: if $\varphi \vee \psi \in X$, then $\varphi \in X$ or $\psi \in X$. The accessibility relation is inclusion.

Let $\langle F, < \rangle$ be a Kripke frame. A point $x \in F$ is called *reflexive* if $x < x$, otherwise it is *irreflexive*. We recall that $\leq$ denotes the reflexive closure of $<$. The preorder $\leq$ induces an equivalence relation $x \sim y$ iff $x \leq y \leq x$; its equivalence classes are called *clusters*. (In an intuitionistic frame, all points are reflexive, and all clusters are singletons.) For any subset X of F, we put

$$X{\uparrow} = \{y \in F;\ \exists x \in X\, x < y\},$$
$$X\underline{\uparrow} = \{y \in F;\ \exists x \in X\, x \leq y\}.$$

A point $x \in F$ is an *irreflexive tight predecessor* of X if $x{\uparrow} = X\underline{\uparrow}$, and it is a *reflexive tight predecessor* of X if $x{\uparrow} = \{x\} \cup X\underline{\uparrow}$. Notice that when $X = \{x\}$ is a reflexive singleton, x is both a reflexive and an irreflexive tight predecessor of X; in particular, irreflexive tight predecessors do not have to be irreflexive points. If $F = \{x\}\underline{\uparrow}$, then F is called a *rooted frame*, and x is called its root (any $y \sim x$ is also a root). A *generated subframe* of a general frame $\langle F, <, V \rangle$ is a frame $\langle G, \prec, W \rangle$, where $G \subseteq F$ satisfies $G{\uparrow} \subseteq G$, $\prec$ is the restriction of $<$ to G, and

$$W = \{X \cap G;\ X \in V\}.$$

If F is an L-frame, then so is G. Conversely, if every rooted generated subframe of F is an L-frame, then F is also an L-frame. Generated subframes of Kripke frames are Kripke frames. A subset $X \subseteq F$ is an *antichain*, if $x \not< y$ for any distinct $x, y \in X$. The *width* of a rooted frame is the least upper bound on cardinalities of its antichains. In general, the width of a frame is the lub of widths of its rooted generated subframes. A transitive logic L has *finite* (or *bounded*) *width*, if every refined L-frame has finite width. If L has finite width, there actually exists a natural number k such that every refined L-frame has width at most k; the least such k is called the width of the logic. The width of L also coincides with the width of any canonical L-frame in an infinite number of variables (if we do not distinguish infinite cardinalities).

Following [10], we will work with *multiple-conclusion* (or *generalized*) *rules*. These are expressions of the form $\Gamma \,/\, \Delta$, where Γ and Δ are finite sets of formulas (thus syntactically, rules are the same kind of objects as sequents). We will often omit braces in rules, writing $\varphi_1, \dots, \varphi_k \,/\, \psi_1, \dots, \psi_\ell$ instead of $\{\varphi_1, \dots, \varphi_k\} \,/\, \{\psi_1, \dots, \psi_\ell\}$. A *rule system* over a normal modal or intermediate logic L is a set of rules which is closed under substitution, cut, and weakening, and which contains all postulated rules of L (MP, Nec, and axioms). A rule ϱ is *derivable* over L from a set R of rules, if ϱ is included in the smallest rule system over L which contains R. If every rule from R' is derivable from R, and every rule from R' is derivable from

R, we say that the sets of rules R and R' are *equivalent.* A rule $\Gamma \ / \ \Delta$ is *valid* in a general frame $\langle F, <, V\rangle$, if for every admissible valuation $\Vdash$ such that $x \Vdash \varphi$ for all $\varphi \in \Gamma$ and $x \in F$, there exists $\psi \in \Delta$ such that $x \Vdash \psi$ for all $x \in F$.

A rule $\Gamma \ / \ \Delta$ is *L-admissible*, if for every substitution σ such that $\vdash_L \sigma\varphi$ for all $\varphi \in \Gamma$, there exists $\psi \in \Delta$ such that $\vdash_L \sigma\psi$. A set B of L-admissible rules is a *basis* of L-admissible rules, if every L-admissible rule is derivable from B over L. A basis B is *independent*, if no proper subset of B is a basis. A logic L' *inherits* admissible rules of L, if every rule admissible in L is also admissible in L'. Notice that any logic which inherits L-admissible rules must be an extension of L. Bases and inheritance of single-conclusion rules are defined in a similar way. A rule is L-admissible if and only if it is valid in all canonical L-frames, or equivalently, if it is valid in a canonical L-frame over an infinite set of variables.

We define the rules listed in figure 1, where $n, m \in \omega$. We also put $A^\circ = \{A^\circ_{n,m};\ n, m \in \omega\}$, and similarly for the other rules. The next theorem, which characterizes admissible rules of the basic transitive logics, is the starting point of our investigations.

Theorem 2.1 (Iemhoff [7, 8], Jeřábek [10]) *IPC, $K4$, GL, and $S4$ have bases of single-conclusion and multiple-conclusion admissible rules as given in table 2.*

More generally, these rules form a basis of single-conclusion (multiple-conclusion) admissible rules for any logic which inherits single-conclusion (multiple-conclusion) admissible rules of IPC, $K4$, GL, or $S4$.

basis \ logic	*IPC*	*GL*	$S4$	$K4$
multiple-conclusion	V	$A^\bullet$	A°	$A^\bullet + A^\circ$
single-conclusion	v	$a^\bullet$	a°	$a^\bullet + a^\circ$

Table 2: bases of admissible rules for basic transitive logics

IPC and *GL* have no proper extensions which inherit their admissible multiple-conclusion rules. A simple description of all logics inheriting multiple-conclusion admissible rules of $K4$ or $S4$ was given in [11]; in particular, the largest such logics are $K4Grz$ and $S4Grz$, respectively.

The structure of logics inheriting only the single-conclusion rules of the basic transitive logics appears to be more complicated, but at least we have model-theoretic criteria for inheritance of single-conclusion rules: semantic conditions for logics with the finite model property inheriting admissible rules of *IPC*, $S4$, and $K4$ were given by Rybakov (see [13]), Rybakov et al. [15], and Gencer [4]. Using the methods of the present paper, it is easy to extend these criteria to logics without FMP, and to *GL*.

Finally, we remind the reader that the empty set is a finite set, and zero is a fine natural number.

$$\Box q \to \bigvee_{i<n} \Box p_i \Big/ \{\boxdot q \to p_i;\ i<n\} \tag{$A_n^\bullet$}$$

$$\bigwedge_{j<m} (q_j \equiv \Box q_j) \to \bigvee_{i<n} \Box p_i \Big/ \Big\{ \bigwedge_{j<m} \boxdot q_j \to p_i;\ i<n \Big\} \tag{$A_{n,m}^\circ$}$$

$$\boxdot(q \equiv \Box q) \to \bigvee_{i<n} \Box p_i \Big/ \{\boxdot q \to p_i;\ i<n\} \tag{A_n'}$$

$$\bigwedge_{j<n} (p_j \to q_j) \to \bigvee_{i<n+m} p_i \Big/ \Big\{ \bigwedge_{j<n} (p_j \to q_j) \to p_i;\ i<n+m \Big\} \tag{$V_{n,m}$}$$

$$\Big(\bigvee_{i<n} p_i \to q \Big) \to \bigvee_{i<n} p_i \Big/ \{ q \to p_i;\ i<n \} \tag{V_n'}$$

$$\Box\Big(\Box q \to \bigvee_{i<n} \Box p_i\Big) \vee \Box r \Big/ \bigvee_{i<n} \Box(\boxdot q \to p_i) \vee r \tag{$a_n^\bullet$}$$

$$\Box\Big(\bigwedge_{j<m} (q_j \equiv \Box q_j) \to \bigvee_{i<n} \Box p_i\Big) \vee \Box r \Big/ \bigvee_{i<n} \Box\Big(\bigwedge_{j<m} \boxdot q_j \to p_i\Big) \vee r \tag{$a_{n,m}^\circ$}$$

$$\Box\Big(\boxdot(q \equiv \Box q) \to \bigvee_{i<n} \Box p_i\Big) \vee \Box r \Big/ \bigvee_{i<n} \Box(\boxdot q \to p_i) \vee r \tag{a_n'}$$

$$\Big(\bigwedge_{j<n} (p_j \to q_j) \to \bigvee_{i<n+m} p_i\Big) \vee r \Big/ \bigvee_{i<n+m} \Big(\bigwedge_{j<n} (p_j \to q_j) \to p_i\Big) \vee r \tag{$v_{n,m}$}$$

$$\bigg(\Big(\bigvee_{i<n} p_i \to q\Big) \to \bigvee_{i<n} p_i\bigg) \vee r \Big/ \bigvee_{i<n} (q \to p_i) \vee r \tag{v_n'}$$

$$\Box q \to \bigvee_{i<n} \Box p_i \Big/ \Big\{ \boxdot\Big(q \wedge \bigwedge_{j\neq i} p_j\Big) \to p_i;\ i<n \Big\} \tag{$\Pi_n^\bullet$}$$

$$\boxdot(q \equiv \Box q) \to \bigvee_{i<n} \Box p_i \Big/ \Big\{ \boxdot\Big(q \wedge \bigwedge_{j\neq i} p_j\Big) \to p_i;\ i<n \Big\} \tag{Π_n°}$$

$$\Big(\bigvee_{i<n} p_i \to q\Big) \to \bigvee_{i<n} p_i \Big/ \Big\{ q \wedge \bigwedge_{j\neq i} p_j \to p_i;\ i<n \Big\} \tag{Π_n}$$

$$\Box\Big(\Box q \to \bigvee_{i<n} \Box p_i\Big) \vee \Box r \Big/ \bigvee_{i<n} \Box\bigg(\boxdot\Big(q \wedge \bigwedge_{j\neq i} p_j\Big) \to p_i\bigg) \vee r \tag{$\pi_n^\bullet$}$$

$$\Box\Big(\boxdot(q \equiv \Box q) \to \bigvee_{i<n} \Box p_i\Big) \vee \Box r \Big/ \bigvee_{i<n} \Box\bigg(\boxdot\Big(q \wedge \bigwedge_{j\neq i} p_j\Big) \to p_i\bigg) \vee r \tag{π_n°}$$

$$\bigg(\Big(\bigvee_{i<n} p_i \to q\Big) \to \bigvee_{i<n} p_i\bigg) \vee r \Big/ \bigvee_{i<n} \Big(q \wedge \bigwedge_{j\neq i} p_j \to p_i\Big) \vee r \tag{π_n}$$

Figure 1: our battlefield

3 Construction of independent bases

This section is devoted to the proof of our main theorem:

Theorem 3.1 *Let L be a modal or intermediate logic.*

- (i) *If L inherits admissible multiple-conclusion rules of IPC, $K4$, GL, or $S4$, then it has an independent basis of admissible multiple-conclusion rules.*
- (ii) *If L inherits admissible single-conclusion rules of IPC, $K4$, GL, or $S4$, then it has an independent basis of admissible single-conclusion rules.*

We break the proof of theorem 3.1 into theorems 3.7, 3.10, 3.12, 3.15, and several lemmas.

Lemma 3.2 *The sets of rules V and V' are equivalent over IPC, and likewise v and v' are equivalent over IPC.*

Proof: On the one hand, V'_n follows from the instance of $V_{n,0}$ with $q_i = q$, as $\bigvee_{i<n} p_i \to q$ is equivalent to $\bigwedge_{i<n}(p_i \to q)$, and

$$\bigwedge_{i<n}(p_i \to q) \to p_j \vdash_{IPC} q \to p_j.$$

On the other hand, put $\alpha = \bigwedge_{i<n}(p_i \to q_i)$. We have

$$\vdash_{IPC} (p_i \to \alpha) \to (p_i \to q_i)$$

for all $i < n$, hence

$$\vdash_{IPC} \Big(\bigvee_{i<n} p_i \to \alpha\Big) \to \alpha,$$

which implies

$$\alpha \to \bigvee_{i<n+m} p_i \vdash_{IPC} \Big(\bigvee_{i<n+m} p_i \to \alpha\Big) \to \bigvee_{i<n+m} p_i.$$

An instance of V'_{n+m} thus derives the rule

$$\alpha \to \bigvee_{i<n+m} p_i \Big/ \{\alpha \to p_i;\, i < n+m\},$$

i.e., $V_{n,m}$.

The case of v and v' is analogous. □

Lemma 3.3 *For every $m \in \omega$, there exists a formula $\alpha(\vec{q})$ such that $K4$ proves*

$$\boxdot(\alpha \equiv \Box\alpha) \to \bigwedge_{i<m}(q_i \equiv \Box q_i),$$

$$\bigwedge_{i<m} q_i \to \alpha.$$

In particular, A° is equivalent to A' over $K4$, and a° is equivalent to a' over $K4$.

Proof: Notice that $\boxdot(q \equiv \Box q)$ is equivalent to $(q \equiv \Box q) \wedge ((q \equiv \Box q) \equiv \Box(q \equiv \Box q))$, thus A' is a special case of A°, and a' is a special case of a°. The other direction clearly follows from the existence of α, it thus suffices to prove the first part of the lemma.

We put $M = \{0, \dots, m-1\}$, and for every $X \subseteq M$, let $q^X := \bigwedge_{i \in X} q_i \wedge \bigwedge_{i \notin X} \neg q_i$. For every nonempty $C \subseteq \mathcal{P}(M) \smallsetminus \{M\}$, we fix $f(C) \in C$, and define

$$\alpha_C := \boxdot\Big(q^M \vee \Big(\bigvee_{X \in C} q^X \wedge \bigwedge_{X \in C} \Diamond q^X\Big)\Big),$$

$$\alpha := \big(\Box q^M \to q^M\big) \wedge \bigwedge_{\substack{C \subseteq \mathcal{P}(M) \smallsetminus \{M\} \\ C \neq \emptyset}} \big(\alpha_C \to \neg q^{f(C)}\big).$$

Clearly, $q^M \to \alpha$ is a tautology.

Claim 1 *K4 proves $\boxdot\alpha \to q^M$, thus $\Box\alpha \equiv \Box q^M$, and $\boxdot(\Box\alpha \to \alpha) \to \boxdot\big(\Box q^M \to q^M\big)$.*

Proof: Let F be a finite transitive Kripke model, and $x \in F$ such that $x \nVdash q^M$. Fix a $y \geq x$, $y \nVdash q^M$ such that q^M holds in all points above y's cluster. If y is irreflexive, then $y \Vdash \neg q^M \wedge \Box q^M$, thus $y \nVdash \alpha$. If y is reflexive, let c be the cluster of y, and define

$$C := \big\{X \subsetneq M;\ \exists z \in c\ z \Vdash q^X\big\}.$$

Clearly, C is a nonempty subset of $\mathcal{P}(M) \smallsetminus \{M\}$, and every element of c satisfies α_C. Moreover, there exists a $z \in c$ such that $z \Vdash q^{f(C)}$, thus $z \geq x$ and $z \nVdash \alpha$. $\Box$ (Claim 1)

Claim 2 *K4 proves*

$$\boxdot(\alpha \to \Box\alpha) \to \boxdot\big(\alpha_C \to q^M\big)$$

for every $C \subseteq \mathcal{P}(M) \smallsetminus \{M\}$ such that $|C| \geq 2$.

Proof: Let F be a finite transitive Kripke model, and $x \in F$ such that $x \Vdash \alpha_C \wedge \neg q^M$. Let $X \in C \smallsetminus \{f(C)\}$. By the definition of α_C, there exist $z > y \geq x$ such that $y \Vdash q^X$ and $z \Vdash q^{f(C)}$. Clearly $z \nVdash \alpha$, thus $y \nVdash \Box\alpha$. It suffices to verify $y \Vdash \alpha$. We have $y \Vdash \Box q^M \to q^M$, as $z \nVdash q^M$. Trivially $y \Vdash \alpha_C \to \neg q^{f(C)}$. We claim that $y \Vdash \neg\alpha_D$ for every $C \neq D$: if there exists a $Y \in C \smallsetminus D$, then $y \Vdash \alpha_C \wedge \neg q^M$ implies $y \Vdash \Diamond q^Y$, but α_D implies $\Box\neg q^Y$. The case $Y \in D \smallsetminus C$ is symmetric. $\Box$ (Claim 2)

Claim 3 *K4 proves*

$$\boxdot(\alpha \equiv \Box\alpha) \to \bigvee_{X \in \mathcal{P}(M) \smallsetminus \{M\}} \alpha_{\{X\}} \vee \boxdot q^M.$$

Proof: Put

$$\beta := \boxdot(\alpha \equiv \Box\alpha) \wedge \bigwedge_{X \in \mathcal{P}(M) \smallsetminus \{M\}} \neg\alpha_{\{X\}}.$$

We have

$$\vdash \beta \to \bigwedge_{\substack{C \subseteq \mathcal{P}(M) \smallsetminus \{M\} \\ C \neq \emptyset}} \big(\alpha_C \to \neg q^{f(C)}\big)$$

by claim 2, and the definition of β. As $\beta \to \big(\Box q^M \to q^M\big)$ by claim 1, we obtain $\beta \to \alpha$. Using $\beta \to (\alpha \to \Box\alpha)$, we have $\beta \to \boxdot\alpha$, hence $\beta \to \boxdot q^M$ by claim 1. $\Box$ (Claim 3)

To finish the proof of the lemma, consider $X \subsetneq M$, and notice that $\alpha_{\{X\}}$ implies

$$\bigwedge_{i \in X} \boxdot q_i \wedge \bigwedge_{i \notin X} \boxdot \big(q_i \equiv q^M\big).$$

Claim 1 gives $\vdash \boxdot(\alpha \equiv \Box\alpha) \to \big(q^M \equiv \Box q^M\big)$, hence

$$\vdash \boxdot(\alpha \equiv \Box\alpha) \wedge \alpha_{\{X\}} \to \bigwedge_{i<m} (q_i \equiv \Box q_i).$$

As X was arbitrary, claim 3 implies

$$\vdash \boxdot(\alpha \equiv \Box\alpha) \to \bigwedge_{i<m} (q_i \equiv \Box q_i).$$

□

Lemma 3.4 *The sets of rules $\Pi^\bullet$, Π°, Π, $\pi^\bullet$, π°, and π are equivalent to $A^\bullet$, A°, V, $a^\bullet$, a°, and v, respectively.*

Proof: We will consider $A^\bullet$, the other cases are analogous (modulo lemmas 3.2 and 3.3). Clearly $A^\bullet$ derives $\Pi^\bullet$, we will show that $\{\Pi^\bullet_m;\, m \leq n\}$ derives $A^\bullet_n$ by induction on n. The cases $n = 0$ and $n = 1$ are trivial, let $n \geq 2$. We work "inside" the rule system $K4 + \{\Pi^\bullet_m;\, m \leq n\}$. Assume

$$\Box q \to \bigvee_{i<n} \Box p_i.$$

By $\Pi^\bullet_n$, we have

$$(*) \qquad \boxdot\Big(q \wedge \bigwedge_{j \neq i} p_j\Big) \to p_i$$

for some $i < n$. For every $j \neq i$, we apply the induction hypothesis to

$$\Box q \to \Box(p_i \vee \boxdot p_j) \vee \bigvee_{k \neq i,j} \Box p_k.$$

We obtain either $\boxdot q \to p_k$ for some k, in which case we are done, or the formula

$$\boxdot q \to p_i \vee \boxdot p_j.$$

As j was arbitrary, we have

$$\boxdot q \to p_i \vee \boxdot \bigwedge_{j \neq i} p_j,$$

and $(*)$ implies $\boxdot q \to p_i$. □

Definition 3.5 Let $\langle U, < \rangle$ be the Kripke frame constructed by the following procedure:

- start with the empty frame,
- whenever X is a finite antichain in U, adjoin to U a reflexive and an irreflexive tight predecessor of X, unless it already has one.

(We remind the reader that reflexive singletons are their own tight predecessors.)

Lemma 3.6 *If a consistent logic L inherits admissible single-conclusion rules of $K4$, then $\langle U, <\rangle$ is an L-frame.*

Proof: $K4$ admits the rule $\varphi \,/\, \bot$ whenever φ is a variable-free formula unprovable in $K4$, thus $K4$ and L have the same variable-free fragment. Let V be the set of all subsets of U definable by a variable-free formula. The general frame $\langle U, <, V\rangle$ is refined, and its dual is the free $K4$-algebra over the empty set of generators (see e.g. [2]), thus $\langle U, <, V\rangle$ is an L-frame. Every rooted generated subframe of $\langle U, <, V\rangle$ is finite and refined, therefore it is a Kripke frame. It follows that all rooted generated subframes of the Kripke frame $\langle U, <\rangle$ are L-frames, thus $\langle U, <\rangle$ itself is also an L-frame. $\square$

Theorem 3.7

(*i*) *If L inherits multiple-conclusion admissible rules of $K4$, then $\Pi^\bullet + \Pi^\circ$ is an independent basis of L-admissible multiple-conclusion rules.*

(*ii*) *If L is a consistent logic inheriting admissible single-conclusion rules of $K4$, then $\pi^\bullet + \pi^\circ$ is an independent basis of L-admissible single-conclusion rules.*

Proof: The given sets of rules are bases by theorem 2.1, and lemma 3.4, it thus suffices to show their independence over L.

Let $n \in \omega$ and $* \in \{\bullet, \circ\}$. Fix an antichain $X = \{x_i;\, i < n\}$ of n irreflexive points in $\langle U, <\rangle$, let t be its irreflexive (if $* = \bullet$) or reflexive (if $* = \circ$) tight predecessor, and define the Kripke frame $F = \{x \in U;\, x \not\leq t\}$. F is a generated subframe of U, thus it validates L by lemma 3.6. Let $m \in \omega$ and $\vartriangle \in \{\bullet, \circ\}$ be such that $m \neq n$ or $\vartriangle \neq *$, we will show that $\Pi^\vartriangle_m$ is valid in F. Let $\Vdash$ be a valuation in F which refutes all conclusions of $\Pi^\vartriangle_m$, and let $y_i \in F$, $i < m$ be such that $y_i \Vdash \boxdot\big(q \wedge \bigwedge_{j \neq i} p_j\big) \wedge \neg p_i$. Then $Y = \{y_i;\, i < m\}$ is an antichain, and $Y \neq X$ or $\vartriangle \neq *$, thus F contains an irreflexive (if $\vartriangle = \bullet$) or a reflexive (if $\vartriangle = \circ$) tight predecessor y of Y. We have $y \nVdash \bigvee_{i<m} \Box p_i$, but $y \Vdash \Box q$ (if $\vartriangle = \bullet$) or $y \Vdash \boxdot(q \equiv \Box q)$ (if $\vartriangle = \circ$), thus $\Vdash$ refutes the assumption of $\Pi^\vartriangle_m$. The modal disjunction property (DP) rule $\Box p \vee \Box q \,/\, p, q$ also holds in F, because it is downwards directed: if $u \nVdash p$ and $v \nVdash q$ for some valuation $\Vdash$ and $u, v \in F$, we can find $w \in F$ such that $w < u, v$, hence $w \nVdash \Box p \vee \Box q$. It follows that the rule $\pi^\vartriangle_m$ is valid in F, as it is derivable from $\Pi^\vartriangle_m$ and DP.

On the other hand, the rules Π^*_n and π^*_n fail in F. Again, the two rules are equivalent over DP, it thus suffices to refute Π^*_n. We define

$$x \Vdash p_i \Leftrightarrow x \neq x_i.$$

If $* = \bullet$, we put

$$x \Vdash q \Leftrightarrow x \in X\!\uparrow.$$

Clearly $x_i \Vdash \boxdot\big(q \wedge \bigwedge_{j \neq i} p_j\big) \wedge \neg p_i$, we claim that $\Box q \to \bigvee_{i<n} \Box p_i$ holds in every $x \in F$. Indeed, if $x \Vdash \Box q$, then every successor of x belongs to $X\!\uparrow$; as x is not an irreflexive tight predecessor of X, we must have $x \not< x_i$ for some $i < n$, thus $x \Vdash \Box p_i$.

If $* = \circ$, we define

$$x \Vdash q \Leftrightarrow x \in X\updownarrow \vee \exists y > x\,(y \neq x \wedge y \notin X\updownarrow).$$

We have $x_i \Vdash \boxdot\big(q \wedge \bigwedge_{j\neq i} p_j\big) \wedge \neg p_i$ as before, we will verify that $(q \equiv \Box q) \to \bigvee_{i<n} \Box p_i$ holds everywhere in F. Assume $x \nVdash \bigvee_{i<n} \Box p_i$, which means $x < x_i$ for every $i < n$. If x is the irreflexive tight predecessor of X, then $x \Vdash \Box q \wedge \neg q$. Otherwise x is neither a reflexive nor an irreflexive tight predecessor of X, thus there exists $y > x$ such that $y \neq x$ and $y \notin X\updownarrow$. If we take y maximal with this property, then $y \nVdash q$. Clearly $x \Vdash q$, thus $x \Vdash q \wedge \neg\Box q$. $\Box$

The proof of theorem 3.7 relied essentially on the fact that the 0-generated canonical frame of $K4$ has infinite width. IPC, GL, and $S4$ lack this convenient property, which will make the proof of theorem 3.1 for their extensions more difficult. We will take GL first.

Lemma 3.8 *Let L be a transitive modal logic which admits $a^\bullet$, and C a canonical L-frame in an arbitrary number of variables. If $X = \{x_i;\, i < n\}$ is a finite subset of C and $x < X$, there exists an irreflexive tight predecessor $t \in C$ of X, and $z \in C$ such that $z < x, t$.*

Proof: Put

$$a = \Big\{\bigwedge_{i<n} \Diamond\varphi_i \wedge \Box\psi;\ \forall i < n\, \varphi_i \wedge \boxdot\psi \in x_i\Big\},$$
$$b = \Diamond x \cup \Diamond a.$$

We claim that b is L-consistent: if not, there exist $\varphi_i \in x_i$, $\chi \in x$, and ψ such that $\boxdot\psi \in \bigcap_{i<n} x_i$, and

$$\vdash_L \Box\Big(\Box\psi \to \bigvee_{i<n} \Box\neg\varphi_i\Big) \vee \Box\neg\chi,$$

because x and a are closed under conjunction. However, we have $x < x_i \nVdash \boxdot\psi \to \neg\varphi_i$ for every $i < n$, hence the formula

$$\bigvee_{i<n} \Box\big(\boxdot\psi \to \neg\varphi_i\big) \vee \neg\chi$$

is refuted in x, contradicting the admissibility of $a^\bullet$ in L.

Pick $z \in C$ such that $z \supseteq b$. We have $z < x$, as $\Diamond x \subseteq z$. Moreover, $z \supseteq \Diamond a$ implies that there exists a $t > z$ such that $t \supseteq a$. We claim that t is a tight predecessor of X. On the one hand, $t \supseteq \Diamond x_i$, thus $t < x_i$. On the other hand, let $u \notin X\updownarrow$. For every $i < n$, there exists a ψ_i such that $\boxdot\psi_i \in x_i$ and $\neg\psi_i \in u$. Put $\psi = \bigvee_{i<n} \psi_i$: we have $\neg\psi \in u$, but $\boxdot\psi \in \bigcap_{i<n} x_i$, thus $\Box\psi \in t$ and $t \nless u$. $\Box$

Remark 3.9 It is not hard to show that the condition in lemma 3.8 holds for any descriptive frame which validates $a^\bullet$, and conversely, any frame (not necessarily descriptive) which satisfies the condition validates $a^\bullet$. Similar considerations also work for lemmas 3.11 and 3.14.

Theorem 3.10

(*i*) $\Pi^\bullet$ *is an independent basis of multiple-conclusion rules admissible in GL.*

(*ii*) *If L is a consistent logic inheriting GL-admissible single-conclusion rules, then either* $\{\pi^\bullet_n;\ n\in\omega\}$ *is an independent basis of L-admissible single-conclusion rules, or* $L = GL.3$.

Proof: We will skip the proof of (*i*), which is similar to (*ii*), but much easier. Assume that $L\supseteq GL$ and L admits $a^\bullet$. Consider first the case when L has width 1, i.e., $L\supseteq GL.3$. As every proper extension of $GL.3$ proves $\Box^n\bot$ for some $n\in\omega$ and GL admits $\Box^n\bot\ /\ \bot$, either $L = GL.3$, or L is inconsistent.

Assume that L has width at least 2. Then $\{\pi^\bullet_n;\ n\in\omega\}$ is a basis of L-admissible rules by theorem 2.1, and lemma 3.4. Fix $n\in\omega$, we will show that $\pi^\bullet_n$ is independent on $L+\{\pi^\bullet_m;\ m\neq n\}$. If $n=0$, the irreflexive singleton is a model of $L+\{\pi^\bullet_n;\ n>0\}$, and refutes $\pi^\bullet_0$. Assume $n\geq 2$ (we will take care of $n=1$ later). Let $\langle C,<,V\rangle$ be the canonical L-frame in ω variables, and $\Vdash_c$ its generating valuation. By assumption, C contains a two-element antichain visible from a single point, and a repeated application of lemma 3.8 shows that C has rooted subframes of arbitrary large width. Let $\{y_i;\ i<n\}$ be an antichain visible from a point x. For any $i\neq j$, $y_j\not\leq y_i$ implies that there exist formulas α^j_i such that $y_j\Vdash_c \boxdot\alpha^j_i$, and $y_i\nVdash_c \alpha^j_i$. Put $\alpha_i := \bigvee_{j\neq i}\alpha^j_i$. Then $y_i\Vdash_c \bigwedge_{j\neq i}\boxdot\alpha_j\wedge\neg\alpha_i$, and as $C\Vdash_c GL$, there exists $x_i\geq y_i$ such that

$$(*)\qquad x_i\Vdash_c \bigwedge_{j\neq i}\boxdot\alpha_j\wedge\Box\alpha_i\wedge\neg\alpha_i.$$

The set $X=\{x_i;\ i<n\}$ is an antichain of n distinct points. Put $F_0 = X{\uparrow}$, and by induction on k, let F_{k+1} consist of F_k, together with all $u\in C$ which are irreflexive tight predecessors of a finite antichain $Y\subseteq F_k$ distinct from X. Put $F=\bigcup_{k\in\omega}F_k$, and define

$$W=\{A\subseteq F;\ \exists B\in V\ A\cap F_0 = B\cap F_0\}.$$

Claim 1 *L is valid in the frame* $\langle F,<,W\rangle$.

Proof: It suffices to show that every rooted generated subframe of $\langle F,<,W\rangle$ validates L. Let $u\in F$, we will show that the subframes of $\langle F,<,W\rangle$ and $\langle C,<,V\rangle$ generated by u coincide. It suffices to prove that every subset of $u{\uparrow}\smallsetminus F_0$ is definable in $u{\uparrow}$. By the construction of F, $u{\uparrow}\smallsetminus F_0$ is finite. For every $i<n$ and $v\in u{\uparrow}\smallsetminus F_0$, there exists a formula $\beta_{i,v}$ such that $x_i\Vdash_c \boxdot\beta_{i,v}$ and $v\Vdash_c\neg\beta_{i,v}$, because $x_i\not\leq v$. We take $\beta=\bigvee_i\bigwedge_v\beta_{i,v}$, and observe that $\neg\beta$ defines $u{\uparrow}\smallsetminus F_0$ in $u{\uparrow}$. As C is refined, we can separate elements of $u{\uparrow}\smallsetminus F_0$ from each other, thus every $v\in u{\uparrow}\smallsetminus F_0$ is definable in $u{\uparrow}$. $\Box$ (Claim 1)

Claim 2 *The rules $\pi^\bullet_m$, $m \neq n$ are valid in F.*

Proof: Let $\Vdash$ be a valuation in F, and $u \in F$ which refutes the conclusion of $\pi^\bullet_m$. We have $u \nVdash r$, and for every $i < m$, there exist $u_i > u$ such that $u_i \Vdash \boxdot\big(q \wedge \bigwedge_{j \neq i} p_j\big) \wedge \neg p_i$. By lemma 3.8, there exist $z, t \in C$ such that t is an irreflexive tight predecessor of $U = \{u_i;\, i < m\}$, and $z < t, u$. As U is an antichain of size $m \neq n$, $t \in F$. We claim that we can choose $z \in F$ as well. We may assume that z is a tight predecessor of $\{u, t\}$ by another application of lemma 3.8. Then $z \in F$ unless $n = 2$, and $\{u, t\} = \{x_0, x_1\}$. In that case we find a tight predecessor t' of $\{x_0\}$, and a tight predecessor z of $\{x_1, t'\}$. By $(*)$, x_0 is irreflexive, thus $t' \notin F_0$, and $t', z \in F$.

The choice of t and z ensures that $t \Vdash \Box q \wedge \bigwedge_{i<m} \neg\Box p_i$, and $z \nVdash \Box r$, thus the assumption of $\pi^\bullet_m$ is refuted in z. $\Box$ (Claim 2)

It remains to refute $\pi^\bullet_n$ in $\langle F, <, W\rangle$. We define a valuation $\Vdash$ on F by

$$\begin{aligned} &u \Vdash p_i \Leftrightarrow u \neq x_i,\\ &u \Vdash q \Leftrightarrow u \in F_0,\\ &u \nVdash r. \end{aligned}$$

The points x_i are definable in F_0 by $(*)$, thus $\Vdash$ is indeed an admissible valuation in $\langle F, <, W\rangle$. The assumption of $\pi^\bullet_n$ is valid in F under $\Vdash$, because $\Box q \wedge \neg \bigvee_{i<n} \Box p_i$ can hold only in an irreflexive tight predecessor of X, which does not exist in F. Clearly $x_i \Vdash \boxdot\big(q \wedge \bigwedge_{j\neq i} p_j\big) \wedge \neg p_i$. By the proof of claim 2, there exist $z \in F$ such that $z < x_i$ for every $i < n$. Then the conclusion of $\pi^\bullet_n$ fails in z.

Finally, let us return to the case $n = 1$. The construction above almost works, the only place where it could fail is in claim 2 when $x_0 = u \leq t$: there is no guarantee that x_0 has a predecessor in F. We avoid this problem as follows. We construct an antichain $X' = \{x'_0, x'_1\}$ as in the case $n = 2$, we find a tight predecessor x_0 of X', and proceed with $X = \{x_0\}$ as before. As x'_0 and x'_1 are definable in $X'\!\updownarrow$ and C is refined, x_0 is definable in F_0. F contains a tight predecessor y of $\{x'_0\}$, and a tight predecessor z of $\{y, x_0\}$, as y and x_0 are incomparable. $\Box$

Lemma 3.11 *Let L be an intermediate logic which admits v, and C a canonical L-frame. If $X = \{x_i;\, i < n\}$ is a finite subset of C and $x \leq X$, there exists a reflexive tight predecessor t of X, and z such that $z \leq x, t$.*

Proof: Put

$$\begin{aligned} a &= \Big\{\varphi \to \psi;\, \varphi \notin \bigcap_i x_i, \psi \in \bigcap_i x_i\Big\},\\ b &= \Big\{\bigvee_i \varphi_i;\, \forall i < n\, \varphi_i \notin x_i\Big\},\\ c &= \Big\{\Big(\Big(\bigvee_i \varphi_i \to \psi\Big) \to \bigvee_i \varphi_i\Big) \vee \chi;\, \psi \in \bigcap_i x_i, \chi \notin x, \forall i < n\, \varphi_i \notin x_i\Big\}. \end{aligned}$$

The sets x and x_i have the disjunction property, and $\bigcap_i x_i$ is closed under conjunction, thus c is closed under disjunction in the following sense: if $\alpha, \alpha' \in c$, where

$$\alpha = \Big(\Big(\bigvee_i \varphi_i \to \psi\Big) \to \bigvee_i \varphi_i\Big) \vee \chi,$$
$$\alpha' = \Big(\Big(\bigvee_i \varphi'_i \to \psi'\Big) \to \bigvee_i \varphi'_i\Big) \vee \chi',$$

then $\vdash \alpha \vee \alpha' \to \beta$ and $\beta \in c$, where

$$\beta = \Big(\Big(\bigvee_i (\varphi_i \vee \varphi'_i) \to \psi \wedge \psi'\Big) \to \bigvee_i (\varphi_i \vee \varphi'_i)\Big) \vee (\chi \vee \chi').$$

If $\alpha \in c$, then α does not hold in C, because $\bigvee_i(\psi \to \varphi_i) \vee \chi$ fails in x, and C validates v'. By Zorn's lemma, there exists a maximal set z such that $z \nvdash \alpha$ for all $\alpha \in c$. The closure of c under disjunction then guarantees that z has the disjunction property, i.e., $z \in C$ and $z \cap c = \emptyset$.

Clearly $z \leq x$. If $\alpha \in b$, then $z \cup a \nvdash \alpha$. Let $t \supseteq z \cup a$ be maximal with this property. As b is closed under disjunction, t has the disjunction property, thus $t \in C$. We have $z \leq t \leq x_i$ for every i. If $t \lneq u$, there exists a formula $\alpha = \bigvee_i \varphi_i \in b$ such that $u \vdash \alpha$ by the maximality of t. Using the disjunction property of u, we obtain $\varphi_j \in u \smallsetminus \bigcap_i x_i$ for some j. As $u \supseteq a$, we have $u \supseteq \bigcap_i x_i$, and the disjunction property of u implies $u \geq x_i$ for some i. Thus t is a tight predecessor of X. □

Theorem 3.12

(*i*) $\{\Pi_n;\, n \neq 1\}$ *is an independent basis of multiple-conclusion rules admissible in IPC.*

(*ii*) *If L inherits IPC-admissible single-conclusion rules, then either $\{\pi_n;\, n \geq 2\}$ is an independent basis of L-admissible single-conclusion rules, or L has width at most 2, and a finite basis.*

Proof: We will only prove (*ii*). Assume that L admits all *IPC*-admissible rules. The rules π_0 and π_1 are derivable in *IPC*, thus $\{\pi_n;\, n \geq 2\}$ is a basis of L-admissible rules by theorem 2.1 and lemma 3.4. If L has width at most 2, it derives all π_n, $n \geq 3$, thus it has a finite basis $\{\pi_2\}$.

Assume that L has width at least 3, we will show that the basis $\{\pi_n;\, n \geq 2\}$ is independent. Fix $n \geq 3$ (we postpone the case $n = 2$), let $\langle C, \leq, V\rangle$ be the canonical L-frame, and $\Vdash_c$ its generating valuation. As in theorem 3.10, lemma 3.11 implies that C has infinite width, thus let $\{y_i;\, i < n\}$ be an antichain visible from a point x. There are formulas α_i such that $y_i \Vdash_c \bigwedge_{j\neq i} \alpha_j$, and $y_i \nVdash_c \alpha_i$. Let $x_i \supseteq y_i$ be a maximal set such that $x_i \nvdash \alpha_i$. Then x_i is disjunctive, thus $x_i \in C$, and $x_i \geq y_i$.

Let X be the antichain $\{x_i;\, i < n\}$, $F_0 = X\!\uparrow$, and we construct F as the closure of F_0 under reflexive tight predecessors of finite antichains distinct from X, as in theorem 3.10. We define

$$W = \{A \subseteq F;\, \exists B \in V\; A \cap F_0 = B \cap F_0\}.$$

An argument similar to theorem 3.10 shows that $\langle F, \le, W\rangle$ is an L-frame, F is downwards directed, and the rules π_m, $m \neq n$ are valid in F. We define a valuation on F by

$$\begin{aligned} &u \Vdash q \Leftrightarrow u \in F_0, \\ &u \Vdash p_i \Leftrightarrow u \not\le x_i, \\ &u \nVdash r. \end{aligned}$$

The valuation $\Vdash$ is admissible in F by our choice of the points x_i. We have $x_i \Vdash q \wedge \bigwedge_{j\neq i} p_j$, and $x_i \nVdash p_i$. As F is directed, there exists $z \in F$ such that $z \le X$, thus $z \nVdash \bigvee_i (q \wedge \bigwedge_{j\neq i} p_j \to p_i)$. Let $u \nVdash \bigvee_i p_i$, we will verify $u \nVdash \bigvee_i p_i \to q$. The formula $\bigvee_i p_i$ is true in F_0, and $u{\uparrow} \smallsetminus F_0$ is finite, thus there exists a maximal $v \ge u$ such that $v \nVdash \bigvee_i p_i$. This means that v is a predecessor of X, and as it cannot be a tight predecessor, there exists a $w \gneq v$ such that $w \notin F_0$. Then $w \Vdash \bigvee_i p_i$ and $w \nVdash q$.

In the case $n = 2$, we use the same construction, but we must avoid the situation that X has no predecessor in F. We find an antichain $\{x_0, x_1', x_2'\}$ as in the case $n = 3$, and we pick a tight predecessor x_1 of $\{x_1', x_2'\}$. Then we proceed as before. Let t be a tight predecessor of $\{x_0, x_1', x_2'\}$, and u a tight predecessor of $\{t, x_1\}$. We have $t \in F$, and $t \neq x_0$, thus $u \in F$ is a predecessor of $\{x_0, x_1\}$. $\square$

Example 3.13 Unlike theorem 3.10, we cannot reduce the width to 1 in theorem 3.12.

- Let L_1 be the logic of the Kripke frame $F_1 = \{a_n, b_n; n \in \omega\}$, where $a_{n+1} < a_n$, $b_{n+1} < b_n$, $a_{n+1} < b_n$, and $b_{n+2} < a_n$. L_1 has width 2, and inherits single-conclusion *IPC*-admissible rules. L_1 coincides with the set of formulas provable in *IPC* under every substitution using only one variable (cf. [2]).
- Let L_2 be the logic of the Kripke frame $F_2 = \{a_n, b_n; n \in \omega\}$, where $a_{n+1} < a_n$, $b_{n+1} < b_n$, $a_{n+1} < b_n$, and $b_{n+1} < a_n$. Again, L_2 has width 2, and inherits single-conclusion *IPC*-admissible rules. In fact, L_2 is the smallest logic (called V in [8]) in which all *IPC*-admissible rules are derivable.

It can be shown that L_1 is the smallest logic of bounded width inheriting admissible rules of *IPC*.

Lemma 3.14 *Let L be a transitive logic which admits a°, and C a canonical L-frame. If $X = \{x_i; i < n\}$ is a finite subset of C and $x < X$, there exists a reflexive tight predecessor t of X, and z such that $z < x, t$.*

Proof: We proceed similarly to lemmas 3.8 and 3.11. For simplicity, we assume that $L \supseteq S4$. Put

$$a = \Big\{ \bigwedge_j (\varphi_j \to \Box\varphi_j) \wedge \bigwedge_{i<n} \neg\Box\psi_i;\ \Box\varphi_j \in \bigcap_i x_i, \psi_i \notin x_i \Big\}.$$

As x and a are closed under conjunction, and C validates a°, the set $\Diamond x \cup \Diamond a$ is consistent, hence there exists a $z \in C$ such that $z \supseteq \Diamond x \cup \Diamond a$. Then $z \le x$, and as a is closed under conjunction, there exists $t \ge z$ such that $t \supseteq a$. We have $t \le x_i$. If $t \lneq u$, but $u \ngeq x_i$ for all i, there are formulas φ_i and ψ such that $\Box\varphi_i \in x_i$, $\neg\varphi_i \in u$, $\psi \in t$, and $\neg\psi \in u$. Put $\chi = \bigvee_i \varphi_i \vee \psi$: then $\Box\chi \in \bigcap_i x_i$ and $\chi \in t$, thus $\Box\chi \in t$, but $\neg\chi \in u$, a contradiction. $\square$

Theorem 3.15

(*i*) *If L inherits multiple-conclusion admissible rules of $S4$, then $\{\Pi^\circ_n;\, n \neq 1\}$ is an independent basis of L-admissible multiple-conclusion rules.*

(*ii*) *If L inherits $S4$-admissible single-conclusion rules, then exactly one of the following cases holds:*

- *$L \nvdash S4.1$, and $\{\pi^\circ_n;\, n \neq 1\}$ is an independent basis of single-conclusion rules admissible in L,*
- *$L \vdash S4.1$, and $\{\pi^\circ_n;\, n \geq 2\}$ is an independent basis,*
- *L has width at most 2, and a finite basis.*

Proof: We will concentrate on (*ii*). If L admits all $S4$-admissible rules, then π° is a basis of L-admissible rules by theorem 2.1 and lemma 3.4. The rule π°_1 is derivable in $S4$. The rule π°_0 is dependent on $\{\pi^\circ_n;\, n > 0\}$ iff $L \vdash S4.1$: on the one hand, π°_0 is derivable in $S4.1$. On the other hand, if $L \nvdash S4.1$, then the two-element cluster is an L-frame (cf. [2]), validates all π°_n, $n > 0$, and refutes π°_0.

Assume that L has width at least 3, let $\langle C, \leq, V\rangle$ be the canonical L-frame, and $n \geq 3$ (the case $n = 2$ can be handled by the same trick as in theorems 3.10 and 3.12). As in the proof of 3.12, there exists an antichain $\{y_i;\, i < n\}$ in C visible from a point x, and formulas α_i such that $y_i \Vdash_c \bigwedge_{j\neq i} \Box\alpha_j \wedge \neg\alpha_i$. Let T_i be a maximal set of formulas such that $\{\psi;\, \Box\psi \in y_i\} \subseteq T_i$ and $\Box T_i \nvdash \alpha_i$, and pick $x_i \in C$ such that $\Box T_i \cup \{\neg\alpha_i\} \subseteq x_i$. Then $x_i \geq y_i$, $x_i \Vdash_c \bigwedge_{j\neq i} \Box\alpha_j \wedge \neg\alpha_i$, and α_i holds in every $u \geq x_i$ such that $u \nleq x_i$.

Put $X = \{x_i;\, i < n\}$, $F_0 = X{\uparrow}$, let F be the closure of F_0 under reflexive tight predecessors of finite antichains Z such that $Z{\uparrow} \neq F_0$, and let $W = \{A \subseteq F;\, \exists B \in V\; A \cap F_0 = B \cap F_0\}$. (The condition $Z{\uparrow} = F_0$ is equivalent to $Z = \{z_i;\, i < n\}$ and $z_i \sim x_i$ for all $i < n$. Such a Z has the same tight predecessors as X.)

As in theorem 3.10 or 3.12, $\langle F, \leq, W\rangle$ is an L-frame, and validates $\{\pi^\circ_m;\, m \neq n\}$. Define a valuation on F by

$$
\begin{aligned}
&u \Vdash q \Leftrightarrow u \in F_0 \vee \exists v \geq u\,(v \neq u \wedge v \notin F_0),\\
&u \Vdash p_i \Leftrightarrow \neg(x_i \leq u \leq x_i),\\
&u \nVdash r.
\end{aligned}
$$

The valuation $\Vdash$ is admissible in F, because the cluster of x_i is definable in F_0 by $\neg\Box\alpha_i$. If z is any predecessor of X in F, then z refutes the conclusion of π°_n. Assume $u \nVdash \bigvee_i \Box p_i$. Then $u \notin F_0$ and u is a predecessor of X. As it is not a tight predecessor, there exists $v \gneq u$ such that $v \notin F_0$, which implies $u \Vdash q$. We may pick v maximal possible, as $u{\uparrow} \smallsetminus F_0$ finite. The cluster of v is simple, thus $v \nVdash q$, and $u \nVdash \Box q$. □

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