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# Log-space Constructible Universal Traversal Sequences for Cycles of Length $O(n^{4.03})$

(Preliminary version)

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**Abstract**

The paper presents a simple construction of polynomial length universal traversal sequences for cycles. These universal traversal sequences are log-space (even $NC^1$) constructible and are of length $O(n^{4.03})$. Our result improves the previously known upper-bound $O(n^{4.76})$ for log-space constructible universal traversal sequences for cycles.

## 1 Introduction

One of the major problems in computer science is the graph $s, t$-connectivity problem. It is well known that a directed version of this problem is complete for nondeterministic log-space ($NL$), and its undirected version is complete for symmetric log-space ($SL$), which is defined to be the class of problems log-space reducible to undirected $s, t$-connectivity. It has been conjectured that symmetric log-space is equal to deterministic log-space ($SL = L$).

One approach to proving that conjecture is to use universal traversal sequences introduced by Cook (see [AKL+].) A traversal sequence for a $d$-regular graph is a sequence of numbers from $\{1, \ldots, d\}$, which directs us in traversing the graph. For given $d$ and $n$, a universal traversal sequence for $d$-regular graphs of size $n$ is a traversal sequence which completely traverses any $d$-regular graph of size $n$ starting at any vertex. [AKL+] gave a probabilistic argument for existence of universal traversal sequences of polynomial length in $d$ and $n$. In a sequence of papers [N], [NSW], [ATWZ] it was shown using derandomization of that argument that $SL \subseteq DSPACE(\log^{4/3} n)$.

A more direct approach to proving $SL = L$ is to construct an explicit log-space constructible universal traversal sequence for $d$-regular graphs, where $d \geq 3$. The steps in this direction are explicit constructions of universal traversal sequences for specific classes of graphs. For 2-regular graphs (cycles), [BBK+], and [B] constructed a universal traversal sequences of length $n^{O(\log n)}$, which were later superseded by log-space constructible[1] universal traversal sequence for cycles of length $O(n^{4.76})$ by Istrail [I]. In this paper we present a simple log-space construction of universal

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[1] Feigenbaum and Reingold [J. Feigenbaum, N. Reingold, *Universal Traversal Sequences*, American Mathematical Monthly, 101 (1994), pp. 262-265] claim without any further argument that Istrail's construction is not log-space constructible. However, it seems to us to be possible to show the log-space constructibility of Istrail's construction by an argument similar to one in Section 4.3.



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traversal sequences for cycles of length $O(n^{4.03})$ (which we believe can be improved up to $O(n^{3.99})$ by setting parameters of the construction optimally.) The currently known best lower bound $\Omega(n^{1.43})$ and non-constructive upper bound $O(n^3)$ on the length of universal traversal sequences for cycles are due to [BT] and [A], respectively (see [BT] for comprehensive overview.) The other studied classes of graphs are a class of complete graphs for which [KPS] presented $n^{O(\log n)}$ universal traversal sequences, and a class of expanders for which [HW] presented $n^{O(d \log d)}$ universal traversal sequences. The best construction of universal traversal sequences for 3-regular graphs is based on Nisan's pseudo-random generator [N]; these sequences are of length $n^{O(\log n)}$.

## 1.1 Overview of The Construction

Our construction has two parts:

1. The first part is a construction of an $n^{O(\log n)}$ universal traversal sequence for cycles of length $n$. This part uses essentially the same idea as a construction of Bridgland [B]. It is a recursive construction in which at every iteration a universal traversal sequence for cycles of length $cn$, $c \geq 2$, is constructed from a universal traversal sequence for cycles of length $n$. Every iteration consists of two dual stages. The depth of the recursion is $O(\log n)$ and the factor by which the generated universal traversal sequence is expanded at each stage is $O(n)$.

2. The second part of the construction involves again two dual stages that are aimed at reducing the expansion factor to $O(1)$, and they interleave the stages of the first part.

At both stages of the first part the traversed graph is reduced to a smaller one, whereas at both stages of the second part the graph is expanded a little bit by inserting new vertices. After applying all four stages the graph is smaller by at least a constant factor.

The main contribution of this paper lies in the construction of the second part where we show that we can modify the traversed graph by inserting new vertices so that this modification is transparent for traversal sequences. This idea could be possibly useful in construction of universal traversal sequences for 3-regular graphs.

The previous construction by Istrail was also based on Bridgland's construction. Istrail's construction also reduced the expansion factor to a constant as our does but using slightly different (ad hoc) approach. Our construction seems more natural and better motivated hence, much simpler to understand.

Our paper is organized as follows. Section 2 contains definitions and preliminaries. Section 3 contains the first part of the construction yielding an $n^{O(\log n)}$ universal traversal sequence. Section 4 contains the second part of the construction including the algorithm. Section 5 contains the analysis of the length of the universal traversal sequence that is produced.

# 2 Preliminaries

Let $G = (V, E)$ be an undirected cycle. Let $V = \{v_1, \ldots, v_n\}$ so that $E = \{(v_i, v_{i+1});\ 1 \leq i < n\} \cup \{(v_1, v_n)\}$. We may look at every edge $(u, v)$ in $G$ as on a pair of two directed edges $(u, v)$ and $(v, u)$. For $1 \leq i < n$, the directed edges $(v_i, v_{i+1})$ and $(v_n, v_1)$ are called the *right edges*, and $(v_{i+1}, v_i)$ and $(v_1, v_n)$ are called the *left edges*. Similarly, $v_{i+1}$ is called a right neighbor of $v_i$, and $v_i$ is called a left neighbor of $v_{i+1}$ ($v_n$ is a left neighbor of $v_1$, and vice versa.) At every vertex in $G$, let one of the outgoing edges be labeled by 1 and the other one by 0. We label every vertex in $G$ according to the label of its right outgoing edge (Fig. 1.)

A 0-1 sequence $b(v_1), b(v_2), \ldots, b(v_n)$ uniquely describes $G$, where $b(v_i)$ is the label of vertex $v_i$. We call such a sequence *a binary representation* of the graph or *a graph sequence*. Two graph sequences are considered to be the same if one of them is a cyclic shift of the other one. Hence, we actually consider any graph sequence to be wrapped around (cyclic). We consider all the graphs with the same binary representation to be the same. We identify every graph sequence with its corresponding graph (actually a class of the same graphs), and every graph with its sequence. We will refer to the digits of a graph sequence as *vertices*. (We will use term vertex also for any digit in a general 0-1 sequence.)

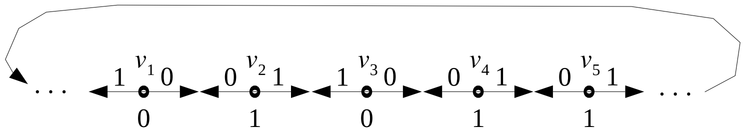


**Figure 1.**

*A traversal sequence* is a 0-1 sequence. "*To run a traversal sequence $t$ on a graph $G$ starting at a vertex $v$*" means to walk in $G$ starting at $v$ and following edges labeled consistently with $t$. For example, if we run a sequence 10100 on $G$ starting at $v_2$ (Fig. 1), we visit vertices $v_2, v_3, v_4, v_5, v_4, v_3$ in this order. Note, if the label of a vertex in the traversal sequence is identical to the label of the traversed vertex then we are going to the right, otherwise to the left. Because we identify a graph with its graph sequence we may also talk about running a traversal sequence on a 0-1 sequence. It should be always clear from the context which sequence is a traversal sequence and which one is a graph sequence.

Note, in traversing a graph sequence by a traversal sequence there are two kinds of symmetry. If we negate both, the traversal sequence and the graph sequence, we will run exactly as before. If we reverse the graph sequence (the first vertex becomes the last, and so on) and then we negate it, any traversal sequence will run exactly in opposite direction than before. (This corresponds to changing left-right orientation of the underlying graph.)

Let $s$ be a 0-1 sequence. A 0-*run* (1-*run*) in $s$ is any maximal part of $s$ consisting only of 0's (1's). The type of the run, 0-run or 1-run, is called *a color* of that run. The leftmost and rightmost vertices of the run are called *border vertices*. The border vertices of a run of length 1 coincide. A 0-1 sequence consisting only of runs of length $i$ and $j$ is $(i, j)$-sequence e.g., 000 111 000 111111 000 is a $(3, 6)$-sequence. A sequence consisting of a single run is called *monochromatic*.

A 0-1 sequence $t$ *completely traverses* $s$ starting at vertex $v$ if during traversal of $s$ by $t$ starting at $v$ we visit all the vertices of $s$. The 0-1 sequence $t$ *strongly traverses* $s$ starting at $v$ if it completely traverses $s$ starting at $v$ and for every vertex there is a visit to that vertex which comes from left and continues to the right, or it comes from right and continues to the left i.e., the vertex is entered using one edge and left using the other one. (In the singular case of a one vertex cycle $s$, $t$ strongly traverses $s$ if $t$ contains at least two consecutive vertices of the same color.) A 0-1 sequence $t$ is a *(strong) universal traversal sequence* (UTS) for cycles of length $n$ if for any cycle $G$ of length $n$, $t$ completely (strongly) traverses $G$ starting at any vertex.

**Proposition 1** *Let $n \geq 2$ be an integer.*

1. *Any (strong) universal traversal sequence for cycles of length $n + 1$ is a (strong) universal traversal sequence for cycles of length $n$.*

2. *Any universal traversal sequence for cycles of length $n + 2$ is a strong universal traversal sequence for cycles of length $n$.*

**Proof:** We leave the proof of the first claim as an easy exercise and we prove the second one. Assume that for some UTS $t$ for cycles of length $n + 2$ there is a graph $G$ on $n$ vertices and a vertex $v$ in $G$ such that $G$ is not strongly traversed by $t$ starting at $v$. By the first part of the proposition, $G$ is traversed completely by $t$ starting at $v$. Thus, there is a vertex $u$ in $G$ which is not strongly traversed i.e., it is always visited in the way that it is left using the same edge by which it was entered. Replace $u$ in graph $G$ by three interconnected vertices labeled identically to $u$. If $u \neq v$ then clearly the middle inserted vertex is never visited by $t$ starting at $v$. If $u = v$ then the middle inserted vertex is never visited by $t$ if $t$ is started at either the rightmost or the leftmost inserted vertex, depending whether the first digit of $t$ has the same color as $u$. Both cases give a contradiction. $\square$

# 3 An $n^{O(\log n)}$ Universal Traversal Sequence

## 3.1 A Parity Contraction

Consider running a 1-run $t$ of length $l$ on a 1-run $r_1$ of length $n \leq l$ starting at a left border vertex of $r_1$. Assume that $r_1$ is followed by another 0-run $r_0$ (111…111000…000). It is easy to see that if the parity of $l$ and $n$ are the same then we end up at the left border vertex of $r_0$, otherwise we end up at the right border vertex of $r_1$. The first $n-1$ digits of $t$ bring us to the right border vertex of $r_1$ and the remaining digits alternate us between the right border vertex of $r_1$ and the first vertex of $r_0$. By symmetry, we get symmetric behavior for running a 0-run on a 0-run starting at the left border vertex, and for running a 1-run on a 0-run and a 0-run on a 1-run starting at the right border vertices. Hence, if the parity of the length of the traversed run corresponds to the parity of the length of the traversal run we end up at the border vertex of the next run, otherwise we end up at the opposite border vertex of the same run.

This motivates the following definition. Let $s$ be a non-monochromatic graph sequence. *A parity contraction* $s_\oplus$ of $s$ is a (1,2)-sequence obtained from $s$ by replacing every run of even length by a run of length 2 of the same color, and by replacing every run of odd length by a run of length 1 of the same color. (Keep in mind that sequence $s$ is cyclic.) For example, …111 000 11 0000 1111 0 11 0…is parity contracted to …1 0 11 00 11 0 11 0….

More formally, let $s$ be the same as $r_1 r_2 \cdots r_k$, where every $r_i$ is a monochromatic run and for every $1 \leq i < n$, $r_i$ has an opposite color than $r_{i+1}$, and $r_1$ and $r_n$ have different colors. For every $1 \leq i \leq n$, let $r'_i$ be a run of the same color as $r_i$, and let the length of $r'_i$ be one if $r_i$ is of an odd length and two otherwise. Then the parity contraction of $s$ is a sequence $r'_1 r'_2 \cdots r'_k$.

There is a natural mapping of border vertices of original runs to border vertices of reduced runs. In the case of odd length runs, both border vertices are mapped to the same vertex.

Observe that for traversing a parity contraction it suffices to consider (1,2)-sequences.

**Proposition 2** *Let $s$ be a non-monochromatic graph sequence of length $l$, and let $s_\oplus$ be its parity contraction. Let $v$ be a border vertex in $s$ and $v_\oplus$ the corresponding border vertex in $s_\oplus$. If a traversal (1,2)-sequence $t_\oplus$ strongly traverses $s_\oplus$ starting at $v_\oplus$ then $t$, obtained from $t_\oplus$ by inflating every run by $l + (l \bmod 2)$ vertices, strongly traverses $s$ starting at $v$.*

**Proof:** First observe that $t$ visits all runs in $s$ in the same order as $t_\oplus$ in $s_\oplus$. This is because after every run in $t$ and $t_\oplus$, respectively, we are in $s$ and $s_\oplus$, respectively, at the corresponding border vertices. The runs in $s_\oplus$ were obtained by removing even number of vertices from runs in $s$, and the runs in $t_\oplus$ could be obtained by removing even number of vertices from runs in $t$. The number of vertices in every run of $t$ guarantees that every run in $s$ is traversed completely starting at a border vertex and going in direction of the run. Because $t_\oplus$ strongly traverses $s_\oplus$, during traversal of $s$ by $t$ we enter at least one border vertex of each run in $s$ and continue to traverse the whole run. During that traversal we strongly traverse all the vertices of that run. $\square$

### 3.2 A Pair Contraction

Let $s$ be a (1,2)-sequence. We call a 0-run (1-run) of length 2 in $s$ a 00-*pair* (11-*pair*). A 0-run and a 1-run of length 1 are *singletons* and any maximal part of $s$ consisting of singletons is 01-*run*. (A 01-run does not necessary start by 0 and end by 1.) Let us consider a graph sequence $s$ containing a 01-run followed by a 11-pair and another 01-run: 01010 11 0101010101 ..., and let us consider a traversal sequence $t = $ 01010101010 11 01010101.... Starting $t$ at the first vertex of $s$, we reach the left vertex $v_L$ of 11-pair after 5 digits of $t$. The next 1 in $t$ brings us to the right vertex $v_R$ of the 11-pair in $s$, the following 0 in $t$ takes us back to the vertex $v_L$, and so on. When 11-pair in $t$ comes we get to the vertex following the 11-pair in $s$ and we continue to run to the right.

If we would use $t$ containing a 00-pair instead of the 11-pair, $t = $ 0101010101 00 101010101..., again we would first reach the 11-pair in $s$, but then we would bounce on that 11-pair using the 00-pair in $t$, and we would go back to the left. By symmetry, the two cases of running $t$ from the right have the similar behavior.

Thus, if $s$ is a graph (1,2)-sequence and we traverse $s$ by a (1,2)-sequence $t$ then whenever we reach a pair in $s$ during that traversal we stay at that pair until a pair appears in $t$ and if the color of the pairs in $t$ and $s$ is the same then after that pair in $t$, the traversal goes to the right neighbor of the pair in $s$ otherwise to the left one. Even if several pairs are next to each other in $s$ not separated by any 01-run, this behavior of the traversal can be observed on each of these pairs.

This motivates the following construction. Let $s$ be a (1,2)-sequence containing at least one pair. *A pair contraction* $s_c$ of $s$ is a 0-1 sequence obtained from $s$ by first removing all singletons, and then replacing every pair by one vertex of the same color e.g., 01010 11 010 11 01 00 10... is contracted to 110.... Note, the length of $s_c$ is at most half of the length of $s$.

**Proposition 3** *Let $s_c$ be a pair contraction of a (1,2)-sequence $s$ of length $l$. Let $t$ be a traversal sequence obtained from a strong traversal sequence $t_c$ of $s_c$ starting at $v_c$ by replacing every* 1 *by* $011(01)^{\lceil l/2 \rceil}$*, and every* 0 *by* $001(01)^{\lceil l/2 \rceil}$*. Then $t$ is a strong traversal sequence for $s$ starting at the left vertex of a pair corresponding to $v_c$, if $v_c$ is colored* 0*, and at the right vertex of that pair otherwise.*

**Proof:** Because any 01-run in $s$ must be shorter than $l$, the 01-runs between consecutive pairs in $t$ always bring us from one pair in $s$ to another. It is easy to see that $t$ visits pairs in $s$ in the same order as $t_c$ visits corresponding vertices in $s_c$. Thus, all pairs in $s$ get visited by $t$ started at the vertex specified in the proposition. Because $t_c$ strongly traverses $s_c$, every 01-run in $s$ must be entirely traversed either from left or from right (strong traversal implies traversal of all edges in $s_c$). Thus $t$ is a complete traversal sequence of $s$ starting at the appropriate vertex. Observe that it is actually a strong traversal sequence. □

### 3.3 Putting Them Together

Propositions 2 and 3 give us a way to construct a universal traversal sequence of length $n^{O(\log n)}$. Having constructed a strong universal traversal sequence $t_n$ for cycles of length $n$, we may construct a strong UTS $t_{2n}$ for cycles of length $2n$ using a construction of Proposition 3 and then construction of Proposition 2 as follows.

We first replace every 1 in $t_n$ by $011(01)^n$ and every 0 in $t_n$ by $001(01)^n$ as in Proposition 3 to get $t_{2n}^{-3}$. Then we prepend $(01)^n$ in front of $t_{2n}^{-3}$ to get $t_{2n}^{-2}$, which is a (1,2)-sequence. Now, we expand every run in $t_{2n}^{-2}$ by $2n = 2n + (2n \bmod 2)$ vertices as in Proposition 2 to get $t_{2n}^{-1}$. Finally, we prepend $1^{2n}$ in front of $t_{2n}^{-1}$ to get $t_{2n}$. We call the prepended sequences $(01)^n$ and $1^{2n}$ *loaders*.

We claim that $t_{2n}$ is a strong UTS for cycles of length $2n$. Let $s$ be a graph sequence of length $2n$, let $s_\oplus$ be its parity contraction (assuming $s$ is not monochromatic), and let $s_c$ be the pair contraction of $s_\oplus$ (assuming $s_\oplus$ contains at least one pair.) Clearly, $|s_c| \leq n$, so $t_n$ strongly traverses $s_c$ starting at any vertex.

Observe that $t_{2n}^{-2}$ strongly traverses $s_\oplus$ starting at any vertex. That is because during traversal of $s_\oplus$ by $t_{2n}^{-2}$, loader $(01)^n$ brings us to the pair "closest" to the starting vertex in $s_\oplus$, and then by Proposition 3 applied on $s_\oplus$, $s_c$, $t_{2n}^{-3}$, and $t_n$, sequence $s_\oplus$ is strongly traversed by $t_{2n}^{-3}$. (If there is no pair in $s_\oplus$ then the loader of $t_{2n}^{-2}$ strongly traverses $s_\oplus$ by itself.)

Similarly, $t_{2n}$ is a strong traversal sequence for $s$ starting at any vertex. During a traversal of $s$ by $t_{2n}$ starting at any vertex loader $1^{2n}$ brings us to a border vertex "closest" to the starting vertex in $s$ (or strongly traverses $s$), and then by Proposition 2 applied on $s$, $s_\oplus$, $t_{2n}^{-1}$ and $t_{2n}^{-2}$, sequence $s$ is strongly traversed by $t_{2n}^{-1}$. Hence, $t_{2n}$ is a strong UTS for cycles of length $2n$.

We need a strong UTS for cycles of length (let's say) 5 to start this recursive construction of UTS's. Such a sequence can be obtained from any UTS for cycles of length 7 by Proposition 1.

Essentially the same construction of $n^{O(\log n)}$ UTS appears in [B].

# 4 An $O(n^c)$ Universal Traversal Sequence

A disadvantage of the previous construction is the super-polynomial length of the produced UTS. The reason for this length is a pessimism of the construction in two directions. The construction is pessimistic about the length of runs at each step, and about the number of rounds the construction has to be repeated.

A possible way how to reduce the overall length of the resulting UTS is to protect us against appearance of long runs. The way how we achieve this is by introducing two new stages in the construction which break down long runs. One of these stages takes place before parity contraction and the other one before pair contraction. The former stage reduces the length of 0-runs and 1-runs to a constant length, the latter one reduces the length of 01-runs to a constant length. A consequence of that run length reduction is an expansion of a traversal sequence in Propositions 2 and 3 just by a constant factor.

Both run breaking stages split long runs by inserting extra vertices in them. This actually increases the length of the graph sequence but in the consecutive contraction stages these increases are eliminated. The insertion is done in a way which ensures that any generated traversal sequence behaves on the new graph sequence and on the old one in the same way, i.e., the insertion seems to be transparent for the traversal sequence.

## 4.1 1-run and 0-run Breaking

Let us first explicitly define the property of sequences which we implicitly used in Propositions 2 and 3. Let $\mathcal{T}$ be a class of 0-1 sequences, and let $\sqsubseteq$ be a transitive and reflexive binary relation on $\mathcal{T}$. We say that $\sqsubseteq$ is *a prefix relation* if $\forall x, y \in \mathcal{T}$, $x \sqsubseteq y$ implies $x$ is a prefix of $y$. E.g., let $\mathcal{T}_{3,6}$ denote the class of all (3,6)-sequences. We define a prefix relation $\sqsubseteq$ on $\mathcal{T}_{3,6}$ in the following way: $\forall x, y \in \mathcal{T}_{3,6}$, $x \sqsubseteq y$ iff $x$ is a prefix of $y$ which can be obtained from $y$ by removing several (or possibly none) of the last runs in $y$. (Hence, $000 \sqsubseteq 000\ 111111$, but $000\ 111 \not\sqsubseteq 000\ 111111$.) For a prefix relation $\sqsubseteq$ we naturally define a relation $\sqsubset$ by ($x \sqsubset y \Leftrightarrow x \sqsubseteq y$ & $x \neq y$).

**Definition 4** *Let $s$ and $s'$ be graph sequences. Let sequences $r$ and $r'$ be parts of $s$ and $s'$, respectively. Let $v$ and $v'$ be vertices in $r$ and $r'$, respectively. Let $v_l$ and $v_r$ ($v'_l$ and $v'_r$) be vertices in $s$ ($s'$). (Vertices $v$ and $v'$ are called input vertices and the other vertices are called output vertices. Think about them as in Fig. 2.) Let $\mathcal{T}$ be a class of traversal sequences, and $\sqsubseteq_{\mathcal{T}}$ be a prefix relation on $\mathcal{T}$. Let, for every $t \in \mathcal{T}$, $v_t$ ($v'_t$) denote a vertex reached by running $t$ on $s$ ($s'$) starting at $v$ ($v'$). We say that $t \in \mathcal{T}$ behaves the same on $r$ and $r'$ with respect to input and output vertices if:*

1. $\forall t' \sqsubseteq_{\mathcal{T}} t$, $v_{t'}$ *is in* $r$ *and* $v'_{t'}$ *is in* $r'$, *or*

2. $\exists t_0 \sqsubseteq_{\mathcal{T}} t$ *such that* ($v_{t_0} = v_l$ *and* $v'_{t_0} = v'_l$) *or* ($v_{t_0} = v_r$ *and* $v'_{t_0} = v'_r$), *and* $\forall t' \sqsubset_{\mathcal{T}} t_0$, $v_{t'}$ *is in* $r$ *and* $v'_{t'}$ *is in* $r'$.

The rough meaning of this definition is that $t$ behaves the same on $r$ and $r'$ if $t$ keeps us in $r$ as long as it keeps us in $r'$ and then it leaves $r$ and $r'$ at the same time on corresponding vertices. As soon as $t$ brings us outside of $r$ and $r'$ we do not worry about it anymore. In that, we are not looking at positions in $s$ and $s'$ after every single digit of $t$ but rather after a bigger units given by $\sqsubseteq$.

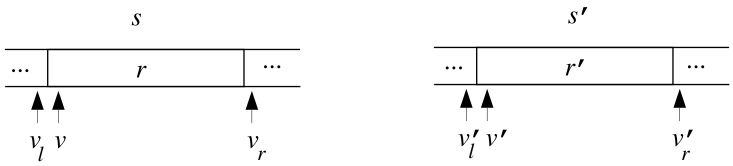


**Figure 2.**

The following proposition describes behavior of (3,6)-sequences on short runs. The proof of that proposition is just an easy case by case analysis, and we leave it as an exercise.

**Proposition 5** *1. Any (3,6)-sequence started at left border vertices or right border vertices behaves the same on runs* 11, 1111 *and* 111111, *where the output vertices are vertices of the other color that are adjacent to these runs in graph sequences containing these runs. The same holds for* 00, 0000 *and* 000000.

*2. Any (3,6)-sequence started at left border vertices or right border vertices behaves the same on runs* 1 *and* 111, *where output vertices are as above. The same holds for* 0 *and* 000.

We are going to use the following simple proposition for $m$ equal 3 and 5.

**Proposition 6** *Let $m \geq 3$ be an odd integer. Then for any $l > m$ there is a $k$ and even $d$ such that $2 \leq d \leq 2m$ and $l = mk + d$.*

Let $s$ be a graph sequence. We replace every run of length five and every run of length longer than six in $s$ by another sequence to obtain a graph $s'$ with all runs of short length. The replacement is done in the following way. Let $r$ be a 0-run or 1-run of length $l = 3k + d \in \{5, 7, 8, 9, \ldots\}$ in $s$, where $k$ and $d$ are from the previous proposition. If $r$ is a 0-run then we replace it by $r' = (000\ 01)^k 0^d$ otherwise $r$ is a 1-run and we replace it by $r' = (111\ 10)^k 1^d$. We call this operation *0-run and 1-run breaking* (or simply *1-run breaking*.) We refer to $r$ as to an *original* sequence and to $r'$ as to a *stuffed* sequence.

We want to show that that any (3,6)-sequence that (strongly) completely traverses $s$ starting at a border vertex of some run (strongly) completely traverses also $s'$ starting at the corresponding border vertex. To show that we will show that any (3,6)-sequence traverses $s$ and $s'$ in a similar way. In particular, we will show that any (3,6)-sequence enters any run in $s$ at the same time as it enters the corresponding stuffed sequence in $s'$, and also it leaves them at the same time at corresponding locations. We will use the following lemma to show that.

**Lemma 7** *Let $s'$ be obtained from a non-monochromatic graph sequence $s$ by 1-run breaking. Let $r$ be a run in $s$ and $r'$ its corresponding (stuffed) sequence in $s'$. Let $v$ be the leftmost vertex of $r$, and let $v'$ be the leftmost vertex of $r'$. Let $v_l$ ($v'_l$) be the left neighboring vertex of $v$ ($v'$), and let $v_r$ ($v'_r$) be the vertex to the right of the rightmost vertex of $r$ ($r'$) (Fig. 3.) Then every $t \in \mathcal{T}_{3,6}$ behaves the same on $r$ and $r'$ with respect to vertices $v, v_l, v_r$ and $v', v'_l, v'_r$.*

Notice that in the case of graph $s$ containing only two runs $v_l$ may actually coincide with $v_r$. To establish this lemma we'll show a correspondence between traversals of sequences $r$ and $r'$ by $t \in \mathcal{T}_{3,6}$.

**Proof:** Let the sequences and vertices be as in the statement of the lemma and let $l = |r|$. If $l \leq 4$ or $l = 6$ then $r = r'$ and the lemma is trivial. So let us assume $l = 5$ or $l \geq 7$ and assume $r$ is a 1-run (the case of a 0-run would be similar, just interchange 0's and 1's in the proof). We are going to number vertices in $r$ from left to right by $\{0, \ldots, l-1\}$ and to refer to vertices in $r'$ as in Figure 3 i.e., there are vertices labeled $0_R, 1_L, 1_M, 1_R, 2_L, \ldots, k_M, k_R, (k+1)_L$ in $r'$. (Figure 3 explicitly shows the right end of $r'$ for $d = 2, 4, 6$.)

If the first run of $t$ is a 0-run then after running this run on $r$ and $r'$ we either end up at $v_l$ and $v_l'$, respectively, or back at $v$ and $v'$, respectively. In the former case the lemma is proven for $t$. Thus, for the rest of the proof assume that $t$ starts with 1-run (which actually will always be the case in application of this lemma).

For any $\tau \in \mathcal{T}_{3,6}$, define $o_0^\tau, o_1^\tau, e_0^\tau, e_1^\tau$ to be the number of odd 0-runs, odd 1-runs, even 0-runs, and even 1-runs in $\tau$, respectively. Further, define $p(\tau) = (o_1^\tau + 2e_1^\tau) - (o_0^\tau + 2e_0^\tau)$. (Note, $3p(\tau)$ is a difference between the number of ones and zeroes in $\tau$.) Further, let $[\tau]$ denote the last run in $\tau$, let $\tau - 1$ denote a sequence obtained from $\tau$ by removing $[\tau]$, and let $l(t)$ denote the number of runs in $\tau$.

Lemma 7 is a consequence of the following lemmas which capture a correspondence between traversal of $r$ and $r'$ by $t$. These lemmas essentially say that during the traversal of $r$ and $r'$, we are at vertex $3i$ in $r$ iff we are at vertex $i_L$, $i_R$, or $(i \pm 1)_M$ in $r'$.

**Lemma 8** *Let $t \in \mathcal{T}_{3,6}$ be such that $t = \epsilon$ or $t$ starts with a 1-run. If $\forall t' \sqsubseteq t;\ 0 \leq p(t') \leq k+1$ then the following hold:*

1. *$\forall t' \sqsubseteq t$, if $p(t') = 0$ then $[t']$ is a 0-run, and if $p(t') = k+1$ then $[t']$ is a 1-run,*
2. *if $d = 4, 6$ then we are at a vertex numbered $3p(t)$ in $r$ after running $t$ on $r$ starting at $v$,*
3. *if $d = 2$ we are at vertex $3p(t) - \delta(t)$ in $r$ after running $t$ on $r$ starting at $v$, where $\delta(t) \in \{0, 2\}$, and $\delta(t) = 2$ iff ($\exists t' \sqsubseteq t;\ p(t') = k+1$ and $\forall t''$ such that $t' \sqsubseteq t'' \sqsubseteq t$, $p(t'') > 0$).*

**Proof:** Let us prove the first part. Clearly, $\forall \epsilon \neq t' \sqsubseteq t$, $p(t') \neq p(t'-1)$. By assumption, $\forall t' \sqsubseteq t$, $0 \leq p(t') \leq k+1$, so $p(t') = 0$ implies $p(t'-1) > 0$, hence $[t']$ has to be a 0-run. Similarly, $p(t') = k+1$ implies $p(t'-1) < k+1$, so $[t']$ has to be a 1-run.

For the second part, if $d = 4, 6$ then $3(k+1) \leq l-1$, and given that $0 \leq p(t') \leq k+1$, for all $t' \sqsubseteq t$, we clearly are at vertex numbered $3p(t)$ in $r$ after running $t$ starting at $v$ ($v$ is numbered by 0.)

Let us consider the last part i.e., $d = 2$. We prove the claim by induction on the number $c(t)$ of $t' \sqsubseteq t$ such that $p(t') = 0$ or $k+1$. For $c(t) = 1$ the claim is obvious because $p(\epsilon) = 0$, so $\forall \epsilon \neq t' \sqsubseteq t$, $0 < p(t') < k+1$ and $\delta(t) = 0$, thus $0 < 3p(t') - \delta(t') < l-1$.

For $m > 1$, assume that the claim holds for any $t \in \mathcal{T}_{3,6}$ with $c(t) < m$, and let us consider $t \in \mathcal{T}_{3,6}$ such that $c(t) = m$. There is a $t' \sqsubseteq t$ such that $c(t'-1) = m-1$ and $c(t') = m$. Clearly, $p(t') = 0$ or $k+1$. Denote $t'-1$ by $t''$.

Consider the case where $p(t') = k+1$. Note, $\delta(t') = 2$. By the above argument $p(t'') < k+1$, and also $p(t'') \geq k-1$, hence $p(t'')$ is $k-1$ or $k$, depending on whether $[t']$ are 6 ones or 3 ones. If $\delta(t'') = 2$ then we are at vertex $3(k-1) - 2$ (if $[t'] = 1^6$) or $3k - 2$ (if $[t'] = 1^3$) in $r$ after running $t''$ starting at $v$, and we are at vertex $3(k+1) - 2 = 3p(t') - \delta(t')$ after running the following $[t']$.

If $\delta(t'') = 0$ then we are at vertex $3(k-1)$ or $3k$, resp., in $r$ after running $t''$ starting at $v$. We are going to run $[t']$, i.e., 6 or 3 ones, resp., from that vertex. In the case of the last run of $t'$ being 6 ones, after the first five out of them we reach a vertex $v_r$, and by the last one we return to vertex $l-1$ of $r$. The similar thing happens for $[t']$ being 3 ones. Hence, after running $t'$ on $r$ starting at $v$ we are at vertex numbered $l-1 = 3p(t') - \delta(t')$ in $r$.

For every $t_i \in T$ such that $t' \sqsubset t_i \sqsubseteq t$, $0 < p(t_i) < k+1$, hence $0 < 3p(t_i) - \delta(t_i) < l-1$, and we are at vertex $3p(t_i) - \delta(t_i)$ after running $t_i$ starting at $v$. This is true in particular for $t_i = t$.

Now consider $p(t') = 0$. Note, $\delta(t') = 0$. By similar argument as the previous one, after running $t''$ starting at $v$ we are at vertex $3 \cdot 1 - \delta(t'')$ (if $[t'] = 0^3$) or $3 \cdot 2 - \delta(t'')$ (if $[t'] = 0^6$.) If $\delta(t'') = 0$ then after running $[t']$ from the vertex to which we get by $t''$ we are at vertex numbered 0 in $r$. If $\delta(t'') = 2$ then after running $t''$ we are at vertex 1 or 4, resp. By running $[t']$ and bouncing on vertex $v_l$, we get to vertex numbered 0 in $r$. Note, $3p(t') - \delta(t') = 0$. By the same argument as before, any $t' \sqsubset t_i \sqsubseteq t$ brings us to vertex numbered $3p(t_i) - \delta(t_i)$ in $r$. In particular, $t$ brings us to vertex $3p(t) - \delta(t)$ starting at $v$. □

By the definition of to behave the same, to establish Lemma 7 we just need to consider $t$ such that, for all $t' \sqsubset t$, after running $t'$ on $r$ and $r'$, respectively, we are at vertices inside of $r$ and $r'$, respectively.

**Lemma 9** *Let $\epsilon \neq t \in \mathcal{T}_{3,6}$ start with a 1-run. The following are equivalent:*

1. *$\forall t' \sqsubset t$, we are at a vertex belonging to $r$ after running $t'$ on $r$ starting at $v$, and we are at a vertex outside of $r$ after running $t$ on $r$,*
2. *$\forall t' \sqsubset t$; $0 \leq p(t') \leq k+1$ and (($p(t-1) = 1$ and $[t]$ is an even 0-run), or ($p(t-1) = k$ and $[t]$ is an even 1-run.))*

**Proof:** By the previous lemma, (2) ⇒ (1). Let us prove (1) ⇒ (2). If $t$ brings us outside of $r$ then, by the previous lemma, $\exists t_m \sqsubseteq t$ such that $p(t_m) < 0$ or $p(t_m) > k+1$. Take such shortest $t_m$ i.e., $t_m$ such that $\forall t' \sqsubset t_m$; $0 \leq p(t') \leq k+1$. The previous lemma applies for $t_m - 1$.

If $p(t_m) < 0$ then $p(t_m - 1) \leq 1$ and $[t_m]$ has to be a 0-run, hence $[t_m - 1]$ is a 1-run. Thus $p(t_m - 1) \neq 0$ (by the previous lemma), so $p(t_m - 1) = 1$ and $[t_m]$ is an even 0-run. By the previous lemma, $t_m$ brings us to vertex 3 or $3 - \delta(t_m)$. In both cases $t_m$ brings us to vertex $v_l$.

If $p(t_m) > k+1$ then $[t_m]$ is a 1-run, $[t_m - 1]$ is an 0-run, thus $p(t_m - 1) = k$, so $[t_m]$ is an even 1-run, and $t_m$ brings us to vertex $v_r$. Hence by the opposite direction of this lemma applied on $t_m$, $t_m = t$, and the lemma is proven. □

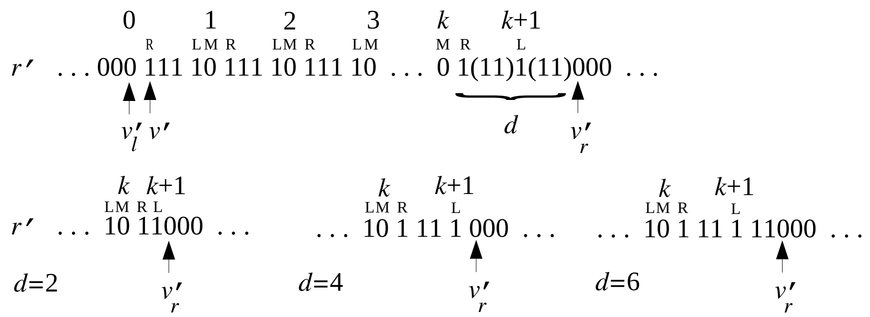


**Figure 3.**

Let us now prove similar two lemmas for traversing $r'$.

**Lemma 10** *Let $\epsilon \neq t \in \mathcal{T}_{3,6}$ start with a 1-run. If $\forall t' \sqsubseteq t$; $0 \leq p(t') \leq k+1$ then by running $t$ on $r'$ starting at $v$:*

- *we get to vertex $(p(t))_L$ iff $[t]$ is an odd 1-run,*

- *we get to vertex $(p(t))_R$ iff $[t]$ is an odd 0-run,*
- *we get to vertex $(i)_M$ iff ($i = p(t) - 1$ and $[t]$ is an even 1-run) or ($i = p(t) + 1$ and $[t]$ is an even 0-run). Furthermore, $0 < i < k + 1$.*

**Proof:** The proof is by induction on $l(t)$. We can easily verify that claim is true for any $t$ such that $l(t) = 1$. For $m > 1$, let us assume that claim is true for any $t$, such that $l(t) < m$, and let us prove it for $t$ such that $l(t) = m$. Let $t' = t - 1$. By induction hypothesis the lemma is true for $t'$.

If after running $t'$ on $r'$ starting at $v'$ we are at $(p(t'))_L$ then $[t]$ has to be a 0-run. If $[t]$ is an odd 0-run then we get to $(p(t') - 1)_R$ by running $t$ starting at $v'$, if $[t]$ is an even 0-run we get to $(p(t') - 1)_M$ by running $t$ starting at $v$. In the former case $p(t) = p(t') - 1$, in the latter one $p(t) = p(t') - 2$ and $(i)_M = (p(t') - 1)_M = (p(t) + 1)_M$, hence $0 < i < k + 1$.

If after running $t'$ we are at $(p(t'))_R$ then $[t]$ has to be a 1-run, and if it is an odd run then we get to $(p(t))_L$ by running $t$ on $r'$, otherwise we get to $(p(t) - 1)_M = (p(t') + 1)_M = (i)_M$ and $0 < i < k + 1$.

If after $t'$ we are at $(i)_M$ then either $i = p(t') - 1$ and $[t]$ is a 0-run, or $i = p(t') + 1$ and $[t]$ is a 1-run. If $[t]$ is an even 0-run then $p(t) = p(t') - 2$ and we are at $(i)_M = (p(t) + 1)_M$ after $t$ ($0 < i < k + 1$,) if $[t]$ is an odd 0-run then $p(t) = p(t') - 1$ and we are at $(i)_R = (p(t))_R$. If $[t]$ is an even 1-run we are at $(p(t) - 1)_M = (i)_M$ after running $t$ and $0 < i < k + 1$, otherwise we are at $(p(t))_L$.

Notice that there are no other possibilities. That establishes the lemma. □

**Lemma 11** *Let $\epsilon \neq t \in \mathcal{T}_{3,6}$ start with a 1-run. The following are equivalent:*

1. *$\forall t' \sqsubset t$, we are at a vertex belonging to $r'$ after running $t'$ on $r'$ starting at $v'$, and we are at a vertex outside of $r'$ after running $t$ on $r'$,*
2. *$\forall t' \sqsubset t$; $0 \leq p(t') \leq k + 1$ and (($p(t - 1) = 1$ and $[t]$ is an even 0-run), or ($p(t - 1) = k$ and $[t]$ is an even 1-run.))*

**Proof:** Again, (2) ⇒ (1) follows from the previous lemma. (Note, if $p(t - 1) = 1$ and $[t]$ is an 0-run then $[t - 1]$ can be neither an even 1-run nor a 0-run. The former case would imply that $p((t - 1) - 1) < 0$, the latter one that $[t]$ is a 1-run. Hence, after running $t - 1$ we are at $1_L$. Similarly, if $p(t - 1) = k$ and $[t]$ is an even 1-run then we are at $k_R$.)

(1) ⇒ (2) follows by a similar argument as lemma 9. Let $t_m \sqsubseteq t$ be such that $p(t_m) < 0$ or $p(t_m) > k + 1$, and $\forall t' \sqsubset t_m$; $0 \leq p(t') \leq k + 1$. (Such $t_m$ must exists by the previous lemma.)

If $p(t_m) < 0$ then $p(t_m - 1) \leq 1$, $[t_m]$ is a 0-run, $[t_m - 1]$ is a 1-run, hence $p(t_m - 1) = 1$, $[t_m - 1]$ must be an odd 1-run and $[t_m]$ must be an even 0-run. Thus, $t_m$ brings us to vertex $v'_l$.

Similarly, if $p(t_m) > k + 1$ then $[t_m]$ is a 1-run, $p(t_m - 1) = k$, $[t_m - 1]$ is an odd 0-run and $t_m$ is an even 1-run. Thus, $t_m$ brings us to vertex $v'_r$. Therefore by the opposite direction of this lemma applied on $t_m$, $t_m = t$, and the lemma is proven. □

To conclude Lemma 7 notice, that $p(t)$ depends only on $t$, but neither on $r$ nor on $r'$. Hence, Lemma 7 follows from Lemmas 9 and 11. □

Further, we can establish an equivalent of Lemma 7 for traversal from right, i.e., claim that any (3,6)-sequence $t$ behaves the same on original sequence $r$ and stuffed sequence $r'$ starting at their rightmost vertices (instead of the leftmost ones), where output vertices are as in Lemma 7.

The proof of such a claim would be entirely symmetric to the proof of Lemma 7 except for the fact that the rightmost run of the stuffed sequence has length 2, 4, or 6 depending on $d$. (Proposition 5 tells us that this really does not make any difference for traversal by (3,6)-sequences.)

Vertices in $r$ would be numbered by $\{0, \ldots, l - 1\}$ from the right, and labels of vertices in $r'$ would be interchanged in the following way: $(k+1)_L \leftrightarrow 0_R$, $(i)_L \leftrightarrow (k+1-i)_R$, $(i)_R \leftrightarrow (k+1-i)_L$,

$(i)_M \leftrightarrow (k+1-i)_M$, for $1 \le i \le k$ (Fig. 4.) Then, Lemmas 8, 9, and 11 would be exactly the same. The only difference would be in Lemma 10 for the case of $d = 6$. Because vertex $v'$ is not labeled by $0_R$, we would need to assume w.l.o.g. that the first run of $t$ is an even run to get to a labeled vertex. (The case of $t$ beginning with an odd run can be reduced to the case of $t$ beginning with an even run.)

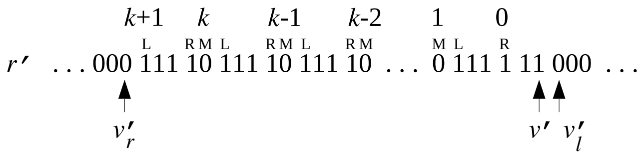


**Figure 4.**

We may conclude that any (3,6)-sequence behaves the same on the original and stuffed sequences if started at the first or last vertices and with respect to output vertices being the neighboring ones.

Let us state a lemma which relates traversal of $s$ and $s'$ obtained from $s$ by 1-run breaking.

**Lemma 12** *Let $s'$ be obtained from a non-monochromatic graph sequence $s$ by 1-run breaking. Let $v$ be a border vertex of some run in $s$ and let $v'$ be its corresponding vertex in $s'$. Let (3,6)-sequence $t$ strongly traverse $s'$ starting at $v'$. Then $t$, starting at $v$, completely traverses $s$, and all vertices in $s$ except possibly $v$ are strongly traversed.*

**Proof:** Because $t$ behaves the same on original runs in $s$ as on the corresponding stuffed runs in $s'$, $t$ traverses all runs in $s$ in the same order as it traverses the corresponding stuffed sequences in $s'$. Thus, we only need to argue that for any run $r$ in $s$ and its corresponding stuffed sequence $r'$ in $s'$, if all vertices in $r'$ are strongly traversed during traversal of $s'$ by $t$, then all vertices in $r$ but $v$ are strongly traversed during traversal of $s$ by $t$, too.

If $r = r'$ then there is nothing to prove. (Note, bouncing at a vertex during traversal of a neighboring stuffed sequence does not constitute a complete traversal of that vertex, so there must be some run in $t$ which starts its running in $r$ and $r'$, respectively.) Let us assume $r \neq r'$ and let $|r| = l = 3k + d$, for $d \in \{2, 4, 6\}$.

If there is a (3,6)-subsequence[2] $t'$ of $t$ which during traversal of $s'$ by $t$ starts to run on $r'$ at the leftmost vertex of $r'$ and which leaves $r'$ at the right neighboring vertex of $r'$, or which runs on $r'$ starting at the rightmost vertex and which leaves $r'$ at the left neighboring vertex of $r'$, then, by Lemmas 8 and 10, during traversal of $s$ by $t$, subsequence $t'$ traverses $r$ in exactly the same manner i.e., it starts in $r$ at one side and leaves $r$ on the other side. Clearly, all the vertices of $r$ are strongly traversed during traversal by $t'$. (The possible exception is if $t'$ is a prefix of $t$ and $t'$ starts running at $v$ then $v$ may not be strongly traversed by $t'$.)

If such a (3,6)-subsequence $t'$ does not exist, let $t_l$ be a (3,6)-subsequence of $t$ which during traversal of $s'$ by $t$ starts its traversal of $r'$ at the leftmost vertex of $r'$ and traverses $r'$ furthest to the right among all such subsequences. Similarly, let $t_r$ be a (3,6)-subsequence which traverses $r'$ from right furthest to the left. It is not obvious that $t_r$ and $t_l$ exist. We will show that $t_l$ always exists, and that $t_r$ exists if $d \neq 2$.

Suppose that $t_l$ does not exist. Vertex $0_R$ in $r'$ has to be strongly traversed, hence it has to be strongly traversed from the right. That means, in $t_r$ there must be a run of length 6 running from $1_L$. But that run ends outside of $r'$, thus, $t_r$ has a property of sequence $t'$ which is a contradiction. Therefore $t_l$ always exists. For $d \in \{4, 6\}$, $t_r$ has to exist by similar argument. (If $d = 2$, the

[2] $t'$ is obtained from $t$ by removing complete runs from the beginning of $t$ and from the end of $t$.

argument for existence of $t_r$ fails because the rightmost vertex of $r'$ might be strongly traversed from the left by $t_l$ using a run of length 3.)

Let $u_l$ be the furthest labeled vertex to the right strongly traversed by $t_l$ in $r'$. If $u_l$ is $0_R$ or $1_L$ then $t_r$ has to exist to strongly traverse $1_M$. By Lemmas 8 and 10, $t_l$ traverses vertices $0, \ldots, 3$ in $r$, so it strongly traverses vertices $0, \ldots, 2$ in $r$. By Lemmas 8 and 10 modified for traversal from right, $t_r$ has to visit vertices $0, \ldots, 3(i+1)$ numbered from right in $r$, where $i = (k+1) - 1$, in order to traverse vertex $1_M$ (which is s vertex $(i)_M$ labeled from right). (Lemmas 8 and 10 are applicable because $t_l$ and $t_r$ have to start with a run of appropriate color.) Hence, $t_r$ strongly traverses $3(k+1)$ vertices from right in $r$, and $t_l$ along with $t_r$ strongly traverse all vertices.

If $u_l$ is neither $0_R$ nor $1_L$, let $i_l$ be the largest $i$ such that $t_l$ strongly traverses $(i)_M$ during its traversal in $r'$. Thus, $t_l$ has to strongly traverse vertices $0, \ldots, 3(i_l+1)-1$ in $r$. If $i_l = k$ and $d = 2$ then all vertices in $r$ are traversed strongly by $t_l$. If $i_l = k$ and $d = 4, 6$, then $t_r$ has to exist and must strongly traverse at least three rightmost vertices of $r$. Hence, $t_l$ and $t_r$ strongly traverse altogether $3(k+2) \geq l$ vertices in $r$ (for $d = 4$ some of them twice), therefore all vertices in $r$ are strongly traversed.

If $i_l < k$ then $t_r$ must exist, and let $i_r$ be the largest $i$ such that $(i)_M$ is reached by $t_r$ in $r'$ (labeled right-to-left.) Clearly, $(k+1) - i_r \leq i_l + 1$. By Lemmas 8 and 10, $t_l$ strongly traverses vertices $0, \ldots, 3(i_l+1)-1$ in $r$ (labeled left-to-right) and $t_r$ strongly traverses vertices $0, \ldots, 3(i_r+1)-1$ in $r$ (labeled right-to-left). Thus, they strongly traverse at least $3(i_l+1) + 3((k+1)-(i_l+1)+1) \geq l$ vertices in $r$.

Note, in all these cases the first and last vertices of $r$ are traversed strongly because $t$ has to run "around" $s$ to get from one end of $r$ to the other one in between $t_l$ and $t_r$.

Hence, in all cases, all vertices in $s$ except possibly $v$ are strongly traversed. □

Let $s$ and $s'$ be as in the statement of Lemma 12. Let $t$ be a traversal (3,6)-sequence which strongly traverses $s'$ starting at any vertex. Let sequence $t'$ be obtained from $t$ by prepending a run $w$ (called a "loader") of length $|s|$ of opposite color than the first run of $t$. Then, $t'$ strongly traverses $s$ starting at any vertex.

Clearly, if we run $t'$ on $s$ starting at any vertex then $w$ brings us to a border vertex $v$ in $s$ and $t$ starts to run from that border vertex. By the previous lemma, all vertices in $s$ are strongly traversed by $t$ but vertex $v$. However, $v$ is also traversed strongly because of $w$ followed by $t$.

We conclude that stuffing long runs of 0's and 1's is transparent for (3,6)-traversal sequences. We presented the analysis explicitly for (3,6)-sequences but essentially identical analysis can be carried out for any $(2i+1, 4i+2)$-sequence, for $i \geq 1$, where we stuff long runs every $(2i+1)$ vertices. The reason for choosing (3,6)-sequence in this step is the optimality of resulting sequence with respect to other possible choices.

Let $s'$ be a (1,2)-sequence and $t'$ be a strong traversal sequence of $s'$ starting at a vertex $v'$. We say that $t'$ *extra strongly traverses* $s'$ if every pair in $s'$ is traversed by $t'$ in such a way that it is entered from outside at one border vertex, walked back and forth on it and then left through the other border vertex.

**Lemma 13** *Let $s$ be a non-monochromatic graph sequence with every run of length from $\{1, 2, 3, 4, 6\}$. Let $s_\oplus$ be the parity contraction of $s$. Let $v$ be a border vertex in $s$, and $v_\oplus$ its corresponding vertex in $s_\oplus$. Let $t_\oplus$ be an extra strong traversal (1,2)-sequence of $s_\oplus$ starting at $v_\oplus$. Then $t$, obtained from $t_\oplus$ by replacing every run of length 1 by run of length 3, and every run of length 2 by run of length 6, completely traverses $s$ starting at $v$.*

**Proof:** First observe that $t$ started on $s$ at $v$ traverses all runs of $s$ in the same order as $t_\oplus$ started on $s_\oplus$ at $v_\oplus$. This is because any (3,6)-sequence has the same behavior on runs of length 2,4 and 6, and on runs of length 1 and 3, and these are the only lengths that may appear in $s$.

Now notice that $t$ traverses $s$ completely. It is easy to see that because of the strong traversal of $s_\oplus$ by $t_\oplus$, all vertices in $s$ of runs no longer than 4 are traversed completely (and strongly) (use the same argument as in Proposition 2.)

Let us consider a run $r$ in $s$ of length 6. Its parity contraction is a pair $r_\oplus$ in $s_\oplus$, vertices of which are extra strongly traversed. Hence, there is a run of length 6 in $t$ which runs starting at one border vertex of $r$, traverses all vertices of $r$ and leaves on the other end of $r$. Clearly, all vertices in $r$ are strongly traversed. $\square$

Note, if $s'$ is obtained from a non-monochromatic graph sequence $s$ by 1-run breaking, and $s_\oplus$ is obtained from $s'$ by parity contraction, then the length of $s_\oplus$ is at most the length of $s$. (During 1-run breaking we insert 10 and 01 after every three 1's and 0's, respectively. Two of these vertices are always removed by parity contraction.)

Let $t_\oplus$ be a (1,2)-sequence which extra strongly traverses any graph (1,2)-sequence of length $|s|$ starting at any vertex. W.l.o.g. the first vertex in $t_\oplus$ has color 0. Let $t'$ be obtained from $t_\oplus$ by the construction from Lemma 13, and let $t$ be obtained from $t'$ by prepending a loader $1^{|s|}$. Then $t$ strongly traverses $s$ starting at any vertex. Moreover, $t$ is a strong UTS for cycles of length $|s|$. (If $t$ is started at any vertex in $s$ then it first reaches a border vertex in $s$ by the loader (or $s$ is strongly traversed if $s$ is monochromatic.) Call that vertex $v$. There is a vertex $v'$ in $s'$ corresponding to $v$, and a vertex $v_\oplus$ in $s_\oplus$ corresponding to $v'$. By Lemma 13, $s'$ is strongly traversed by $t'$ starting at $v'$, therefore, $s$ is strongly traversed by $t$ starting at any vertex, using Lemma 12 and a remark following it.)

## 4.2 01-run Breaking

This stage is aimed at breaking down long 01-runs. The method used here for breaking long 01-runs shares the same spirit with the previous method of breaking 0-runs and 1-runs. Because of that the substance of the proof of correctness of this stage is identical to the proof of correctness of the previous stage but the objects are different. Hence, instead of presenting here the full proof, which would be merely a rephrasing of previous one, we point out the common elements among these two stages and state the appropriate lemmas. Let us first describe this stage.

A (5,10)-01-*sequence* is a (1,2)-sequence in which every two consecutive pairs are separated by a 01-run and every 01-run is of length five or ten. Note, a 01-run between consecutive pairs of the same color always has odd length, and a 01-run between pairs of different colors always has even length. Therefore, in any (5,10)-01-sequence pairs of the same color are separated by 01-runs of length five and pairs of different colors are separated by 01-runs of length ten.

Let $s$ be a graph (1,2)-sequence containing at least one pair. We replace every 01-run in $s$ of length $l \in \{7, 9, 11, 12, 13, 14, \ldots\}$ by a stuffed sequence $r'$ according to the following table. Let $l = 5k + d$ as in Proposition 6. Let $r_0 = 01010$, $r_1 = 10101$, $r'_0 = 01010\ 100100$, and $r'_1 = 10101\ 011011$.

| $k$ even | $k$ odd |
|---|---|
| $(r_0r_1)^{k/2}(01)^{d/2} \rightarrow (r'_0r'_1)^{k/2}(01)^{d/2}$ | $(r_0r_1)^{(k-1)/2}r_0(10)^{d/2} \rightarrow (r'_0r'_1)^{(k-1)/2}r'_0(10)^{d/2}$ |
| $(r_1r_0)^{k/2}(10)^{d/2} \rightarrow (r'_1r'_0)^{k/2}(10)^{d/2}$ | $(r_1r_0)^{(k-1)/2}r_1(01)^{d/2} \rightarrow (r'_1r'_0)^{(k-1)/2}r'_1(01)^{d/2}$ |

Hence, we insert 011011 and 100100, alternatively, every five vertices in $r$. Observe that the first stuffing that is inserted in $r$ is 011011 iff the pair ending $r$ on the left is a 00-pair, and the last stuffing that is inserted in $r$ is 011011 iff the pair ending $r$ on right is a 00-pair. Thus, all 01-runs in stuffed sequence $r'$ are of even length less or equal to ten.

Similarly to the previous 1-run breaking stage we want to argue that a (5,10)-01-sequence $t$ behaves the same on the original and on the stuffed sequence with respect to particular input and output vertices.

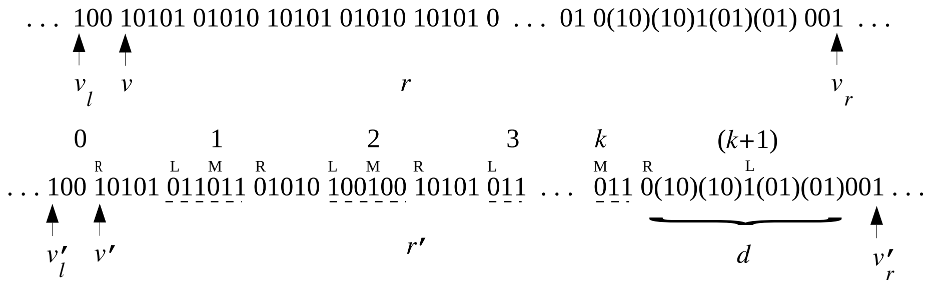


**Figure 5.**

Let $\mathcal{T}_{5,10}$ be the class of (5,10)-01-sequences starting with a 01-run and ending with a pair. Let $\epsilon \in \mathcal{T}_{5,10}$. Let a prefix relation $\sqsubseteq$ on $\mathcal{T}_{5,10}$ be defined by $x \sqsubseteq y$ iff $x$ is a prefix of $y$.

**Lemma 14 (≈7)** *Let $s'$ be obtained from a graph (1,2)-sequence $s$, that contains at least one pair, by 01-run breaking. Let $r$ be a 01-run in $s$ and $r'$ its corresponding (stuffed) sequence in $s'$. Let $v$ be the leftmost vertex of $r$, and let $v'$ be the leftmost vertex of $r'$. Let $v_l$ ($v'_l$) be the vertex immediately to the left of the pair that is neighboring $r$ ($r'$) on the left, and let $v_r$ ($v'_r$) be the vertex just to the right of the pair that is neighboring $r$ ($r'$) on the right (Fig. 5.) Then every $t \in \mathcal{T}_{5,10}$ behaves the same on $r$ and $r'$ with respect to vertices $v, v_l, v_r$ and $v', v'_l, v'_r$.*

A proof of this lemma goes along the same lines as the proof of Lemma 7. Therefore, we will not present the full proof but rather we will present statements of the necessary lemmas which correspond to lemmas in the proof of Lemma 7. Lemma 14 follows from them in a similar way.

**Proof of Lemma 14 (*sketch*):** For the proof, we assume that $r \neq r'$, otherwise the lemma is trivial, we set $l = |r|$, and $l = 5k + d$ as in Proposition 6. We number vertices in $r$ from left to right by $\{0, 1, \ldots, l-1\}$, and we refer to vertices in $r'$ as in Figure 5 i.e., there are vertices labeled $0_R, 1_L, 1_M, 1_R, 2_L, \ldots, k_M, k_R, (k+1)_L$ in $r'$. (The vertex labeled $(k+1)_L$ is the sixth vertex from the left of the last 01-run, for $d \in \{6, 8, 10\}$, and it is the rightmost vertex of that 01-run otherwise.) For the rest of the proof we may assume that $t$ and $r$ start with the vertex of color 1. (This corresponds to assumption that $r$ is a 1-run and $t$ starts with a 1-run in the proof of Lemma 7.)

For any $\tau \in \mathcal{T}_{5,10}$ let us call any 01-run in $\tau$, which is preceded by an odd number of pairs, *an even positioned 01-run* (e.p. 01-run) and let us call a 01-run, which is preceded by an even number of pairs, *an odd positioned 01-run* (o.p. 01-run.) (Even positioned 01-runs correspond to 0-runs, and odd positioned 01-runs correspond to 1-runs in $t$ in the proof of Lemma 7.) Let $[\tau]$ denote the last 01-run in $\tau$, let $\tau - 1$ denote a sequence which is obtained from $\tau$ by removing $[\tau]$ and the last pair in $\tau$, and let $l(t)$ denote the number of 01-runs in $\tau$.

Define $o_0^\tau, o_1^\tau, e_0^\tau, e_1^\tau$ to be the number of odd e.p. 01-runs, odd o.p. 01-runs, even e.p. 01-runs, and even o.p. 01-runs in $\tau$, respectively. (Notice, $l(\tau) = o_0^\tau + o_1^\tau + e_0^\tau + e_1^\tau$.) Further, define $p(\tau) = (o_1^\tau + 2e_1^\tau) - (o_0^\tau + 2e_0^\tau)$.

Now we state the lemmas which describe a traversal of $r$ by $t$.

**Lemma 15 (≈8)** *Let $t \in \mathcal{T}_{5,10}$ be such that $t = \epsilon$ or $t$ starts with a vertex of color 1. Let $\forall t' \sqsubseteq t;\ 0 \leq p(t') \leq k+1$. Then the following hold:*

1. *$\forall t' \sqsubseteq t$, if $p(t') = 0$ then $[t']$ is an e.p. 01-run, and if $p(t') = k+1$ then $[t']$ is an o.p. 01-run,*
2. *if $d = 6, 8, 10$ then we are at a vertex numbered $5p(t)$ in $r$ after running $t$ on $r$ starting at $v$,*

3. *if* $d = 2, 4$ *then we are at vertex* $5p(t) - \delta_d(t)$ *in* $r$ *after running* $t$ *starting at* $v$*, where* $\delta_d(t) \in \{0, 6-d\}$*, and* $\delta_d(t) \neq 0$ *iff* $(\exists t' \sqsubseteq t;\ p(t') = k+1$ *and* $\forall t''$ *such that* $t' \sqsubseteq t'' \sqsubseteq t$*,* $p(t'') > 0)$*.*

The first two parts of this lemma are easy to verify. You can show the third part of the lemma by an induction on the number of $t' \sqsubseteq t$ such that $p(t') = 0$ or $k+1$, similarly to the proof of Lemma 8.

**Lemma 16 (≈9)** *Let* $\epsilon \neq t \in \mathcal{T}_{5,10}$ *start with a vertex of color 1. The following are equivalent:*

1. $\forall t' \sqsubset t$*, we are at a vertex belonging to* $r$ *after running* $t'$ *on* $r$ *starting at* $v$*, and we are at a vertex outside of* $r$ *after running* $t$ *on* $r$*,*

2. $\forall t' \sqsubset t;\ 0 \leq p(t') \leq k+1$ *and ((*$p(t-1) = 1$ *and* $[t]$ *is an even e.p. 01-run), or (*$p(t-1) = k$ *and* $[t]$ *is an even o.p. 01-run.))*

The proof of this lemma is based on the previous lemma and is essentially identical to the proof of Lemma 9.

The following two lemmas describe a traversal of $r'$ by $t$.

**Lemma 17 (≈10)** *Let* $\epsilon \neq t \in \mathcal{T}_{5,10}$ *start with a vertex of color 1. If* $\forall t' \sqsubseteq t;\ 0 \leq p(t') \leq k+1$ *then by running* $t$ *on* $r'$ *starting at* $v$*:*

- *we get to vertex* $(p(t))_L$ *iff* $[t]$ *is an odd o.p. 01-run,*
- *we get to vertex* $(p(t))_R$ *iff* $[t]$ *is an odd e.p. 01-run,*
- *we get to vertex* $(i)_M$ *iff (*$i = p(t) - 1$ *and* $[t]$ *is an even o.p. 01-run) or (*$i = p(t) + 1$ *and* $[t]$ *is an even e.p. 01-run). Furthermore,* $0 < i < k+1$*.*

This lemma can be proven by an induction on $l(t)$, similarly to the proof of Lemma 10.

**Lemma 18 (≈11)** *Let* $\epsilon \neq t \in \mathcal{T}_{5,10}$ *start with a vertex of color 1. The following are equivalent:*

1. $\forall t' \sqsubset t$*, we are at a vertex belonging to* $r'$ *after running* $t'$ *on* $r'$ *starting at* $v'$*, and we are at a vertex outside of* $r'$ *after running* $t$ *on* $r'$*,*

2. $\forall t' \sqsubset t;\ 0 \leq p(t') \leq k+1$ *and ((*$p(t-1) = 1$ *and* $[t]$ *is an even e.p. 01-run), or (*$p(t-1) = k$ *and* $[t]$ *is an even o.p. 01-run.))*

The proof of this lemma is based on the previous lemma and is essentially identical to the proof of Lemma 11. The conclusion of the proof of Lemma 14 is now the same as of the proof of Lemma 7. □

We state the following proposition a proof of which is trivial hence, omitted.

**Proposition 19 (≈5)** 1. *Any sequence in* $\mathcal{T}_{5,10}$ *started at the leftmost vertices or rightmost vertices behaves the same on sequences* $10$*,* $10\,10$*,* $10\,10\,10$*,* $10\,10\,10\,10$*, and* $10\,10\,10\,10\,10$*, where the output vertices are vertices that are neighbors of the surrounding pairs in graph sequences containing these sequences (similarly to Lemma 7.) The same holds for negated sequences.*

2. *Similarly, any sequence in* $\mathcal{T}_{5,10}$ *behaves the same on sequences* $1$*,* $101$ *and* $10101$*, where input and output vertices are as above. The same holds for negated sequences.*

Similarly to the case of Lemma 7 we could now state a lemma that would deal with traversal of $r$ and $r'$ by $t$ starting at the rightmost vertices of $r$ and $r'$, respectively. All Lemmas 15 – 18 can be shown to be valid for such a traversal if $r$ and $r'$ are numbered from the right instead of from the left. The only minor difference would be in Lemma 17. For the case of $d = 8$ or 10, it would be necessary to assume (w.l.o.g.) that $t$ begins with a 01-run of length ten. (This requirement is similar to the requirement that $t$ begin with an even 1-run in the modification of Lemma 10 for traversals starting at the rightmost vertices of $r$ and $r'$, respectively, which is necessary because the traversal does not start at a labeled vertex.)

Based on the previous lemmas it is possible to establish the following statement.

**Lemma 20 (≈12)** *Let $s'$ be obtained from a graph (1,2)-sequence $s$ containing at least one pair by 01-run breaking. Let $t$ be a traversal (5,10)-01-sequence ending with a 01-run of length ten. Let $v$ be the left vertex of some pair in $s$ if that pair has the same color as the first vertex of $t$, or let $v$ be the right vertex of that pair otherwise. Let $v'$ be the vertex corresponding to $v$ in $s'$. If $t$ extra strongly traverses $s'$ starting at $v'$ then $t$ extra strongly traverses $s$ starting at $v$.*

**Proof:** First observe that $t$ traverses $s$ in similar way as $s'$, in particular, a pair in $s$ is extra strongly traversed during traversal by $t$ whenever its corresponding pair in $s'$ is extra strongly traversed. The first pair in $t$ brings us in $s$ and $s'$, respectively, outside of the pair containing $v$ and $v'$, respectively. In both sequences we get either to the left neighbor of that pair or to the right neighbor, and that vertex is either a vertex of a pair or a vertex of a 01-run in both sequences. (A 01-run breaking never inserts a pair next to another pair, so such a pair must exist in both sequences.) In the former case we can apply an induction on the length of $t$ to conclude that $t$ traverses $s$ and $s'$ in similar way. In the latter case we can use the fact that a (5,10)-01-sequence starting with a 01-run and ending with a pair i.e., a sequence from $\mathcal{T}_{5,10}$, behaves the same on a 01-run and its corresponding stuffed sequence, and then again use the induction on the length of $t$.

Hence, all pairs in $s$ are extra strongly traversed by $t$ starting at $v$. Using an argument similar to the one in the proof of Lemma 12, because all vertices in $s'$ are strongly traversed by $t$, all vertices in 01-runs in $s$ are strongly traversed, too. ($t$ starts the traversal on a vertex belonging to a pair which is strongly traversed by $t$, so the exception of the starting vertex not to be strongly traversed does not occur here.) □

We conclude this section with the following lemma.

**Lemma 21** *Let $s$ be a (1,2)-sequence containing at least one pair with every 01-run having length in $\{1, 2, 3, 4, 5, 6, 8, 10\}$, and let $s_c$ be the pair contraction of $s$. Let $t_c$ be a strong traversal sequence of $s_c$ starting at the vertex $v_c$ corresponding to some pair in $s$. Let $v$ be the left vertex of that pair if the color of that pair is the same as the color of the first vertex in $t_c$, or let $v$ be the right vertex of that pair otherwise. Let $t$ be obtained from $t_c$ using the following rules:*

| *vertex* | *replace by* |
|---|---|
| *0 followed by 0* | *00 10101* |
| *0 followed by 1 (or nothing)* | *00 1010101010* |
| *1 followed by 1* | *11 01010* |
| *1 followed by 0 (or nothing)* | *11 0101010101* |

*Then $t$ extra strongly traverses $s$ starting at $v$.*

**Proof:** We want to argue that $t$ traverses pairs in $s$ in a similar order as $t_c$ traverses vertices in $s_c$, in particular, if $v_{a_0}, v_{a_2}, \ldots, v_{a_l}$ is a sequence of vertices in $s_c$ as traversed by $t_c$, $l = |t_c|$, then a sequence of pairs in $s$ as traversed by $t$ can be written as $p_{a_{i_0}}, p_{a_{i_2}}, \ldots, p_{a_{i_{l'}}}$, where pair $p_i$

corresponds to vertex $v_i$ in $s_c$, $i_0 = 0$, for all $1 \le j \le l'$, $i_j = i_{j-1} + 1$ or $i_{j-1} + 2$, and $i_{l'} = l$ or $l - 1$.

Let us consider the following cases occurring during the traversal of $s$ and $s_c$ by $t$ and $t_c$, respectively. In all four considered cases let us assume that we reached a vertex $u_c$ colored 1 in $s_c$ and its corresponding pair in $s$ during these traversals. Let us further assume that the next digit in $t_c$ is 1 and the digit following it has color $c$. Hence we assume, in $t$ there is going to be $11(01)^5$ or 11 01010 (depending on $c$.) Let $u_c^R$ denote the right neighbor of $u_c$ in $s_c$.

If $u_c^R$ has color 1 then pairs corresponding to $u_c$ and $u_c^R$ in $s$ are separated by a 01-run of odd length, i.e., of length at most five. Thus the digit 1 in $t_c$ brings us to $u_c^R$, and the 11-pair followed by a 01-run in $t$ brings us to a pair corresponding to $u_c^R$ in $s$.

If $u_c^R$ has color 0 then pairs corresponding to $u_c$ and $u_c^R$ in $s$ are separated by a 01-run of even length at most ten (or they are separated by an empty string.) If $c$ is 0 then there is $11(01)^5$ followed by a 00-pair in $t$. In that case, the digit 1 in $t_c$ brings us to vertex $u_c^R$ in $s_c$, and $11(01)^5$ in $t$ brings us to the pair corresponding to $u_c^R$ in $s$. If $c$ is 1 then there is $11\,01010\,11(01)^5$ or 11 01010 11 01010 in $t$. Digit 1 followed by the other digit 1 in $t_c$ brings us first to $u_c^R$ and then back to $u_c$ in $s$. Similarly, $11\,01010\,11(01)^5$ and 11 01010 11 01010 in $t$ bring us to the pair corresponding to $u_c$ in $s$, where the pair in $s$ corresponding to $u_c^R$ may (but need not) be visited during that.

If $u_c$ and $t_c$ are of different colors than that we have considered the behavior is symmetric.

Hence we may conclude that pairs in $s$ are traversed in the same order as the corresponding vertices in $s_c$, but some pairs in $s$ may occasionally not be visited by $t$ during the traversal. However, if a vertex in $s_c$ is traversed strongly then the corresponding pair in $s$ is (extra strongly) traversed at the same time. Because all vertices in $s_c$ are strongly traversed, all pairs in $s$ are extra strongly traversed.

The extra strong traversal of pairs in $s$ means that we have to reach every pair from the left or right, respectively, and then leave it to the right or left, respectively. In particular, if a pair in $s$ has neighboring 01-runs, at least five of the neighboring vertices (or fewer if the 01-runs are shorter than five) in each of these runs are traversed during the extra strong traversal of that pair. Actually, these vertices are all traversed strongly because the 01-runs in $t$, which traverse them, are preceded and followed by pairs. Because every 01-run in $s$ has length at most ten and has neighboring pairs on both sides, all vertices in 01-runs are traversed strongly. Hence, $s$ is extra strongly traversed. $\square$

Note, if $s'$ is obtained from a graph (1,2)-sequence $s$ containing at least one pair by a 01-run breaking, and $s_c$ is obtained from $s'$ by a pair contraction, then $|s| \ge 2|s_c|$. (During the 01-run breaking we insert 2 pairs at most every five singletons, where all singletons are removed by the pair contraction, and every pair is reduced by half.)

Let $t_c$ be a UTS for cycles of length $\lceil |s|/2 \rceil$. W.l.o.g. the first vertex in $t_c$ has color 1. Let $t'$ be obtained from $t_c$ by the construction from Lemma 21, and let $t$ be obtained from $t'$ by prepending a loader $0(10)^{\lfloor |s|/2 \rfloor}$. Then $t$ extra strongly traverses $s$ starting at any vertex. Moreover, $t$ extra strongly traverses any graph (1,2)-sequence of length $|s|$. (If $t$ is started at any vertex in $s$ then it first reaches some pair in $s$ by the loader. Let $v$ be a vertex of that pair to which we get by the loader. There is a vertex $v'$ in $s'$ corresponding to $v$, and a vertex $v_c$ in $s_c$ corresponding to the pair containing $v$. By Lemma 21, $s'$ is extra strongly traversed by $t'$ starting at $v'$, therefore, by Lemma 20, $s$ is extra strongly traversed by $t'$ starting at $v$, hence is extra strongly traversed by $t$.)

We choose here to stuff 01-runs every 5 singletons. Similarly to the 1-run breaking stage, we could choose to stuff 01-runs every $2i+1$ singletons, for $i \ge 1$, and use for traversal a $(2i+1, 4i+2)$-01-sequence.

## 4.3 The Construction

Lemmas 21, 20, 13, and 12 combine to give a construction that converts a strong UTS for cycles of length $m$ into a strong UTS for cycles of length $2m$. This is presented explicitly in step 3 of the algorithm below.

Here is the algorithm constructing a UTS for cycles of length $n$. Let $k = \lceil \log_2 n \rceil$.

**The Algorithm:**

1. Let $t_0$ be a strong UTS for cycles of length 6, that starts with a vertex of color 1.
2. Apply the construction of Proposition 3 to $t_0$ to obtain a sequence $t'_1$, and attach in front of $t'_1$ sequence $0(10)^6$ to get $t_1$.
3. For $i = 1, \ldots, k-1$, construct a sequence $t'_{i+1}$ from $t_i$ by applying the following rules:

| | replace by |
|---|---|
| a 00-pair | $[00\,10101]^5\,00(10)^5$ |
| singleton 0 | $[00\,10101]^2\,00(10)^5$ |
| a 11-pair | $[11\,01010]^5\,11(01)^5$ |
| singleton 1 | $[11\,01010]^2\,11(01)^5$ |

   Prepend loader $0(10)^{6\cdot 2^i}[11\,01010]^{6\cdot 2^i}11(01)^5$ in front of $t'_{i+1}$ to get a sequence $t_{i+1}$. (This step is a composition of the constructions from Lemmas 13 and 21.)

4. Apply the construction of Lemma 13 to $t_k$ to obtain a sequence $t'$, and attach sequence $1^n$ in front of $t'$ to get a sequence $t$.

Sequence $t$ produced by this algorithm is a universal traversal sequence for cycles of length $n$. There is a uniform $NC^1$ circuit that, given $1^n$, will construct the UTS for cycles of length $n$.

Let us analyze this algorithm. Note that $t_1, \ldots, t_k$ are (1,2)-sequences. Also note that the sequences by which we replace individual pairs and singletons start and end by vertices of the same color as the replaced pairs and singletons. That means that there never appears a new pair consisting of one vertex from one replacement sequence and the other one from the next sequence. Thus, the replacement process is essentially "context free". We can describe the replacement process by context free grammar in which there is a single variable for a whole pair of each color and a single variable for a singleton of each color. Hence, we can associate with the output sequence a derivation tree implicitly created by the algorithm.

That derivation tree has essentially degree bounded by 41, except for the nodes that correspond to loaders and the root corresponding to $t_1$. Note that loaders may appear only on the left side of the tree and the parse tree leading to the loaders can have degree 2. Hence, we may represent a path from root to any leaf by a vector $< a_1, a_2, \ldots, a_{k+1}, m >$, where $0 \le a_1 \le l_{t_1}$, $a_2, \ldots, a_k \in \{0, 1, \ldots, 41\}$, $a_{k+1} \in \{0, \ldots, 6\}$, and $1 \le m \le 48 \cdot 2^k + 12$, where $l_{t_1}$ denotes the length of $t_1$. The $a_1, \ldots, a_{k+1}$ denote the index number of the descendant node among its siblings, where $a_i = 0$ has a special meaning. If $a_1, \ldots, a_j$ are all zeros, for some $j$, and all the other $a_i$'s are non-zero then that vector corresponds to a path going through $m$-th descendant of a loader at $(j+1)$-th level.

Observe that there are many vectors which actually do not correspond to any path. Any vector where $a_i \neq 0$ and $a_{i+1} = 0$, for some $i$, is illegal. Also not every node necessarily has 41 descendants, singletons have only 23 descendants. In any case we can test the legality of a given vector in $NC^1$ with respect to $n$ (the length of the vector is logarithmic in $n$, hence a deterministic traversal of the vector is even $DLOGTIME(n)$.)

Thus, there is an $NC^1$ algorithm which computes the path vector for any leaf of the derivation tree given index $i$ of the leaf. (For every vector we can compute by brute force the number of invalid vectors lexicographically preceding it. By comparing this number with $i$ we may decide which vector is the one which we are looking for. The lexicographical order has to appropriately take $m$ into account.) From the path vector we may easily compute the value of $i$-th bit by $DLOGTIME(n)$ algorithm.

# 5 The Analysis

Let $f(i)$ denote the number of nodes at the $i$-th level in the derivation tree associated with a traversal sequence produced by the algorithm. (Every internal node in that tree corresponds either to a pair or to a singleton, leaves correspond to single vertices.) A rough analysis of the algorithm gives us a recurrence $f(i+1) \leq 41f(i)+48\cdot 2^i+12$. By solving this recurrence, at level $k=\lceil \log_2 n\rceil$ of the tree there are $O(n^{\log_2 41})=O(n^{5.358})$ nodes. At step 4 of the algorithm every node of the tree is expanded to 3 or 6 vertices. Thus the total length of the universal traversal sequence generated by the algorithm is $O(n^{5.358})$.

By slight modification of the algorithm and more careful analysis we may show that only $k'=2\lceil \log_5 n\rceil \leq 2+\log_{\sqrt{5}} n$ iterations of step 3 are necessary, and that the number $f'(i)$ of nodes at the $i$-th level of the derivation tree associated with the produced sequence satisfies recurrence $f'(i+1) \leq 25.635f'(i)+48\cdot 4^i+12$. By solving the recurrence we get that the length of the produced sequence is $O(n^{\log_{\sqrt{5}} 25.635})=O(n^{4.031})$.

## 5.1 The Modified Algorithm

We have to slightly modify the algorithm to get the better bound. The modification is to replace $2^i$ by $4^i$ in the loader of $t_{i+1}$ in step 3 of the algorithm, so to use the loader $0(10)^{6\cdot 4^i}[11\,01010]^{6\cdot 4^i}11(01)^5$ instead. This modification is necessary because we want to argue that a graph sequence shrinks to $1/\sqrt{5}$ of its length by contractions (to 1/5 every two iterations) and not only to 1/2. The loader has to be long enough to traverse the whole graph sequence, thus its length has to correspond to the length of the traversed sequences at the given iteration of the algorithm $((\sqrt{5})^i \leq 4^i)$.[3]

**Proposition 22** *For $i=1,\ldots,k$, let $p_i$ and $s_i$ denote the number of pairs and singletons, respectively, in $t_i$. Then $\frac{p_i}{p_i+s_i} < \frac{6}{41}$.*

**Proof:** By construction, $t'_1$ contains one pair per 7 singletons. The loader attached to $t'_1$ to get $t_1$ can only lower the ratio of pairs. Thus, $\frac{p_1}{p_1+s_1} \leq \frac{1}{8} < \frac{6}{41}$.

For $i=1,\ldots,k-1$, $t'_{i+1}$ is obtained from $t_i$ by replacing singletons and pairs according to rules specified in step 3 of the algorithm. The ratio of pairs is $\frac{6}{41}$ in the replacement for pairs, and $\frac{3}{23}$ in the replacement for singletons. The ratio of pairs is at most $\frac{1}{8}$ in the loader attached to $t'_{i+1}$ to get $t_{i+1}$. Hence, $\frac{p_{i+1}}{p_{i+1}+s_{i+1}} < \frac{6}{41}$. □

By the proposition, $t_i$ contains fraction of at most $\frac{6}{41}$ of pairs, $i=1,\ldots,k$. That means that at the $i$-th level in the derivation tree there is a fraction of at most $\frac{6}{41}$ of nodes having 41 descendants, whereas the remaining nodes have only 23 descendants. Thus on average, there are at most $41\cdot 6/41+23\cdot 35/41 \doteq 25.635$ of descendants per node. Thus we obtain the recurrence $f'(i+1) \leq 25.635f'(i)+48\cdot 4^i+12$ for the number of nodes at the $i$-th level.

[3] We could use $3^i$ instead of $4^i$ as well, but for the sake of uniformity it is better to use $4^i$.

### 5.1.1 The Shrinking Factor

We want to show that a graph sequence $s_c^2$ obtained from an arbitrary graph sequence $s$ by repeating the sequence of operations (1-run breaking, parity contraction, 01-run breaking, pair contraction) twice has length at most one-fifth of the length of $s$. We are going to use the following lemmas.

**Lemma 23** *Let $s$ be a graph sequence and let a (1,2)-sequence $s_\oplus$ be obtained from $s$ by a 1-run breaking and a parity contraction. Then there is a 1-1 mapping of vertices in $s_\oplus$ into vertices of $s$, which preserves order of vertices and which maps every pair in $s_\oplus$ to two neighboring vertices of the same color in $s$.*

By preserving the order of vertices we mean that images of any three vertices are in the same left-to-right order as their pre-images. A proof of this lemma follows from Figure 6-a),b),c). Notice, any vertex colored 1 in $s_\oplus$, which comes from 01 stuffing of some 0-run, is always surrounded by 00-pairs in $s_\oplus$, hence it cannot form a pair with neighboring vertices. Similarly, any vertex colored 0 coming from 10 stuffing of some 1-run cannot form a pair with its neighbors. The cases a) and b) in Figure 6 occur only for runs having length in $\{1, 2, 3, 4, 6\}$.

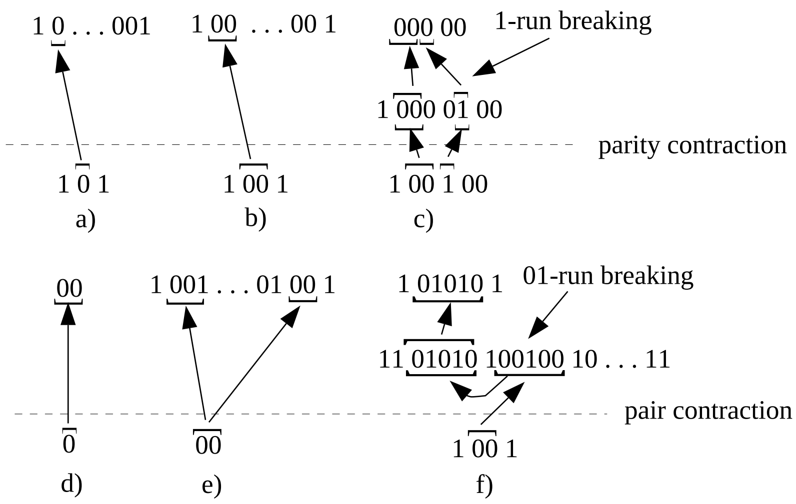


**Figure 6.**

**Lemma 24** *Let $s_\oplus$ be a (1,2)-graph sequence and let a sequence $s_c$ be obtained from $s_\oplus$ by a 01-run breaking and a pair contraction. Let some vertices in $s_c$ be grouped into non-overlapping pairs with adjacent vertices of the same color, and let the other vertices in $s_c$ be left single.*

*Then there is a map which maps every pair in $s_c$ to five vertices in $s_\oplus$, and every single vertex in $s_c$ to two vertices in $s_\oplus$, such that no vertex in $s_\oplus$ is in an image of two distinct objects from $s_c$. Moreover, every single vertex in $s_c$ which does not come from a contraction of a pair inserted by the 01-run breaking is mapped to a pair in $s_\oplus$.*

**Proof:** Let us consider a pair of vertices $v_1, v_2$ in $s_c$. Each of these two vertices comes from the contraction of some pair. Vertices $v_1$ and $v_2$ have the same color so the pairs they come from are of the same color, too. Each of these pairs may either be present in $s_\oplus$ or be inserted by 01-run breaking.

Let us consider the case when both pairs come from $s_\oplus$. Because these pairs are of the same color they must be separated by at least one vertex of different color in $s_\oplus$. Clearly, between these pairs there cannot be any other pair, so they are separated by 01-run of length at least one (but no more than five otherwise 01-run breaking would insert other pairs between them.) We may associate pair $v_1, v_2$ with vertices as in Figure 6-e).

Let us consider the case that one of $v_1$ and $v_2$ comes from contraction of a pair inserted by 01-run breaking. Then we claim that the other one also comes from a pair inserted by 01-run breaking, moreover they both come from the same 011011 or 100100. W.l.o.g let us assume that $v_1$ and $v_2$ have color 0, so the one which comes from 01-run breaking comes from the contraction of a pair from an inserted 100100. We know that we alternate between 011011 and 100100 during 01-run breaking, and that the pair ending the broken 01-run on the left side has a different color than pairs in the first inserted 011011 or 100100, respectively, and also the pair ending the broken 01-run on the right side has a different color than the pairs in the last inserted 011011 or 100100, respectively. Hence, two vertices that come from the contraction of pairs from 100100 are surrounded by vertices of the opposite color, hence the two vertices obtained by the contraction of 100100 are $v_1$ and $v_2$. We may associate inserted 011011 and 100100, respectively, with five preceding vertices coming from the broken 01-run in $s_\oplus$ (Fig. 6-f).) Hence, we have associated every pair in $s_c$ with a five vertices in $s_\oplus$.

Let us consider single vertices in $s_c$. Any single vertex $v$ in $s_c$ comes from a contraction of some pair. If that pair comes from $s_\oplus$, we associate $v$ with that pair (Fig. 6-d).) Otherwise the pair comes from 011011 or 100100 inserted by 01-run breaking. We may associate $v$ with the left two or right two vertices of the five vertices preceding the inserted 011011 and 100100, respectively, depending on whether $v$ comes from the left or the right pair of the inserted 011011 and 100100, respectively. □

We are going to use the previous lemmas to prove the following one.

**Lemma 25** *Let $s$ be a graph sequence, and let $s_c^2$ be obtained from $s$ by repeating the sequence of operations (1-run breaking, parity contraction, 01-run breaking, pair contraction) two times. Then the length of $s_c^2$ is at most on- fifth of length of $s$.*

**Proof:** We are going to show that we may assign to every vertex from $s_c^2$ five vertices from $s$ so that no vertex in $s$ is assigned to more than one vertex in $s_c^2$. Let us denote by $s_\oplus^1$ a sequence obtained from $s$ by 1-run breaking and parity contraction, let $s_c^1$ denote a sequence obtained from $s_\oplus^1$ by 01-run breaking and pair contraction, and let $s_\oplus^2$ denote a sequence obtained from $s_c^1$ by 1-run breaking and parity contraction.

Some vertices in $s_c^2$ may come from contraction of pairs which were inserted during the last 01-run breaking. Group these vertices into pairs with their neighboring vertices of the same color (which originated from the same insertion), and leave the other vertices single. By Lemma 24 we may assign to every pair from $s_c^2$ five vertices in $s_\oplus^2$, and to every single vertex from $s_c^2$ a pair of vertices of the same color $s_\oplus^2$. By Lemma 23 any five vertices from $s_\oplus^2$ may be mapped to five vertices in $s_c^1$, and any pair from $s_\oplus^2$ to a pair of the same color from $s_c^1$. Let us keep together these pairs from $s_c^1$, and leave single the remaining vertices from $s_c^1$. By Lemma 24 every pair from $s_c^1$ may be mapped to five vertices from $s_\oplus^1$, and every single vertex from $s_c^1$ to two vertices from $s_\oplus^1$. Thus, we may associate with every vertex from $s_c^2$ five vertices from $s_\oplus^1$, and to every pair from $s_c^2$ ten vertices from $s_\oplus^1$. By one more application of Lemma 23 we get a map which maps single vertices from $s_c^2$ to five vertices from $s$, and pairs from $s_c^2$ to ten vertices from $s$. Hence, we may assign to every vertex from $s_c^2$ five vertices from $s$ so that no vertex from $s$ is assigned to more than one vertex from $s_c^2$. Thus, $s$ has at least five times more vertices than $s_c^2$. □

### 5.1.2 Correctness of the Modified Algorithm

The correctness of the modified algorithm follows from the following lemma.

**Lemma 26** *Let $t$ be a strong UTS for cycles of length $n \geq 1$. Let $t^2$ be obtained from $t$ by repeating the sequence of operations (construction of Lemma 21, attaching a loader $0(10)^{6\cdot 4^{j-1}}$, construction of Lemma 13, attaching a loader $1^{6\cdot 4^j}$) two times, where $j = 2\lceil \log_5 n \rceil + 1$ during the first repetition, and $j = 2\lceil \log_5 n \rceil + 2$ during the second one. Then $t^2$ is a strong UTS for cycles of length $5n$.*

**Proof:** Let $s$ be a graph sequence of length $5n$, let $s_c^1$ be obtained by application of 1-run breaking, parity contraction, 01-run breaking and pair contraction on $s$, and let $s_c^2$ be obtained by application of that sequence of operations on $s_c^1$. Let $t^1$ denote a traversal sequence obtained from $t$ after the first repetition of operations in the statement of the lemma.

By Lemma 25 sequence $s_c^2$ has the length of at most $n$, so that it is strongly traversed by $t$ starting at any vertex. By Lemmas 21 and 13, sequence $s_c^1$ is strongly traversed by $t^1$ starting at any vertex, given that attached loaders are longer than $s_c^1$. The length of $s_c^1$ is at most $5n/2$, the length of the first loader is $12 \cdot 4^{2\lceil \log_5 n \rceil} + 1 \geq 12 \cdot 16^{\log_5 n} \geq 5n/2$, and the length of the second loader is $6 \cdot 4^{2\lceil \log_5 n \rceil + 1} \geq 6 \cdot 16^{\log_5 n} \geq 5n/2$. Because $t^1$ is a strong traversal sequence for $s_c^1$ starting at any vertex, $t^2$ is a strong traversal sequence for $s_c^2$ starting at any vertex by similar argument. Thus, $t^2$ is a strong traversal sequence for cycles of length $5n$. □

### 5.2 Further Improvements

The previous analysis of the shrinking factor is tight. It is easy to see that a long 01-run shrinks to almost exactly one-fifth of its size every two rounds. 01-runs are really a bottleneck of the shrinking. If we would choose to stuff 01-runs every 7 singletons instead of five during 01-run breaking, and we would use (7,14)-01-sequences for traversal, then 01-run breaking would not be a bottleneck, anymore. The 01-runs would shrink at rate 7 per two rounds i.e., $\sqrt{7} \doteq 2.645$ per round on average.

In that setting it could be possible to show by iterating the previous analysis over several rounds using modifications of Lemma 23 and 24 that the shrinking factor converges to $1 + \sqrt{2}$ per round. In particular, for any $\epsilon > 0$, there is a $k \geq 1$ such that any graph sequence shrinks to $(1 + \sqrt{2} - \epsilon)^{-k}$ of its original length during $k$ rounds of contractions.

The recurrence describing the algorithm using (7,14)-01-sequences would be $f(i+1) \leq 33.618 f(i) + 48 \cdot 4^i + 12$, where $55 \cdot 6/55 + 31 \cdot 49/55 \doteq 33.618$. For $\epsilon$ small enough, the solution of the recurrence is $O(n^{3.989})$.

The details of the analysis are sufficiently tedious that we do not claim this as a theorem, but merely give an indication of how this further improvement can be obtained.

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