On Traversal and Exploration Sequences

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Abstract

Traversal sequences were defined in [AKL+] as a tool for the study of undirected s-t-connectivity. [K1] defines a new variant of traversal sequences, *exploration sequences*, with certain advantages over the earlier notion of traversal sequences. An exact relationship of these two notions was not known.

In this paper we establish a relationship between these two concepts, in particular, we show that universal traversal sequences can be efficiently converted into universal exploration sequences. We also study conversion of universal exploration sequences for d'-regular graphs into universal exploration sequences for d-regular graphs. Further, we also show certain self-correcting properties of traversal and exploration sequences and we propose a candidate for a universal exploration sequence.

1 Introduction

The s-t-connectivity problem has drawn a lot of attention in computer science. One direction of study is aimed at investigating the space complexity of this problem. It is known that the directed version of this problem is complete for nondeterministic log-space, and its undirected version is complete for symmetric log-space (SL), which is defined to be the class of problems log-space reducible to undirected s-t-connectivity. It is conjectured that symmetric log-space is equal to deterministic log-space ($SL = L$). The main evidence supporting this conjecture is (a) the existence of short universal traversal sequences, which were proposed by Cook and studied in [A] and [AKL+], and (b) the success of derandomization techniques which, applied to the random walk algorithm, have shown that SL is in $DSPACE(\log^{4/3} n)$, [NSW, SZ, ATWZ].

A traversal sequence is a sequence which guides a walk in a graph. Aleliunas et al. shows in [AKL+] that with high probability a sequence of length $O(d^2n^3 \log n)$ chosen uniformly at random guides a walk in any undirected d-regular connected graph in such a way that all vertices are visited. A sequence with this property is called a *universal traversal sequence* (UTS). If there is a universal traversal sequence

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for 3-regular undirected graphs constructible in log-space then the s-t-connectivity problem is in log-space, hence, $SL = L$ [AKL+].

Explicit constructions of universal traversal sequences were found for various classes of graphs (e.g., 2-regular graphs (*cycles*) [BBK+, B, I, K2], cliques [KPS] and expanders [HW]). The only known explicit constructions of universal traversal sequences of sub-exponential length for 3-regular graphs are based on pseudo-random generators for randomized log-space, [BNS, N, INW]. In particular, Nisan's pseudo-random generator gives a universal traversal sequence for 3-regular graphs of length $n^{O(\log n)}$.

Traversal sequences seem to have certain drawbacks. For instance, the explicit constructions of the universal traversal sequences even for such simple graphs as cycles and cliques are non-trivial. In fact, existing super-linear lower bounds on the length of universal traversal sequences (see [BT, D] for overview of these lower bounds) suggest that this ought to be the case.

[K1] proposed a different notion of traversal sequences, *exploration sequences.* Exploration sequences share many useful properties with traversal sequences, in particular, they give rise to universal exploration sequences. [K1] presents simple constructions of polynomial-length universal exploration sequences for previously studied classes of graphs. These universal exploration sequences are even shorter than the lower bounds on the length of corresponding universal traversal sequences. It is also shown there that many results on traversal sequences translate into results for explorations sequences, including the constructions of universal traversal sequences for 3-regular graphs that are based on pseudo-random generators.

Although traversal and exploration sequences are closely related no exact relationship was established, yet. The known results only suggested that universal exploration sequences might be easier to construct than universal traversal sequences. But is that really the case? In this paper we indicate that this is indeed the case. We prove here that universal traversal sequences can be efficiently converted into universal exploration sequences. Hence, constructing universal exploration sequences is at most as hard as constructing universal traversal sequences. Beside this result our technique allows us to convert universal exploration sequences for d'-regular graphs into universal exploration sequences for d-regular graphs.

The conversion of traversal sequences to exploration sequences may come as a surprise since traversal sequences are memoryless whereas exploration sequences have memory (for instance, they can backtrack). Indeed, on an intuitive level our construction shows that traversal sequences do, in some sense, have "memory" of where they have been – at least on certain graphs. We exploit that in our construction.

It remains open, though, whether universal exploration sequences can be converted into universal traversal sequences. The upper bounds of [K1] and lower bounds of [D] imply that such a conversion of universal exploration sequences into universal traversal sequences cannot be done in linear-time.

In Section 4 we study a candidate for a universal exploration sequence, a starting segment of the infinite concatenation of all the strings in $\{0,1\}^*$ under the lexicographical order. We conjecture that this sequence is indeed a universal exploration sequence and we support our conjecture by experimental evidence. Our experiments leads us to proving a theorem that regards certain self-correcting properties of universal traversal and exploration sequences.

2 Definitions

We will need the following definitions. Let $G(V, E)$ be a graph. For a vertex u of G, the degree of vertex u is defined to be $\deg(u) = |\{v;\ (u, v) \in E\}|$. A *labeling* l of G is a map assigning to every edge $(u, v) \in E$ a label $l(u, v) \in \{0, \ldots, \deg(u)-1\}$, so that for every two distinct edges $(u, v), (u, w)$ incident to u, $l(u, v) \neq l(u, w)$. Labeling l is *consistent* if for every vertex v, and every two distinct edges $\{v, u\}, \{v, w\}$, $l(\{v, u\}, u) \neq l(\{v, w\}, w)$. We are mostly concerned with undirected graphs, i.e., graphs where $(u, v) \in E$ implies $(v, u) \in E$.

Here we define notions of traversal and exploration sequences. Traversal sequences and exploration sequences are very similar. The difference between them is how the edge labels given by the sequences are interpreted. In the case of traversal sequences the label is interpreted absolutely as an edge label; in the case of exploration sequences there is a cyclic ordering of the edges at every vertex and the label is interpreted as a relative shift with respect to the edge by which we entered a given vertex.

We first define traversal sequences. For simplicity, we consider traversal sequences only for undirected graphs where all the vertices are of fixed degree d, i.e., d-regular graphs.

Let $d > 1$ be an integer, $G(V, E)$ be an undirected d-regular graph and l be a labeling of G. Let u, v be vertices of G. We say that $i \in \{0, \ldots, d-1\}$ *takes us from vertex u to vertex v*, if $(u, v) \in E$ and $l(u, v) = i$. We denote this by $u \rightarrow^i v$. For $t \in \{0, \ldots, d-1\}^*$, we extend the notation recursively: if $t = \epsilon$ then $u \rightarrow^t u$, and if $t = t'i$, for $t' \in \{0, \ldots, d-1\}^*$ and $i \in \{0, \ldots, d-1\}$, then $u \rightarrow^t w$, given that $u \rightarrow^{t'} v$ and $v \rightarrow^i w$. We call sequence t a *traversal sequence*. We say that t *visits vertex v starting at u* if there is a prefix t' of t, such that $u \rightarrow^{t'} v$.

Let $\mathcal{G}$ be a class of connected undirected graphs. We say that a traversal sequence t is a *universal traversal sequence for $\mathcal{G}$*, if for every $G \in \mathcal{G}$ and every labeling of G, t visits every vertex in G starting at any vertex u of G.

Next we define exploration sequences. The definition is very similar to traversal sequences although there is a technical difference since exploration sequences walk from an edge to another edge, whereas traversal sequences walk from a vertex to another vertex. We also do not require regularity for graphs traversed by exploration sequences.

Let $G(V, E)$ be an undirected graph and l be a labeling of G. Let (u, v) be an edge of G. We say that $i \in \mathbf{Z}$ *takes us from edge (u, v) to an edge (v, w)*, if $l(v, w) = l(v, u) + i \bmod \deg(v)$. We denote this by $(u, v) \rightarrow^i (v, w)$. (Intuitively, being at an edge (u, v) corresponds to being at v's end of that edge.) For $t \in \mathbf{Z}^*$, we extend the notation recursively: for edges e, f, g, if $t = \epsilon$ then $e \rightarrow^t e$, and if $t = t'i$, for $t' \in \mathbf{Z}^*$ and $i \in \mathbf{Z}$, then $e \rightarrow^t g$, provided that $e \rightarrow^{t'} f$ and $f \rightarrow^i g$. We call sequence t an *exploration sequence*. We say that t *visits a vertex w starting at (u, v)* if there is a prefix t' of t and an edge (w', w) incident to w, such that $(u, v) \rightarrow^{t'} (w', w)$.

Let $\mathcal{G}$ be a class of connected undirected graphs. We say that an exploration sequence t is a *universal exploration sequence for $\mathcal{G}$*, if for every $G \in \mathcal{G}$ and every labeling of G, t visits every vertex in G starting at any edge e of G. Observe than when we consider traversal of a d-regular graph by an exploration sequence we may assume without loss of generality that the exploration sequence is from

$\{0, 1, \ldots, d-1\}^*$.

3 Traversal versus Exploration

This section contains several theorems that regard conversion of universal traversal sequences into universal exploration sequences and also theorems that regard conversion of universal exploration sequences for d'-regular graphs into universal exploration sequences for d-regular graphs. We start with the simple theorem.

Theorem 1 *Let $d, n \geq 3$ be integers. For any traversal sequence $t' \in \{0,1,2\}^*$ that is universal for 3-regular graphs on dn vertices there is an exploration sequence t that is universal for d-regular graphs on n vertices and the length of t is at most the length of t'. Furthermore, t can be computed from t' by TC^0 circuits.*

Proof: In order to prove the theorem we introduce the following graph transformation that will be used also in later proofs. Let G be an undirected graph and l be its labeling. We define a map F_3 that maps (G, l) to (G', l'), where G' is a 3-regular graph and l' is its labeling. Graph G' is obtained from G by replacing every vertex of degree d by a cycle of length d. More precisely, for every vertex v of G, graph G' contains vertices $v_0, v_1, \ldots, v_{\deg(v)-1}$ and for $i = 0, \ldots, \deg(v) - 1$, vertices v_i and $v_{i+1 \bmod \deg(v)}$ are connected in G'(Figure 1). Further, for every edge (u, v) of G, there is an edge $(v_{l(v,u)}, u_{l(u,v)})$ in G'. Labeling l' of G' is defined so that $l'(v_i, v_{i+1 \bmod \deg(v)}) = 1$, $l'(v_{i+1 \bmod \deg(v)}, v_i) = 0$ and $l'(v_{l(v,u)}, u_{l(u,v)}) = 2$.

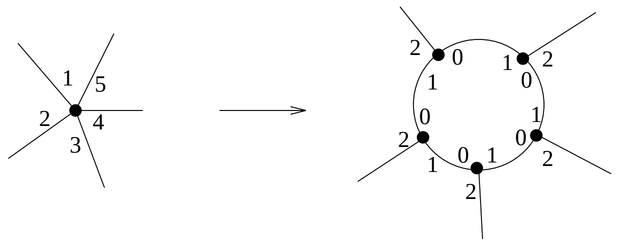

Figure 1:

We define a transformation $\sigma_d : \{0,1,2\}^* \to \{0, 1, \ldots, d-1\}^*$ that transforms traversal sequences into exploration sequences. For $r \in \{0,1\}^*$ define $\mathrm{eval}_d(r) = (|\{j;\ r_j = 1\}| - |\{j;\ r_j = 0\}|) \bmod d$. Let $t' \in \{0,1,2\}^*$ be a traversal sequence and let $t'_1, t'_2, \ldots, t'_{m'} \in \{0,1\}^*$ be such that $t' = t'_1 2 t'_2 2 \cdots 2 t'_{m'}$. Define $\sigma_d(t') = \mathrm{eval}_d(t'_1)\,\mathrm{eval}_d(t'_2) \cdots \mathrm{eval}_d(t'_{m'-1})$. Notice, that $\sigma_d(t')$ can be computed from t' by TC^0 circuits.

Let t' be a universal traversal sequence for d-regular graphs on dn vertices. Set $t = \sigma_d(t')$. We claim that t is a universal exploration sequences for d-regular graphs on n vertices. Let G be any d-regular graph on n vertices and l be its labeling. Let $(G', l') = F_3(G, l)$. Consider the traversal of G by exploration sequence t starting from an edge (u, v) and the traversal of G' by t' starting from $v_{l(v,u)}$. Let $t'_1, t'_2, \ldots, t'_{m'} \in \{0,1\}^*$ be as in the definition of σ_d, i.e., $t' = t'_1 2 t'_2 2 \cdots 2 t'_{m'}$. It is straightforward to prove by induction on k that for $1 \leq k < m'$,

$(u,v) \rightarrow^{\mathrm{eval}_d(t'_1)\cdots\,\mathrm{eval}_d(t'_k)} (z,w)$ if and only if $v_{l(v,u)} \rightarrow^{t'_1 2\cdots t'_k 2} w_{l(w,z)}$. Observe that if t' visits all vertices of G' then t visits all vertices of G. Since G' is a 3-regular graph on dn-vertices, t' visits all vertices of G' starting at any vertex. Hence, t visits all vertices of G starting at any edge. The claim follows. $\square$

We can obtain a more general result.

Theorem 2 *Let $d, n \geq 3$ and $d' \geq 4$ be integers. For any traversal sequence $t' \in \{0,1,\ldots,d'-1\}^*$ that is universal for d'-regular graphs on $(d'+1)dn$ vertices there is an exploration sequence t that is universal for d-regular graphs on n vertices and the length of t is at most the length of t'. Furthermore, t can be computed from t' by a log-space algorithm.*

Proof: The proof is very similar to the previous one, we just need to use a different map $F_{d'}$. For a d-regular graph G and its labeling l define $F_{d'}(G,l) = (G',l')$ as follows. First construct $(G_3, l_3) = F_3(G,l)$, where F_3 is the map from the proof of Theorem 1. For every (undirected) edge (u,v) of G, replace the edge $(u_{l(u,v)}, v_{l(v,u)})$ in G_3 by the graph gadget in Figure 2. By doing so for all the edges of G we obtain a d-regular graph G' on $(d'+1)dn$ vertices.

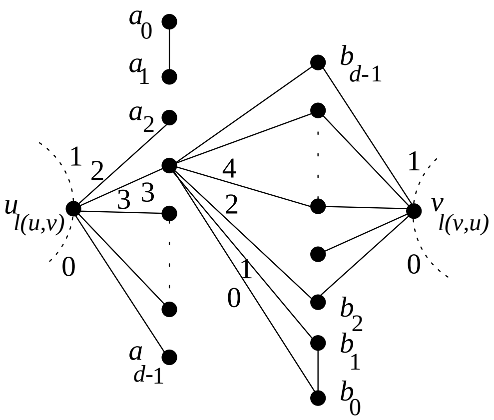

Figure 2:

The gadget in Figure 2 consists of vertices $u_{l(u,v)}$, $v_{l(v,u)}$ and $a_0, \ldots, a_{d'-1}$, $b_0, \ldots, b_{d'-1}$. For $i = 2, \ldots, d'-1$, a_i and $u_{l(u,v)}$ are connected by an edge as well as b_i and $v_{l(v,u)}$, and $l'(a_i, u_{l(u,v)}) = l'(b_i, v_{l(v,u)}) = l'(u_{l(u,v)}, a_i) = l'(v_{l(v,u)}, b_i) = i$. Further, there are edges $(a_0,a_1), (b_0,b_1)$ labeled so that $l'(a_0,a_1) = l'(b_0,b_1) = 0$ and $l'(a_1,a_0) = l'(b_1,b_0) = 1$. Finally, for all $i,j \in \{0,\ldots,d'-1\}$, $i \neq j$, vertices a_i and b_j are connected by an edge and $l'(a_i,b_j) = j$ and $l'(b_j,a_i) = i$. For every vertex v of G and for every $i \in \{0,\ldots,d-1\}$, let $l'(v_i, v_{i+1 \bmod d}) = 1$ and $l'(v_{i+1 \bmod d}, v_i) = 0$.

For a sequence $r \in \{0,\ldots,d'-1\}^*$, let $\mathrm{eval}'_{d'}(r)$ be defined in terms of behavior of r in the gadget of Figure 2, as follows:

$\text{eval}'_{d'}(r) = 0$ if $r = 0$,
$\text{eval}'_{d'}(r) = 1$ if $r = 1$,
$\text{eval}'_{d'}(r) = 2$ if $u_{l(u,v)} \to^r v_{l(v,u)}$ without leaving the gadget,
$\text{eval}'_{d'}(r) = \epsilon$ if $u_{l(u,v)} \to^r u_{l(u,v)}$ without leaving the gadget,
$\text{eval}'_{d'}(r) = \uparrow$ otherwise.

Let traversal sequence t' be $t'_1 t'_2 \cdots t'_{m'}$, where $t'_i \in \{0, \ldots, d'-1\}$. Define $0 = j_0 < j_1 < j_2 < \cdots \le m'$ inductively: $j_0 = 0$ and for $i \ge 0$, let j_{i+1} be the last $j \le m'$ such that $\text{eval}'_{d'}(t'_{j_i+1} t'_{j_i+2} \cdots t'_j) \ne \uparrow$. If there is no such j let j_{i+1} be undefined. Let m be the largest i so that j_i is defined. Define a traversal sequence t_3 to be the concatenation of $\text{eval}'_{d'}(t'_{j_i+1} t'_{j_i+2} \cdots t'_{j_{i+1}})$, for $i = 0, 1, \ldots, m-1$. Clearly, t_3 can be obtained from t' by a log-space algorithm. (In fact, because of the high uniformity of the graph gadget, t_3 can be obtained form t' even by TC^0 circuits.)

Sequence t_3 is in $\{0,1,2\}^*$. Let exploration sequence t be $\sigma_d(t_3)$, where the function σ_d was defined in the proof of Theorem 1. Let (u, v) be an edge of G. Similarly to the proof of Theorem 1, it can be easily verified that if traversal sequence t' visits all vertices in G starting from $v_{l(v,u)}$, then traversal sequence t_3 visits all vertices in G_3 starting from $v_{l(v,u)}$ and further, exploration sequence t visits all vertices of G starting from (u, v). Since, t is a universal traversal sequence for d'-regular graphs on $(d'+1)dn$ vertices, sequence t is a universal exploration sequence for d-regular graphs on n vertices. □

We can generalize Theorem 1 also in the following direction.

Theorem 3 *Let $m \ge 1$ be an integer. For any traversal sequence $t' \in \{0,1,2\}^*$ that is universal for 3-regular graphs on $3m$ vertices there is an exploration sequence $t \in \{-1,0,1\}^*$ that is universal for connected graphs containing m (undirected) edges. The length of t is at most twice the length of t'. Furthermore, t can be computed from t' by AC^0 circuits.*

Proof: The proof is very similar to the proof of Theorem 1. We use a map F'_3 that is similar to map F_3 in the proof of Theorem 1 and that accounts for vertices of degree one and two. Let G be a (possibly irregular) connected graph G on at least two vertices and l be its labeling. F'_3 maps (G, l) to (G', l'), where G' is a 3-regular graph and l' is its labeling. Graph G' is obtained from G by replacing every vertex v of degree $d = \deg(v)$ by a cycle of length $3d$ that consists of vertices $v_0, v_1, \ldots, v_{3d-1}$. For $i = 0, \ldots, 3d-1$, vertices v_i and $v_{i+1 \bmod 3d}$ are connected in G' and $l'(v_i, v_{i+1 \bmod 3d}) = 1$, $l'(v_{i+1 \bmod 3d}, v_i) = 0$. Further, for every edge (u, v) of G, there are edges $(v_{l(v,u)}, u_{l(u,v)})$, $(v_{d+l(v,u)}, u_{\deg(u)+l(u,v)})$, $(v_{2d+l(v,u)}, u_{2\deg(u)+l(u,v)})$, in G', and $l'(v_{k \deg(v)+l(v,u)}, u_{k \deg(u)+l(u,v)}) = 2$, for $k = 0, 1, 2$.

Define $\text{eval}(0) = -1\,0$, $\text{eval}(1) = 1\,0$, $\text{eval}(2) = 0$. Let $t' = t'_1 t'_2 \cdots t'_{m'}$, where $t'_i \in \{0,1,2\}$. Set $t = \text{eval}(t'_1)\,\text{eval}(t'_2) \cdots \text{eval}(t'_{m'})$.

As in the previous proofs one can show that t is a universal exploration sequence for graphs on m (undirected) edges, given that t' is a universal traversal sequence for 3-regular graphs on $3m$ vertices. We leave details of the argument to the reader. □

One can observe that the universal exploration sequence that was obtained in the last theorem visits all the edges of any connected graph on m edges.

The ideas that are contained in the proofs of the previous theorems can be used not only for conversion of universal traversal sequences to universal exploration sequences but also for example for converting universal exploration sequences for d'-regular graphs into universal exploration sequences for d-regular graphs. In particular, the proof of Theorem 1 can be extended to a proof of the following statement.

Theorem 4 *Let $d, n \geq 3$ be integers. For any exploration sequence t' that is universal for 3-regular graphs on dn vertices there is an exploration sequence t that is universal for d-regular graphs on n vertices and the length of t is at most the length of t'. Furthermore, t can be computed from t' by a log-space algorithm.*

Similarly, the proof of Theorem 2 can be used after minor technical modifications to prove the following claim. Since the modifications are rather straightforward and technical we leave details to the reader.

Theorem 5 *Let $d, n \geq 3$ and $d' \geq 4$ be integers. For any exploration sequence t' that is universal for d'-regular graphs on $(d'+1)dn$ vertices there is an exploration sequence t that is universal for d-regular graphs on n vertices and the length of t is at most the length of t'. Furthermore, t can be computed from t' by a log-space algorithm.*

Theorem 3 could be restated in similar way.

4 Self-correcting Properties

In this section we would like to state some conjectures and also show certain self-correcting properties of universal traversal and exploration sequences.

Consider the following experiment. Fix an infinite exploration sequence ω over alphabet $\mathbf{Z}$. Fix a graph G and a starting edge (u, v) in G. Pick a random labeling of graph G. A question: What is the probability p_i that by following ω starting from (u, v) we visit all vertices of G after exactly i steps?

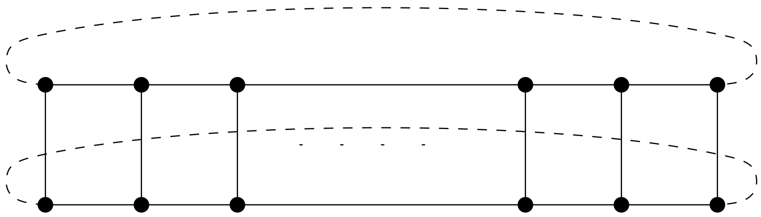

Figure 3: *Ladder* 34, a graph on 2×17 vertices.

Clearly, the probability p_i depends on the sequence ω and also on the graph G. Let ω be the concatenation of all strings in $\{1, 2\}^*$ in lexicographical order. Let us ask the above question about this particular sequence ω and some fixed graph. We do not know how to analyze the induced probability distribution analytically so we performed some experiments. In Figure 4 we plot the probability distribution $\{p_i\}_{i \in \mathbf{N}}$ for the graph in Figure 3.

We performed this experiment also for other graphs (e.g., the two-dimensional grid) and we estimated the probability distribution also for some large graphs by random sampling and we consistently obtained a similar probability distribution.

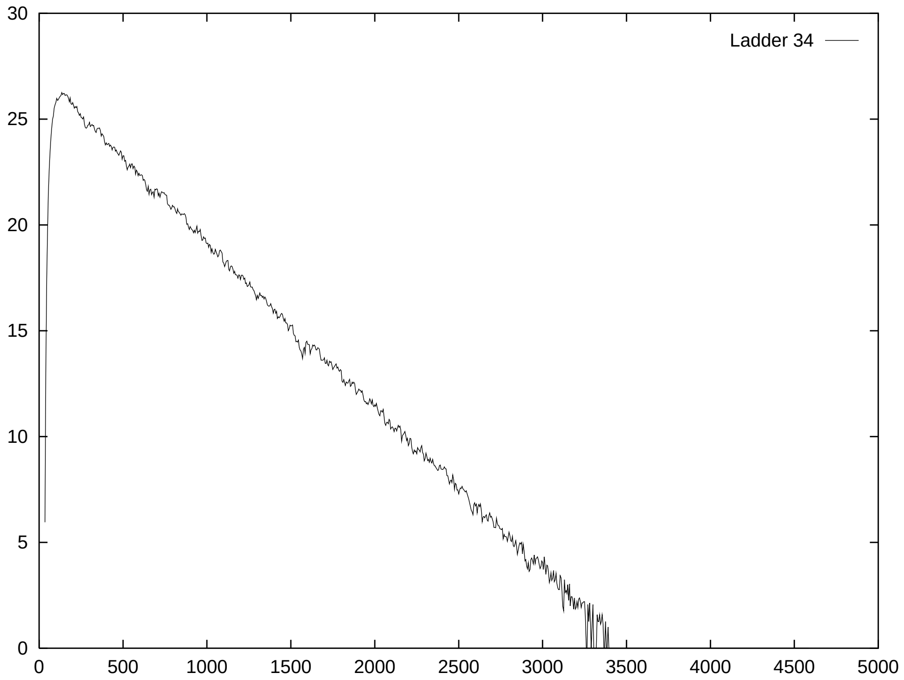

Figure 4: The distribution of p_i for Ladder 34. Horizontal axis: the number of steps i; vertical axis: $\log 2^{34} p_i$. The maximum i such that $p_i > 0$ is 4583 which may not be visible on the graph. (For every vertex there are two possible cyclic orderings of the incident edges hence, in total there are 2^{34} different orderings that were examined.)

The distribution for a graph on n vertices is always tightly concentrated around its mean and it decays exponentially with growing i. The mean of the distribution seems to be approximately $O(n \log n)$ and the maximum non-zero p_i we estimate to be at $i \in O(n^2 \log n)$. The graphs that we have chosen for our experiments were chosen for their large diameter. Clearly, graphs of small diameter (say logarithmic) are easy to traverse.

We believe that our particular exploration sequence ω gives a universal exploration sequence hence, we state the following conjecture.

Conjecture 6 *Let ω be the concatenation of all the strings in $\{1,2\}^*$ in lexicographical order. There exists a polynomial $p(n)$ such that the sequence $t_n = \omega_1 \omega_2 \cdots \omega_{p(n)}$ is a universal exploration sequence for graphs on n vertices.*

The property of sequence ω that we see as relevant is that it does not contain any long subsequence multiple times. No sequence of length $4 \log p(n)$ can appear twice in t_n.

Our experimental results suggest a tight concentration of probability distribution $\{p_i\}_{i \in \mathbf{N}}$ around its mean. However, that may not give enough insight about the behavior of the tail of the distribution and in particular, about the largest i such that $p_i > 0$ which is of our primary interest. Fortunately, the following theorem postulates that the knowledge of the tail of the distribution is not necessary. In fact it gives certain restrictions on the tail of any possible distribution. The theorem could be useful in construction of universal exploration or traversal sequences.

Theorem 7 (Self-correction) *Let ω be an infinite exploration (traversal) sequence. Let $p : \mathbf{N} \to \mathbf{N}$ be a function and $0 < \epsilon \leq 1$ be a constant. If for any $n \geq 4$, any 3-regular connected graph G on n vertices, all labelings of G except for 6^{n-n^ϵ} labelings, and any edge (u,v) of G, the sequence $\omega_1\omega_2\cdots\omega_{p(n)}$ visits all vertices of G starting from (u,v), then $\omega_1\omega_2\cdots\omega_{p(n^{1/\epsilon})}$ is a universal exploration (traversal) sequence for 3-regular graphs on n vertices.*

Proof: The proof is by contradiction. Assume that the hypothesis of the theorem is satisfied but for some n, $\omega_1\omega_2\cdots\omega_{p(n^{1/\epsilon})}$ is not a universal exploration (traversal) sequence for 3-regular connected graphs on n vertices. That means that there is a graph G on n vertices, a labeling l of G and a starting edge (u,v) so that $\omega_1\omega_2\cdots\omega_{p(n^{1/\epsilon})}$ does not visit all vertices of G starting from (u,v). Let w be a vertex of G that is not visited and such that G is still connected when w is removed. Such a vertex clearly exists. Construct a graph G' on $n^{1/\epsilon}$ vertices as follows.

Pick an arbitrary 3-regular connected graph G^f on $n^{1/\epsilon} - n + 2$ vertices. Pick any vertex w^f of G^f. Let (v_1, w), (v_2, w),(v_3, w) be edges incident to w in G and (v_1^f, w^f), (v_2^f, w^f),(v_3^f, w^f) be edges incident to w^f in G^f. Graph G' will have the set of vertices $V(G') = V(G) \cup V(G^f) - \{w, w^f\}$ and the set of edges $E(G') = E(G) \cup E(G^f) \cup \{(v_i, v_i^f), (v_i^f, v_i);\ \ i = 1,2,3\} - \{(v_i, w), (w, v_i), (v_i^f, w^f), (w^f, v_i^f);\ \ i = 1,2,3\}$. Clearly, G' is a connected 3-regular graph on $n^{1/\epsilon}$ vertices. Consider traversal of G' by $\omega_1\omega_2\cdots\omega_{p(n^{1/\epsilon})}$ starting from (u,v) (or (v_i^f, v) if $v_i = u$). There are $6^{n^{1/\epsilon}-n+1}$ labelings of G' that are consistent with l on vertices of G. For none of these labelings does $\omega_1\omega_2\cdots\omega_{p(n^{1/\epsilon})}$ visit all vertices of G', in contradiction to the hypothesis. □

Notice, in the proof we did not put any particular restriction on the graph G^f and hence we could weaken the assumption of the theorem by excluding also a small fraction of graphs from it.

The theorem regards infinite traversal (exploration) sequences. In the usual setting we have a sequence $t_1, t_2, t_3, \ldots$ of traversal sequences, where sequence t_n is used for traversal of graph on n vertices. For the theorem to be applicable in that setting, we can concatenate all the sequences t_n to obtain $\omega = t_1t_2t_3\cdots$. If the length of t_n is upper-bounded by some non-decreasing function $p'(n)$ then we can apply the theorem to ω and $p(n) = np'(n)$. This way we might be losing just a polynomial factor in the length of the universal traversal (exploration) sequence.

We would like to conclude this section with a lesson that we learned from our experiments. The lesson can be put in the following thesis:

The worst case examples are always simple.

As an example we can give the worst case labeling of the graph in Figure 3: the labeling can be concisely described by 11010010000000101-01110111101101011, where each digit corresponds to the orientation of edges at a particular vertex.

(From the perspective of exploration sequences there are just two possible labelings of edges at every vertex, up to an isomorphism.) This phenomenon and the thesis has a natural explanation by means of Kolmogorov complexity: the worst case examples have a very simple algorithmic description. To be precise, the lexicographically first worst case example has a small Kolmogorov complexity, given that the problem for which the example is worst case has a (simple) algorithmic description.

5 Conclusions

We have shown an efficient conversion of universal traversal sequences into universal exploration sequences. It remains open if there is an efficient conversion the other way, i.e., whether one can efficiently convert universal exploration sequences into universal traversal sequences. However, we doubt that there is such a conversion (except the fact that we believe that universal traversal sequences are log-space constructible without any input). It would be also interesting to obtain some analytical results regarding the probability distribution p_i for the sequence ω from Conjecture 6. At this point we do not know how to analyze it. Also, if one cannot prove Conjecture 6, would it be possible to prove it for some larger but sub-exponential function $p(n)$? Such a result would be highly interesting. (For exponential $p(n)$ it holds trivially.) Beside these questions, the major open question is of course the $L = SL$ question.

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