Expanders made elementary:
A combinatorial analysis of an expander construction

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Abstract

We give a fully combinatorial analysis (using edge expansion) of an iterative expander construction due to Reingold, Vadhan, and Wigderson [RVW02]. This shows that both the construction of constant-degree expanders and its analysis can be carried out using only elementary combinatorial tools.

1 Introduction

Expander graphs have become one of the most useful combinatorial objects in theoretical computer science, with many beautiful applications in computer science and mathematics [HLW06], and responsible for several recent breakthroughs in computational complexity [Rei05, Din06]. These graphs have seemingly contradictory properties: sparseness and high connectivity. The high connectivity can be measured in a number of different, but essentially equivalent ways: vertex expansion (every small subset of vertices "expands", i.e., has a larger neighborhood), edge expansion (every small subset of vertices has many edges leaving the set), or fast mixing time (a random walk on a regular expander graph quickly converges to the uniform distribution on vertices).

The existence of expander graphs of constant degree can be argued nonconstructively using a simple probabilistic argument: for any constant $d \geq 3$, a random d-regular graph is almost surely an expander [Pin73]. Constructing such graphs efficiently deterministically is much more difficult. The first explicit constructions were given by Margulis [Mar73] and Gabber and Galil [GG81]. Lubotzky, Phillips, and Sarnak [LPS88] gave a construction of expanders with particularly interesting properties, called Ramanujan graphs.

All of these constructions are algebraic in nature: a graph is defined using a certain algebraic object (e.g., a group). Moreover, the analysis of correctness of the constructions is also algebraic. It relies on the algebraic notion of high connectivity called the *eigenvalue gap* and defined as follows. Consider the adjacency matrix of a given undirected d-regular graph, compute its eigenvalues, and order them according to the absolute value. It can be easily checked that d is the largest value. The difference between d and (the absolute value of) the second largest eigenvalue is the eigenvalue gap. The bigger this eigenvalue gap, the more connected the graph is. From this point of view, a d-regular expander is a graph with the eigenvalue gap at least $\Omega(d)$, i.e., the second largest eigenvalue should be at most some constant fraction of the degree.

A simpler, fully *combinatorial* construction of constant-degree expanders was given by Reingold, Vadhan, and Wigderson [RVW02]. They start with constant-size expander graphs (which can found by brute-force search), and iteratively apply certain graph operations that increase the size of the graph while preserving its expansion property. This way, one can quickly construct an expander graph of any given size.

While the construction of [RVW02] is combinatorial, its analysis is still algebraic and is based on estimating the eigenvalue gap. Alon, Schwartz, and Shapira [ASS07] recently gave a different construction of expanders, which combines algebraically constructed expanders of Alon and Roichman [AR94] with only two applications of a certain graph operation (replacement product), to obtain a constant-degree expander of arbitrary size. They also gave a fully combinatorial analysis of the replacement product operation they used in the second stage of the construction. The full analysis, however, is still algebraic, as it relies on the algebraic construction and the eigenvalue gap analysis of [AR94]. In this respect, the situation in [ASS07] is similar to that in [RVW02] where the analysis of a related graph operation (zig-zag product) can be done in terms of min-entropy, while the analysis of the complete construction is still based on eigenvalues.

1.1 Our results

The main contribution of our paper is a *fully combinatorial analysis* of one of the expander constructions from [RVW02]. Our entire analysis is in terms of edge expansion. Thus, we show that both the construction of expanders and its analysis can be carried out using only elementary combinatorial tools.

The expander construction is also *fully explicit* in the sense that that there is a deterministic polynomial-time algorithm that, given a vertex name v in binary and a number i, outputs the name of the ith neighbor of v in the graph; the full explicitness of our construction follows by the arguments in [RVW02].

1.2 Our motivation

While this fully combinatorial treatment of expanders is interesting in its own right, one of our main motivations for this work comes from bounded arithmetic. Cook and Kolokolova [CK03, CK04] show that it is possible to prove the closure under complementation of the complexity classes P and NL using only reasoning formalizable within these classes; that is, the proofs of closure use only concepts definable within P and NL, respectively. However, it is not known how to formalize the proof of closure for SL [NTS95] within the SL reasoning, since the proof involves AKS sorting networks [AKS83] and formalizing AKS requires proving properties of expander graphs. Reingold's proof of $\mathsf{SL} = \mathsf{L}$ [Rei05] also relies on expanders, and so formalizing his proof also necessitates the formalization of certain properties of expanders (such as the change in expansion with powering and replacement products). The same holds for the derandomized squaring argument of Rozenman and Vadhan [RV05]; there one also needs bounds on the change in expansion, expressed in terms of the eigenvalue gap.

The problem with formalizing such properties of expanders is that, in bounded arithmetic, it is difficult to talk about spectral gap and about eigenvalues in general. It would be much more natural and easier to use the combinatorial notions of expansion such as vertex or edge expansion instead of the spectral gap, since those combinatorial notions can be defined directly in the language of bounded arithmetic. A graph has vertex expansion ϵ if $S \subseteq V \wedge |S| \leq n/2 \rightarrow |\Gamma(S)|/|S| \geq \epsilon$ holds for arbitrary S, where $\Gamma(S)$ is the set of neighbors of S; its edge expansion is defined similarly (the formal definition is given in Section 2 below).

Expander graphs have been used in many recent important results. From the point of view of bounded reverse mathematics, the area that tries to pinpoint the minimal reasoning power needed to prove mathematical theorems, it is very interesting to understand what is the complexity of reasoning required to prove properties of expander graphs, and thus what is the the complexity of reasoning in expander-based proofs such as the known proofs of $\mathsf{SL} = \mathsf{L}$ [Rei05, RV05]. This paper makes a step towards a combinatorial proof of $\mathsf{SL} = \mathsf{L}$ by providing a fully combinatorial analysis of the expander graph construction.

1.3 Relation to previous work

The expander graph construction that we analyze is a variant of the iterative construction of expanders given in [RVW02]. The idea is to start with a constant-size expander graph (found, say, by exhaustive search), and iteratively increase the size of the graph while keeping its expansion larger than some universal constant. The notion of expansion used by [RVW02] is in terms of the eigenvalue gap. To analyze the expansion of the final graph, Reingold, Vadhan, and Wigderson [RVW02] bound the effect of the graph operations they used (graph powering, graph tensoring, and zig-zag product) on the second largest eigenvalue of the adjacency matrix of the resulting graph. The analysis of graph powering (where an edge of the kth power of a graph G is a walk of length k in G) and graph tensoring (where an edge of the tensor product of G and H consists of a pair of edges, one from G and one from H) is immediate from the basic linear algebra. The analysis of the zig-zag product (a way to compose a graph G with a graph H so that the new graph has the degree of H) is technically the most difficult part of the algebraic analysis of the expander construction in [RVW02].

In [ASS07], a graph product replacement (closely related to the zig-zag product) is analyzed in terms of edge expansion, avoiding any mention of the eigenvalue gap. Since product replacement can be used instead of zig-zag product in an iterative expander construction along the lines of [RVW02], this gives a combinatorial analysis of the part of the expander construction. In order to make the entire analysis combinatorial, it suffices to analyze graph powering and graph tensoring also in terms of edge expansion. This is exactly what we do in the present paper.

Our combinatorial analysis of graph tensoring, though subtle, is not very difficult. For the analysis to go through, it turns out necessary to work with graphs that have sufficiently many self-loops around every vertex (at least half the degree). On the other hand, graph powering is much more difficult to analyze combinatorially. Fortunately, here we were able to use the result of Mihail [Mih89] who gave a combinatorial analysis of the mixing time of random walks on expanders in terms of edge expansion. Interestingly, for her proof, she also had to work with graphs that have many self-loops around every vertex. Finally, using Mihail's bounds, we are able to conclude the analysis of graph powering in terms of edge expansion, borrowing some ideas from [AC88].

Remainder of the paper Section 2 contains basic definitions. Our expander construction is defined in Section 3. The analysis of the relevant graph operations in terms of edge expansion is given in Section 4, with the proof of the parameters of our expander construction given in Section 5. Section 6 contains some concluding remarks.

2 Preliminaries

All of our graphs are undirected and regular, possibly with parallel edges and self-loops. The adjacencies of a d-regular graph G are given via its rotation map Rot_G so that, for vertex v of G and an index $i \in [d]$, we have $Rot_G(v,i) = (w,j)$ if w is the ith neighbor of v, and v is the jth neighbor of w; so, in particular, the rotation map induces some fixed numbering of neighbors of a given vertex.

First we define the graph operations that we will use.

- **[Powering]** For a graph $G = (V, E)$ and an integer $k \geq 1$, the kth power G^k is the graph on vertices V where for each walk of length k from a vertex u to a vertex v in G there is an edge $u \sim v$ in G^k.

- **[Tensor product]** For graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, their tensor product $G_1 \otimes G_2$ is the graph on vertices $V_1 \times V_2$, where for every pair of edges $u \sim u'$ in G_1 and $v \sim v'$ in G_2 there is an edge $(u, v) \sim (u', v')$ in $G_1 \otimes G_2$.

- **[Replacement product]** For a D-regular graph $G = (V, E)$ on n vertices and a d-regular graph $H = (V', E')$ on D vertices, the replacement product $G \circ H$ is a $2d$-regular graph on nD vertices $\{(v, i) \mid v \in V,\ 1 \leq i \leq D\}$. The graph $G \circ H$ has the edges $\{(v, i) \sim (v, j) \mid v \in V,\ i \sim j \in E'\}$ as well as, for every edge $v \sim w$ in G such that w is the ith neighbor of v, and v is the jth neighbor of w (i.e., $Rot_G(v, i) = (w, j)$), $G \circ H$ has d parallel edges between (v, i) and (w, j).

- **[Adding self-loops]** For a d-regular graph $G = (V, E)$, the graph $\bigcirc G$ is the $2d$-regular graph obtained from G by adding d parallel self-loops around every vertex of G; note that we count every self-loop around vertex v as one edge $v \sim v$.

For a graph $G = (V, E)$ and a subset $U \subseteq V$ of vertices, we denote by $\bar{U}$ the set $V \setminus U$, and by $E(U, \bar{U})$ the set of edges between U and $\bar{U}$. The *edge expansion* of a d-regular graph $G = (V, E)$ on n vertices is defined as

$$\min_{\emptyset \neq U \subset V,\ |U| \leq n/2} \frac{|E(U, \bar{U})|}{d|U|}.$$

For an n-vertex graph G, its *adjacency matrix* is an $n \times n$ matrix A' whose (i, j)th entry contains the number of edges between vertices i and j in G. For d-regular graphs G, it will be more convenient for us to consider the *normalized adjacency matrix* defined as $\frac{1}{d}A'$. Note that the normalized adjacency matrix A of G is the probability transition matrix for a random walk on G. That is, if π is a probability distribution on vertices of G, then $A\pi$ is the probability distribution induced by one step of a random walk on G starting from a vertex distributed according to π. It is also easy to see that A^k is the normalized adjacency matrix of the graph G^k.

For a vector $v = (v_1, \ldots, v_n)$, its squared ℓ_2-norm is $\|v\|^2 = \sum_{i=1}^n v_i^2$. For two vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$, their inner product is $\langle v, w \rangle = \sum_{i=1}^n v_i w_i$. The *Cauchy-Schwarz inequality* asserts that $(\langle v, w \rangle)^2 \leq \|v\|^2 \|w\|^2$.

We think of vectors as column vectors, and so we use Av to denote the multiplication of a matrix A by a column vector v. However, we will often abuse the notation and also write vA instead of the more proper notation $v^T A$ (where v^T denotes the transpose of v). It will be clear from the context whether a vector is a column or row vector.

3 The expander construction

Here we define the iterative construction of a constant-degree edge expander family that we will analyze using only combinatorial tools. This construction is a variant of the iterative construction given by Reingold, Vadhan, and Wigderson [RVW02].

Let G_0 be a $(2d)$-regular graph of constant size, where d is some constant. Let ϵ_0 be the edge expansion of G_0 such that $\epsilon_0 \geq 1/1296$. Such a graph G_0 exists (by a counting argument) and can be found in constant time, using exhaustive search.

Given this graph G_0, we will define a bigger graph G_1 that is also $(2d)$-regular and has edge expansion at least $1/1296$. In general, given a $(2d)$-regular graph G_i with edge expansion at least $1/1296$, we define G_{i+1} as follows:

$$G_{i+1} = ((\bigcirc((\bigcirc G_i) \otimes (\bigcirc G_i)))^c) \circ H,$$

where c is some constant to be specified later, and H is a d-regular expander graph on $(2(4d)^2)^c$ vertices, with edge expansion at least $1/3$. Again, such a graph can be found using exhaustive search.

Theorem 1. *There is a constant c such that the graph $G_{i+1} = (V_{i+1}, E_{i+1})$ defined from $G_i = (V_i, E_i)$ as above has the following parameters:*

- $|V_{i+1}| = |V_i|^2 D$, *where* $D = (2(4d)^2)^c$,
- *the degree of* G_{i+1} *is* $2d$,
- *the edge expansion of* G_{i+1} *is at least* $1/1296$.

Moreover, there is a deterministic polynomial-time algorithm that, given k (in binary), the name of a vertex v (in binary) of G_k, and an index $i \in [2d]$, outputs the value $Rot_{G_k}(v, i)$ of the rotation map of G_k.

The bounds on the size and the degree of G_{i+1} follow easily from the definitions of the graph operations used to define G_{i+1} from G_i. The lower bound on edge expansion will follow from the lemmas in the subsequent sections that analyze the effect of graph operations on edge expansion. So this part of the proof of Theorem 1 will be given in Section 5.

We now argue the full explicitness of our iterative construction. We need to describe a deterministic polynomial-time algorithm that, given k (in binary), a vertex name v (in binary) of expander graph G_k, and an index $i \in [2d]$, produces the name of the ith neighbor of v (also in binary). This algorithm will be recursive, following the recursive structure of the graph G_k.

First, for any d-regular graph G with rotation map Rot_G, the rotation map of the $2d$-regular graph $\bigcirc G$ is defined as follows:

$$Rot_{\bigcirc G}(v, i) = \begin{cases} Rot_G(v, i) & \text{if } i \leq d \\ (v, i) & \text{if } i > d; \end{cases}$$

that is, we assign the added d self-loops indices between $d+1$ and $2d$. The rotation map of $G \otimes G$ is defined as $Rot_{G\otimes G}((v, w), (i, j)) = ((v', w'), (i', j'))$, where $Rot_G(v, i) = (v', i')$ and $Rot_G(w, j) = (w', j')$. The rotation map of G^c is $Rot_{G^c}(v, (i_1, \ldots, i_c)) = (w, (j_1, \ldots, j_c))$, where w is the vertex reached from v in G by edges $i_1, \ldots, i_c$ using the rotation map Rot_G, and $(j_1, \ldots, j_c)$ describes the same sequence of edges in reverse order from w's point of view (again using Rot_G). Finally, for D-regular graph G and a d-regular graph H on D vertices, the rotation map of the $2d$-regular graph $G \circ H$ is defined as follows:

$$Rot_{G\circ H}((v, i), j) = \begin{cases} ((v, i'), j') \text{ for } Rot_H(i, j) = (i', j') & if j \leq d \\ ((w, i'), j) \text{ for } Rot_G(v, i) = (w, i') & if j > d. \end{cases}$$

We start with our fixed $2d$-regular graph G_0 with some fixed rotation map Rot_{G_0}. We also fix some constant-size d-regular graph H with its rotation map Rot_H. Using the definitions of rotation maps above, we observe that to compute Rot_{G_i} on a given input requires at most $O(c)$ calls to $Rot_{G_{i-1}}$. Unwinding this recurrence, we conclude that a computation of Rot_{G_i} on a given input x can be done in time $|x|^{O(1)}2^{O(i)}$ (since c is a constant).

Now suppose v is an n-bit description of a vertex in G_k, so that G_k is of size about 2^n. Since every application of the tensor product squares the size of the graph, if we start with a constant-size graph G_0, we will get to a graph of size 2^n in $O(\log n)$ iterations. So, we have $k = O(\log n)$. By our observation above, the rotation map of G_k on input v and $i \in [2d]$ is computable in time $n^{O(1)}2^{O(\log n)} = n^{O(1)}$, as required.

4 The effect of graph operations on edge expansion

4.1 Graph powering

Here we analyze the edge expansion of the graph G^k in terms of the edge expansion of G. The main result of this subsection is the following.

Lemma 2. *Let G be a $2d$-regular graph with d self-loops at every vertex. Suppose G has edge expansion ϵ. Then for every integer $k \geq 1$, the powered graph G^k has edge expansion at least $\frac{1-\left(1-\epsilon^2\right)^{k/2}}{2}$.*

Our analysis will be done in two stages. First, we use the result of Mihail [Mih89] that relates the mixing time of a k-step random walk on the graph G to the edge expansion of G, using only combinatorial techniques. Then we relate the edge expansion of G^k to the mixing time of k-step random walk on G, using some ideas from [AC88]. Since both steps use only combinatorial analysis, we get the combinatorial proof of our Lemma 2.

To see why edge expansion is related to the mixing time, consider the following experiment. Let $G = (V, E)$ be a graph on n vertices, and let $U \subset V$ be a subset of at most $n/2$ vertices. Pick a vertex u uniformly at random from U and then pick an edge (u, v) incident to u uniformly at random. Clearly, the probability that v is outside of U is exactly the edge expansion of G. Similarly, if we pick a random vertex u from U and then instead of performing a single step of a random walk we perform k steps, then the probability that we end up at a vertex outside of U is exactly the expansion of U in G^k.

It is well known that the probability distribution induced by taking a random k-step walk on a d-regular graph G tends to the uniform distribution on the vertices of G; the mixing time bounds the distance from the uniform distribution in terms of the number of steps k. Assuming k is large enough so that the random walk ends up in a uniformly distributed vertex of G, we get that the edge expansion of G^k is close to $|\bar{U}|/|V|$, which is at least $1/2$ since $|U| \leq |V|/2$. The closeness of the edge expansion of G^k to $1/2$ is bounded by a function of the mixing time of G.

Mihail [Mih89] gave a combinatorial proof of the following result showing the exponentially fast convergence of a random walk on a regular graph to the uniform distribution.

Lemma 3 ([Mih89]). *Let G be any $2d$-regular graph with d self-loops at every vertex, and with edge expansion ϵ. Let A be the normalized adjacency matrix A of G. Let π be any initial distribution on vertices of G, and let u be the uniform distribution on vertices of G. Then*

$$\|A^k\pi - u\|^2 \leq (1-\epsilon^2)^k\|\pi - u\|^2.$$

The intuition behind the proof of Lemma 3 is the following. Let $e = \pi - u$ be the discrepancy vector between the current probability distribution π and the uniform distribution u. After one step on G, the discrepancy vector becomes $e' = A\pi - u = A\pi - Au = Ae$. The net decrease in the squared ℓ_2-norm of the discrepancy vector $\|e\|^2 - \|e'\|^2$ can be lowerbounded by a quantity proportional to $\sum_{i\sim j\in E}(e_i - e_j)^2$, where the summation is over all edges of G. Note that the net decrease is greater, the more edges there are that connect vertices with significantly different discrepancy values. The high edge expansion of G can then be used to show that there are many such edges in G, and hence one step on G significantly reduces the squared ℓ_2-norm of the initial discrepancy vector. The analysis of a k-step walk follows by induction.

Next we relate the edge expansion of G^k to the mixing time of G. For a subset U of vertices of G, we denote by u_U the probability distribution that is uniform over U, i.e., every vertex in U gets weight $1/|U|$, and every vertex outside of U gets weight 0.

Lemma 4. *Suppose $G = (V, E)$ is a regular graph with normalized adjacency matrix A such that for some $\delta > 0$ the following holds: for every subset $U \subset V$ of size at most $|V|/2$,*

$$\|Au_U - u\|^2 \leq \delta\|u_U - u\|^2,$$

where $u = u_V$ is the uniform distribution on vertices of G. Then G has edge expansion at least $(1 - \sqrt{\delta})/2$.

Before proving Lemma 4, we finish the proof of Lemma 2.

Proof of Lemma 2. Applying Lemma 3 to our $2d$-regular graph G with normalized adjacency matrix A, we get

$$\|A^k u_U - u\|^2 \leq (1 - \epsilon^2)^k \|u_U - u\|^2,$$

for every subset $U \subset V$. Applying Lemma 4 concludes the proof. $\square$

Next we prove Lemma 4.

Proof of Lemma 4. For a given regular graph $G = (V, E)$ on n vertices with normalized adjacency matrix A, estimating its edge expansion is equivalent to lowerbounding the following probability over all nonempty subsets $U \subset V$ of size at most $n/2$:

$$\mathbf{Pr}_{w\in U, w\sim w'\in E}[w' \in \bar{U}].$$

Denoting by u_U the uniform distribution over the set U, and by χ_U the characteristic vector of the set U (i.e., $(\chi_U)_i = 1$ iff $i \in U$), the probability above can be written as

$$\chi_{\bar{U}} A u_U = \langle \chi_{\bar{U}}, Au_U \rangle.$$

This inner product is easy to compute if the distribution u_U were equal to $u = u_V$, the uniform distribution over vertices of G. In that case, the right-hand side of the equality above would simplify to $\langle \chi_{\bar{U}}, Au \rangle = \langle \chi_{\bar{U}}, u \rangle = |\bar{U}|/n$, where we used the easily verifiable fact that $Au = u$ (since G is a regular graph). The latter quantity is at least $1/2$ by our assumption on the size of U being at most $n/2$.

Representing the distribution u_U as the sum of the uniform distribution u and the vector $u_U - u$ yields

$$\langle \chi_{\bar{U}}, Au_U \rangle = \langle \chi_{\bar{U}}, Au \rangle + \langle \chi_{\bar{U}}, A(u_U - u) \rangle = |\bar{U}|/n + \langle \chi_{\bar{U}}, A(u_U - u) \rangle.$$

It remains to upper bound the absolute value of the term $\langle \chi_{\bar{U}}, A(u_U - u) \rangle$.

The idea is to use the Cauchy-Schwarz inequality. However, first we observe that for any two probability distributions π_1 and π_2, and any vector w, the inner product $\langle w, \pi_1 - \pi_2 \rangle$ does not change if we subtract any scalar multiple of the vector u from the vector w. Indeed, for any scalar C, we have $\langle w - Cu, \pi_1 - \pi_2\rangle = \langle w, \pi_1 - \pi_2\rangle - C\langle u, \pi_1 - \pi_2\rangle = \langle w, \pi_1 - \pi_2\rangle$, since the entries in π_1 and in π_2 add up to 1.

So instead of bounding $|\langle \chi_{\bar{U}}, A(u_U - u)\rangle|$, let us upperbound $|\langle \chi_{\bar{U}} - Cu, A(u_U - u)\rangle|$ for suitable C. By Cauchy-Schwarz inequality, this is at most $\|\chi_{\bar{U}} - Cu\| \cdot \|A(u_U - u)\|$. The first factor is the square root of $|\bar{U}|(1-C/n)^2 + |U|C^2/n^2 = |\bar{U}|(n-C)^2/n^2 + |U|C^2/n^2$. If we set $C = |\bar{U}|$, this expression simplifies to $|U||\bar{U}|/n$. Using the assumption of the lemma, the second factor can be upperbounded by $\sqrt{\delta}\|u_U - u\|$. Observing that $\|u_U - u\|^2 = |U|(1/|U| - 1/n)^2 + |\bar{U}|/n^2 = |\bar{U}|/(n|U|)$, we obtain that

$$|\langle \chi_{\bar{U}} - |\bar{U}|u, A(u_U - u)\rangle| \le \sqrt{\frac{|U||\bar{U}|}{n}\frac{\delta|\bar{U}|}{n|U|}} = \sqrt{\delta}|\bar{U}|/n.$$

It follows that

$$\langle \chi_{\bar{U}}, Au_U\rangle \ge |\bar{U}|/n(1 - \sqrt{\delta}) \ge (1 - \sqrt{\delta})/2,$$

since $|\bar{U}| \ge n/2$. □

Remark 5. *The bound on the edge expansion of G^k in terms of the edge expansion of G that we get in Lemma 2 is almost the same as the one would get using the well-known connection between edge expansion and the eigenvalue gap: For a regular graph G with a normalized adjacency matrix A and the second largest eigenvalue λ_2 of A, the edge expansion ϵ of G is such that $(1 - \lambda_2)/2 \le \epsilon \le \sqrt{2(1 - \lambda_2)}$ [Alo86, AM85, Dod84]. Indeed, using the left inequality, we can lowerbound the edge expansion of G^k by $(1 - \lambda_2^k)/2$. Using the right inequality, we get $\lambda_2 \le (1 - \epsilon^2/2)$. So, one gets the edge expansion of G^k to be at least $\frac{1-(1-\epsilon^2/2)^k}{2}$.*

4.2 Tensor product

Here we will show that the tensor product of two regular graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ with edge expansions ϵ_G and ϵ_H, respectively, results in a graph with the edge expansion $\Omega(\min\{\epsilon_G, \epsilon_H\})$. More precisely, we prove the following result (not trying to optimize the constants).

Lemma 6. *Let $G = (V_G, E_G)$ be a d_G-regular graph with $d_G/2$ self-loops at every vertex and $H = (V_H, E_H)$ be a d_H-regular graph with $d_H/2$ self-loops at every vertex. If G has edge expansion ϵ_G and H has edge expansion ϵ_H, then the tensor product graph $G \otimes H$ has edge expansion at least* $0.02\min\{\epsilon_G, \epsilon_H\}$.

Before proving the lemma, we give some intuition. Suppose G is a d_G-regular graph on n_G vertices, and H is d_H-regular graph on n_H vertices. As a "warm-up", consider the special case of a subset of vertices S of the tensor product $G \otimes H$ such that $S = A \times B$. Moreover, assume that $|B| < n_H/2$. Then at least $\epsilon_H d_H|B|$ edges are leaving the set B in graph H. Each of these edges paired up with an edge from A will be an edge leaving $A \times B$ in $G \otimes H$, yielding a total of at least $\epsilon_H d_H|B|d_G|A|$ edges leaving $A \times B$. After normalization (division by $d_G d_H|A||B|$), this yields edge expansion ϵ_H from the set S. In the case, B is larger than $n_H/2$, but A is smaller than $n_G/2$, we can use the edge expansion of A, to obtain the edge expansion at least ϵ_G from S.

For general sets S of vertices in $G \otimes H$, we consider the characteristic matrix of S, which is an $n_G \times n_H$ 0-1 matrix with (i, j)th entry being 1 iff $(i, j) \in S$. We then argue that it is possible to remove some rows or some columns of this matrix so that the resulting matrix has a constant fraction of 1's of the original matrix (i.e., we removed only a constant fraction of vertices from S), and either every row or every column has at most some constant fraction of 1's.

Suppose we have the former case (the other case is treated similarly). That is, we removed some rows of the characteristic matrix of S to obtain a new subset S' that has the form $\{a_1\} \times B_1 \cup \cdots \cup \{a_k\} \times B_k$, where $a_i \in V_G$ and $B_i \subset V_H$, and moreover, each $|B_i|$ is at most some constant fraction of n_H. Then for each B_i, we can use edge expansion of H to argue that ϵ_H fraction of edges from B_i are leaving B_i. Ideally, we would like then to argue that each such edge, when paired up with any edge from vertex a_i, will leave S'. This may not be true, however, as such an edge may go to some vertex in $\{a_j\} \times B_j$. To circumvent this problem, we make an additional assumption that both of our graphs G and H have many self-loops around every vertex (say, half of the degree). In that case, it is easy to argue that each edge leaving B_i in H, when paired up with any self-loop around a_i, yields an edge of $G \otimes H$ that leaves S. Since the number of self-loops around a_i is at least half the degree of G, this yields edge expansion at least $\epsilon_H/2$ from each set $\{a_i\} \times B_i$. Since S' is the union of the pairwise disjoint such sets, we get the edge expansion at least $\epsilon_H/2$ from S'. Finally, since S' contains a constant fraction of vertices from S, we conclude that the edge expansion from S is at least $\Omega(\epsilon_H)$.

We now give the formal proof. We start with a simple averaging result that will allow us to argue the existence of a subset S' of S with required properties.

Claim 7. *Let A be an $n \times m$ 0-1 matrix with δ fraction of ones, for some $0 \leq \delta \leq 1/2$. Then either there is a set of rows containing a total of at least $0.2\delta nm$ ones so that each row contains less than $(5/6)m$ ones, or there is a set of columns containing at least $0.2\delta nm$ ones so that each column contains less than $(5/6)n$ ones.*

Proof. Let a be the number of rows containing at least $5/6$ fraction of ones each, and let b be the number of columns containing at least $5/6$ fraction of ones each. Suppose it is the case that both $a \geq 0.8\delta n$ and $b \geq 0.8\delta m$. For $\alpha = 0.8\delta$, consider an arbitrary subset of exactly αn of such rows, and an arbitrary subset of exactly αm of such columns. The number of ones in the union of these rows and columns is at least $2\alpha nm(5/6)$ minus at most $\alpha^2 nm$ ones that were counted twice. Thus, the total number of ones in A is at least $(5/3)\alpha nm - \alpha^2 nm = \alpha(5/3 - \alpha)nm \geq \alpha(5/3 - 0.4)nm = (38/30) \cdot 0.8\delta nm$, where the last inequality is due to $\delta \leq 1/2$. But, $(38/30) \cdot 0.8 > 1$, contradicting our assumption on the fraction of ones in A being exactly δ.

Hence either a or b must be small. Assume $a < 0.8\delta n$. Then the rows with at least $5/6$ fraction of ones contain less than $0.8\delta nm$ ones from A. This leaves more than $0.2\delta nm$ ones among the rows containing less than $5/6$ fraction of ones each. The case for small b is analogous. □

Proof of Lemma 6. Let S be any subset of vertices of $G \otimes H$ of density $\delta \leq 1/2$. Let A be the 0-1 $n_G \times n_H$ characteristic matrix of S, where $A_{i,j} = 1$ iff $(i, j) \in S$. Apply Claim 7 to A to obtain a submatrix of A containing at least $0.2|S|$ ones in which either each row or each column contains at most $5/6$ fraction of ones. Suppose that the resulting submatrix is obtained by removing some rows of A. That is, the new matrix corresponds to a subset S' of S such that S' is the disjoint union $\{a_1\} \times B_1 \cup \cdots \cup \{a_k\} \times B_k$, for $a_i \in V_G$ and $B_i \subset V_H$, such that $|S'| \geq 0.2|S|$ and the density of every B_i is at most $5/6$.

Observe that every edge leaving B_i in H, when paired with any self-loop of vertex a_i of G, yields an edge of $G \otimes H$ leaving S. Next we lower-bound the number of edges leaving B_i in H. There are two cases to consider. If $|B_i| \leq |V_H|/2$, then there are at least $\epsilon_H d_H |B_i|$ edges leaving B_i. Otherwise, the complement of B_i has density less than half, and so we conclude that the number of edges leaving B_i is at least $\epsilon_H d_H |\bar{B}_i|$. Since $|B_i| + |\bar{B}_i| = n_H$ and $|B_i| < (5/6)n_H$, we obtain that $|\bar{B}_i| > (1/5)|B_i|$. Hence, in both cases, the number of edges leaving B_i is at least $\epsilon_H d_H (|B_i|/5)$.

Thus, for each set $\{a_i\} \times B_i$, we have at least $(d_G/2)\epsilon_H d_H(|B_i|/5)$ edges leaving S. Summing over all i's yields at least $0.1 d_G d_H \epsilon_H |S'| \geq 0.02 d_G d_H \epsilon_H |S|$ edges leaving S in $G \otimes H$, which after normalization gives us edge expansion at least $0.02\epsilon_H$.

The case of S' obtained from S by removing some columns of the matrix A is completely analogous, yielding edge expansion at least $0.02\epsilon_G$. Thus, the edge expansion in $G \otimes H$ is at least $0.02 \min\{\epsilon_G, \epsilon_H\}$, as claimed. □

Remark 8. *Note that the result of Lemma 6 is not true if there are not enough self-loops. For example, consider a bipartite edge expander $G = (L \cup R, E)$ with both sides of the same size. Then the tensor product $G \otimes G$ does not expand at all if one considers the set $(L \times L) \cup (R \times R)$ of half the vertices of $G \otimes G$. Of course, $G \otimes G$ is not connected, but by adding a single self-loop to any vertex of G, we obtain $G \otimes G$ which also does not expand almost at all even though it is connected.*

4.3 Replacement product

The replacement product was combinatorially analyzed in [ASS07].

Lemma 9 ([ASS07])**.** *Let $G = (V_G, E_G)$ be a D-regular graph on n vertices, and let $H = (V_H, E_H)$ be a d-regular graph on D vertices. If G has edge expansion ϵ_G and H has edge expansion ϵ_H, then $G \circ H$ has edge expansion at least $\epsilon_G^2 \epsilon_H/48$.*

The proof idea is to partition a given subset S of vertices of $G \circ H$ into clusters $\{a_1\} \times B_1 \cup \cdots \cup \{a_k\} \times B_k$, where each $a_i \in V_G$ and $B_i \subseteq V_H$. View the clusters that have B_i at most some fraction of $|V_H|$ as light, and the remaining clusters as heavy. For every light cluster, one can use the expansion of H to lowerbound the expansion of B_i (within the copy of H associated with vertex a_i of G). If there are many light clusters, we get a good lower bound on the edge expansion of S. Otherwise, there are many heavy clusters. Using the expansion properties of G, one can argue in this case that there will be many edges between the set of vertices in heavy clusters and the vertices outside S, again obtaining a good lower bound on the expansion of the set S.

For completeness, we give the proof of Lemma 9, along the lines of the proof given in [ASS07].

Proof of Lemma 9. Let $\{1, \ldots, n\}$ be the vertices of G, and let V be the nD vertices of $G \circ H$, where $G \circ H$ has degree $2d$. Let S be a subset of V of size at most $|V|/2$. We view the vertices of $G \circ H$ as partitioned into clusters $C_1, \ldots, C_n$, where each $C_i = \{i\} \times V_H$. We also partition S according to these clusters into *light sets* $L_1, \ldots, L_m$ and *heavy sets* $H_1, \ldots, H_{m'}$, $m + m' \leq n$, where each $|L_i| < (1 - \epsilon_G/4)D$ and $|H_i| \geq (1 - \epsilon_G/4)D$, and each of these sets is obtained as an intersection of S with the corresponding cluster.

For a light set L_i, the number of edges from L_i to $V \setminus S$ is at least $\epsilon_H \epsilon_G d|L_i|/4$. Indeed, either L_i has density at most $1/2$ of its cluster, and then we get that there are at least $\epsilon_H d|L_i|$ edges leaving L_i. Or, the complement of L_i has density at most $1/2$, and we get the number of edges between L_i and its complement within the cluster is at least $\epsilon_H d|\bar{L}_i|$. It can be shown that $|\bar{L}_i| \geq \epsilon_G|L_i|/4$, and so the claim follows.

If $|\bigcup L_i| \geq |S|/6$, then the number of edges between $\bigcup L_i$ and $V \setminus S$ is at least $\epsilon_H \epsilon_G d|S|/24$, and so the edge expansion of S in $G \circ H$ is at least $\epsilon_H \epsilon_G d|S|/(24 * 2d|S|) = \epsilon_H \epsilon_G/48$, as claimed.

Otherwise $|\bigcup H_i| \geq 5|S|/6$. Since each $|H_i| \geq (1 - \epsilon_G/4)D \geq 3d/4$ and $|\bigcup H_i| \leq Dn/2$, we get that $m' \leq 2n/3$. As G expands there are at least $\epsilon_G D \min\{m', n/3\} - \epsilon_G D m'/4 \geq \epsilon_G D m'/4$ vertices in $\bigcup H_i$ that have a neighbor in a cluster that is not heavy. Observe that by the definition of $G \circ H$, all neighbors of vertices in $\bigcup H_i$ in non-heavy clusters are distinct vertices of $G \circ H$.

There are two possibilities. If at least $1/5$ fraction of these neighbors does not belong to $\bigcup L_i$, then there are at least $\epsilon_G Ddm'/20$ edges from $\bigcup H_i$ to $V \setminus S$. Since $m'D \geq |\bigcup H_i| \geq 5|S|/6$, the claimed expansion follows in this case. Otherwise, $|\bigcup L_i| \geq 4\epsilon_G Dm'/20$ (where we used our earlier observation that all the neighbors of $\bigcup H_i$ in other clusters are distinct), and so there are $\epsilon_H \epsilon_G d\epsilon_G Dm'/20$ vertices between $\bigcup L_i$ and $V \setminus S$. Since $m'D \geq 5|S|/6$, the claimed expansion follows in this case as well. $\square$

4.4 Adding self-loops

The following lemma is obvious.

Lemma 10. *If G is a d-regular graph with edge expansion ϵ, then the graph $\bigcirc G$ is $2d$-regular with edge expansion $\epsilon/2$.*

5 Proof of Theorem 1

Now we can finish the proof of Theorem 1. We only need to analyze the edge expansion, which we do by following the sequence of graph operations that define G_{i+1}. Recall that

$$G_{i+1} = ((\bigcirc((\bigcirc G_i) \otimes (\bigcirc G_i)))^c) \circ H,$$

where H has edge expansion at least $1/3$.

Let $\epsilon \geq 1/1296$ be the edge expansion of G_i. First, by Lemma 10, the edge expansion of $\bigcirc G_i$ is at least $\epsilon/2$. By Lemmas 6 and 10, the edge expansion of $\bigcirc((\bigcirc G_i) \otimes (\bigcirc G_i))$ is at least $\epsilon' = 0.005\epsilon$. By Lemma 2, the kth power of this graph has edge expansion at least $\frac{1-(1-(\epsilon')^2)^{k/2}}{2}$. Since ϵ' is at least some constant (independent of the size of G_i), we can choose k to be a sufficiently large constant c so that the edge expansion of the cth power of our graph is at least $1/3$. Finally, by Lemma 9, the edge expansion of the graph G_{i+1} is at least $(1/3)^2 * (1/3)/48 = 1/1296$. This completes the proof.

6 Concluding remarks

One of the motivations for the present work was to try to get a fully combinatorial proof of Reingold's result that undirected graph connectivity is in deterministic logspace [Rei05]. A possible approach was to get a combinatorial analysis of the graph operations used in [Rei05]: graph powering and zig-zag product (or replacement product). The analysis of graph powering given in this paper and the analysis of replacement product given in [ASS07] are not strong enough to achieve that goal.

For a graph G with edge expansion ϵ, our Lemma 2 says that the edge expansion of the graph G^k is approximately $\epsilon^2 k/4$. When ϵ is sub-constant (as may be the case when we start with an arbitrary, not necessarily expander, graph G) but k is constant (which is necessary for Reingold's algorithm), this does not show any improvement in the edge expansion of G^k. Similarly, Lemma 9 only says that the edge expansion of $G \circ H$ is at least $\Omega(\epsilon^2)$ (assuming H is a good edge expander). Since the point of applying replacement product to G and H in (a variant of) Reingold's algorithm would be to create a small-degree graph while not losing more than a constant fraction of the edge expansion of G, Lemma 9 does not help in the case of graphs G with sub-constant edge expansion.

So it is still an interesting open problem whether Reingold's result [Rei05] can be proved using only combinatorial tools. Another problem is to tighten the bounds of Lemmas 2 and 9, if possible.

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