Conference Applications of Mathematics 2013
in honor of the 70th birthday of Karel Segeth.
Jan Brandts, Sergey Korotov, Michal Křížek,
Jakub Šístek, and Tomáš Vejchodský (Eds.),
Institute of Mathematics AS CR, Prague 2013

ON SIMPLICIAL RED REFINEMENT IN THREE AND HIGHER DIMENSIONS

Sergey Korotov[1,2], Michal Křížek[3]

[1] BCAM – Basque Center for Applied Mathematics
Alameda de Mazarredo 14, E–48009 Bilbao
Basque Country, Spain
e-mail: korotov@bcamath.org

[2] IKERBASQUE, Basque Foundation for Science
E–48011, Bilbao, Spain

[3] Institute of Mathematics, Academy of Sciences
Žitná 25, CZ–115 67 Prague 1, Czech Republic
e-mail: krizek@math.cas.cz

Abstract

We show that in dimensions higher than two, the popular "red refinement" technique, commonly used for simplicial mesh refinements and adaptivity in the finite element analysis and practice, never yields subsimplices which are all acute even for an acute father element as opposed to the two-dimensional case. In the three-dimensional case we prove that there exists only one tetrahedron that can be partitioned by red refinement into eight congruent subtetrahedra that are all similar to the original one.

1. Introduction

In his speech at the International Congress of Mathematicians in Paris in 1900, David Hilbert formulated 23 open problems for the 20th century (see [22]). His 18th problem is concerned with tiling space with congruent polytopes [19]. Up to now, we do not know all space-filler polytopes.

In 1923, D. M. Y. Somerville in [21] discovered a special tetrahedral space-filler (which is now called after him the *Sommerville tetrahedron* T_1) having two opposite edges of length 2 and the other four of length $\sqrt{3}$ (see Figure 1). Thus, its mirror image is again T_1. Two of its dihedral angles at edges are right and the other four are $60°$. In Theorem 1 below we show that T_1 is the only one tetrahedron up to similarity (i.e., rotation, translation, and dilatation, but no mirroring) that can be partitioned into 8 congruent subtetrahedra that are all similar to T_1 using a special technique which is called *red refinement* in the numerical analysis community. In such a partition all faces of T_1 are divided by midlines (cf. Figure 3). The tetrahedron T_1 can similarly be partitioned into $27, 64, 125, \dots$ congruent subtetrahedra [13], but in

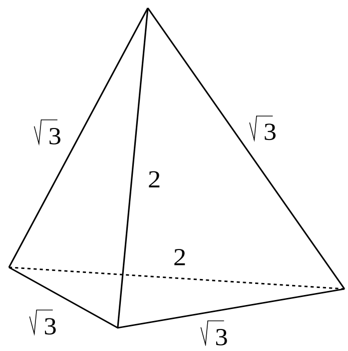

Figure 1: Sommerville tetrahedron T_1.

this work we shall only consider partitions which use the midpoints of edges (for any dimension, i.e. not only for $n = 3$).

For any $n \geq 1$ the convex hull of $n + 1$ points in $\mathbf{R}^n$ that do not lie in one hyperplane is called *n-simplex.* According to [7, p. 231], it is not known whether there exists a 4-simplex that would induce a monohedral tiling of $\mathbf{R}^4$, in general, not face-to-face. In Theorem 3 we prove that no 4-simplex has only Sommerville tetrahedral facets. In this paper we shall consider only face-to-face simplicial partitions of a given n-simplex $S \subset \mathbf{R}^n$, $n = 1, 2, \ldots$, see [3, 4].

If a domain is subdivided into congruent simplices, then we may calculate more easily entries of the stiffness matrix in the finite element method. This saves a lot of CPU time and moreover, some superconvergence phenomena can be achieved [14].

2. Red refinement

First, we will define "red refinement" of a simplex in higher dimension by a technique due to Freudenthal [9]. The term "red refinement" seems to appear first in [1] for two-dimensional triangulations. The regularity of a family of red refinements is investigated in [15] and [23].

The unit hypercube $K = [0, 1]^n$ can be partitioned into $n!$ simplices of dimension n defined as

$$S_\sigma = \{x \in \mathbf{R}^n \,|\, 0 \leq x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \leq 1\}, \tag{1}$$

where σ ranges over all $n!$ permutations of the numbers 1 to n.

The unit hypercube K can also be trivially partitioned into 2^n congruent sub-hypercubes. Each of the sub-hypercubes can be thus partitioned into $n!$ simplices as in (1). This will result in a face-to-face partition of K into $n!2^n$ subsimplices. All the subsimplices that are contained in the reference simplex

$$\hat{S} = \{x \in \mathbf{R}^n \,|\, 0 \leq x_1 \leq \cdots \leq x_n \leq 1\} \tag{2}$$

form a face-to-face partition which will be called to form the *red refinement* of $\hat{S}$. In this case the permutation σ is identity. The partition contains 2^n subsimplices (see Figure 2 for $n = 3$).

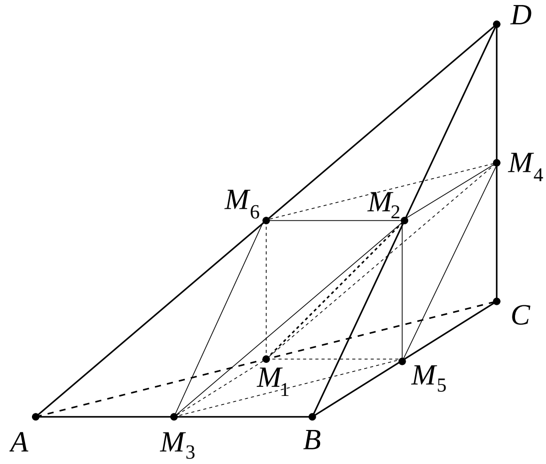

Figure 2: The red refinement of the reference simplex $\hat{S}$.

Definition 1 *Given an arbitrary n-simplex S, the reference n-simplex $\hat{S}$ can be mapped onto S by an affine transformation F. The 2^n subsimplices that form a red refinement of $\hat{S}$ are then mapped by F onto 2^n subsimplices in S, and we will call such a partition as a "red refinement" of S.*

It is clear that the above defined "red refinement" coincides with usual red refinements of triangles and tetrahedra (cf. [1, 13, 17] and Figure 3).

Remark 1 Because of possible permutations of simplex vertices, the red refinement of a given simplex is not uniquely determined except for the case $n = 1, 2$. For example, in the three-dimensional case we have 3 different possibilities how to perform a red refinement, since there are three possibilities to insert a new (interior) edge connecting the midpoints of two opposite edges (cf. [13]). To see this we denote the vertices of the reference tetrahedron $\hat{S}$ by $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (1, 1, 0)$, and $D = (1, 1, 1)$ and let $M_1, \ldots, M_6$ be midpoints of its edges as marked in Figure 2. Now define the affine mapping $F : \hat{S} \to \hat{S}$ so that $F(A) = A$, $F(B) = C$, $F(C) = B$, and $F(D) = D$. Then the line segment M_1M_2 is mapped onto the line segment M_3M_4 yielding a different red refinement of the simplex $\hat{S}$ with the above permutation of vertices. Similarly we can define another affine transformation that maps M_1M_2 to M_5M_6.

Subsimplices that share a vertex with the original simplex are called *exterior* or *corner simplices.*

Remark 2 The $n + 1$ corner subsimplices are obviously similar to the original simplex S for any dimension n. Since F is affine, the volume of each subsimplex in the red refinement is $2^{-n}\mathrm{vol}(S)$ and for each red refinement of S the associated refinements of its lower-dimensional facets are also red. According to [2], the red refinement algorithm produces at most $\frac{n!}{2}$ congruent classes for any initial n-simplex, no matter how many subsequent refinements are performed (see also [23] for $n = 3$). Then the corresponding family of partitions is strongly regular in the sense of Ciarlet [6].

Remark 3 The red refinement of an arbitrary triangle produces only congruent subtriangles. However, the next theorem shows that is not true in the three-dimensional case.

Theorem 1 *There exists only one type of a tetrahedron T (up to similarity) whose red refinement produces eight congruent subtetrahedra similar to T. It is the Sommerville tetrahedron T_1.*

Proof: Let us consider such a tetrahedron T that its red refinement produces eight congruent subtetrahedra similar to T. Its faces are obviously partitioned into four congruent subtriangles. The four exterior subtetrahedra and the four interior subtetrahedra obtained by plane cuts passing through the midlines of its faces are shown in Figure 3. We show that T is similar to the Sommerville tetrahedron T_1.

Let o be the operator that assigns to a given edge of any tetrahedron its opposite edge and let us denote by a, b, c, d, e, f the edges of the front exterior subtetrahedron such that (see the lower part of Figure 3)

$$o(a) = b, \quad o(c) = d, \quad o(e) = f.$$

Parallel edges of the same length are denoted, for simplicity, by the same letters.

The exterior corner subtetrahedra are obviously similar to the original tetrahedron T. Denote by g the inner edge that is surrounded by all four interior subterahedra.

Consider the right interior and right exterior subtetrahedra. Their five edges are a, b, c, d, e. Since these two subtetrahedra are congruent, the remaining sixth edges must have the same length, i.e., $|f| = |g|$. Similarly, for the lower interior and lower exterior subtetrahedra we find that $|e| = |g|$.

Since the regular tetrahedron cannot be divided into eight congruent subtetrahedra, at least two edges of T have a different length. Without loss of generality, we may assume that $|a| \neq |e|$, since e, f, and g are in all cases opposite edges (otherwise we rename the edges a, b, c, and d).

Now consider four cases:

1. Let $|a| \notin \{|b|, |c|, |d|\}$. From the right exterior, right interior, and the lower interior subtetrahedron we see that $o(a) = b$, $o(a) = c$, and $o(a) = d$. Hence, $|b| = |c| = |d|$, since a is obviously mapped only on a during "congruence mapping". Consider the right interior subtetrahedron. If $|b| = |d| = |e| = |g|$, then the four dihedral angles at these edges have the same size. They cannot be nonacute, since any tetrahedron has at least three acute dihedral angles, see [12, p. 727]. Similarly we find that dihedral angles at g are acute for all four interior subtetrahedra, which is a contradiction. Thus, $|b| = |c| = |d| \neq |e| = |f| = |g|$, but then the right interior and

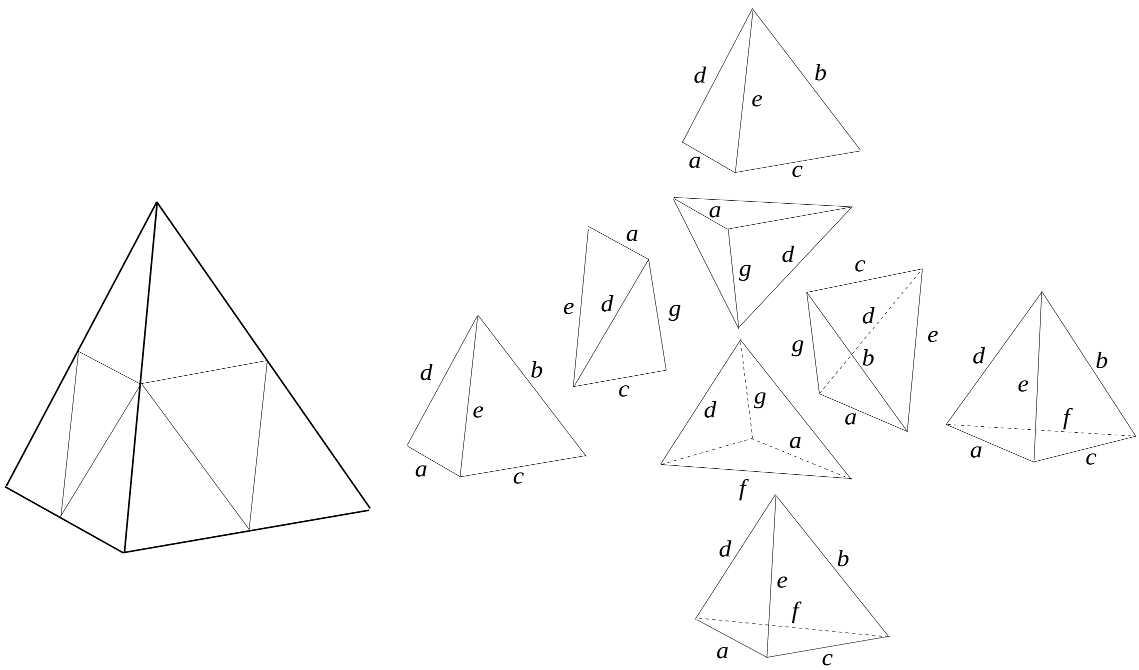

Figure 3: Red refinement of a tetrahedron T by plane cuts through midlines of its faces (left) and its exploded version (right).

right exterior subtetrahedron are not congruent (they are only indirectly congruent up to mirroring), which is a contradiction.

2. So let $|a| = |b|$. Then we easily find that $|b| = |c| = |d|$.

The cases 3. $|a| = |c|$ and 4. $|a| = |d|$ can be treated similarly. Therefore, altogether we obtain

$$|a| = |b| = |c| = |d|, \qquad |e| = |f| = |g|. \tag{3}$$

Due to the mirror image symmetry of T and its eight subtetrahedra, the edge e is perpendicular to the plane passing through the edges f and g. Similarly, the edge f is perpendicular to the plane of symmetry containing e and g. Hence, we find that (see Figure 3)

$$e \perp g \perp f \perp e.$$

Now applying the Parseval equality, we come to

$$(2|a|)^2 = |e|^2 + |g|^2 + |f|^2$$

and thus, (3) implies that

$$2|a| = \sqrt{3}|e|.$$

From this we see that T is the Sommerville tetrahedron T_1 up to similarity (cf. Figure 1) and there is no other type of a tetrahedron that can be partitioned into eight congruent subtetrahedra that are similar to the original one. $\blacksquare$

Red refinement of a tetrahedron that produces congruent subtetrahedra is treated also in [20]. Some authors allow mirroring of congruent tetrahedra. Zhang in [23] presents a different proof of Theorem 1. Dissection of simplices into congruent subsimplices is examined also in [10] and [18].

3. Nonobtuse red refinement

Opposite each vertex of an n-simplex lies a $(n-1)$-dimensional *facet.* For $n = 1$ facets are just points. For $n \geq 1$ the *dihedral angle* α between two facets is defined by means of the inner product of their outward unit normals ν_1 and ν_2,

$$\cos \alpha = -\nu_1 \cdot \nu_2.$$

If $n = 1$ these normals necessarily form an angle of $180°$ and thus $\alpha = 0$. Each simplex in $\mathbf{R}^n$ has $\binom{n+1}{2}$ dihedral angles.

Definition 2 *If all dihedral angles of a given simplex are less than* $90°$ *(less than or equal to* $90°$*) we say that the simplex is acute (nonobtuse).*

For instance, the Sommerville tetrahedron (see Figure 1) is nonobtuse and the regular tetrahedron is acute.

Theorem 2 *If an n-simplex T for $n \geq 2$ is nonobtuse (acute), then any of its lower dimensional facets is also a nonobtuse (acute) simplex.*

For the proof see [8].

Definition 3 *The red refinement is said to be nonobtuse (acute) if all resulting subsimplices are nonobtuse (acute).*

Note that nonobtuse simplicial partitions lead to monotone stiffness matrices when solving elliptic problems by linear finite element methods, see e.g. [5, 11, 16].

Remark 4 We see that the inner diagonal, which is denoted by g in Figure 3 (or M_1M_2 in Figure 2), is surrounded by four tetrahedra. To get a nonobtuse red refinement, it is necessary that all dihedral angles sharing this edge are right. However, another more severe condition comes from the edges, which are denoted by e and f in Figure 3. Here the angle $180°$ is bisected and thus, the corresponding two dihedral angles sharing these edges have to be right. This yields a lot of restrictions on construction of nonobtuse red refinements. For instance, in the red refinement of the regular tetrahedron the dihedral angles at the edge g are all right, but one dihedral angle at edges e and f is greater than $109°$. The red refinement of the (nonobtuse) cube corner terahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, produces angles greater than $125°$ at e and f.

On the other hand, the red refinement of the path simplex yields only path subsimplices in any dimension $n \geq 2$ (cf. Figure 2). The path simplex in its basic position can be stretched or shrinked along any coordinate axis x_i and we still get nonobtuse red refinement. If $n = 3$ then there are six path subtetrahedra T that are congruent with the original path tetrahedron. The remaining two are mirror images of T (see Figure 2 and Remark 2). The red refinement of the Sommerville tetrahedron also produces nonobtuse tetrahedra which follows from Theorem 1. This is due to the fact that the Sommerville tetrahedron is the union of 4 path subtetrahedra. In [12] we introduced the so-called yellow refinement which produces only nonobtuse subtetrahedra provided the initial tetrahedron is nonobtuse and contains the centre of its circumscribed ball.

Remark 5 Consider now a red refinement of a 4-simplex S, i.e., it is partitioned into 16 subsimplices. Then we get a situation which is a little bit difficult to imagine. Namely, we first cut off 5 congruent corner subsimplices that are similar to S. The remaining polytopic domain then has 10 three-dimensional facets and it is partitioned into $16 - 5 = 11$ subsimplices.

Theorem 3 *There is no* 4*-simplex whose three-dimensional facets are all Sommerville tetrahedra.*

Proof: From the well-known Euler-Poincaré formula we find that a 4-simplex has 5 vertices, 10 edges, 10 triangular faces, and there are 5 tetrahedral three-dimensional facets.

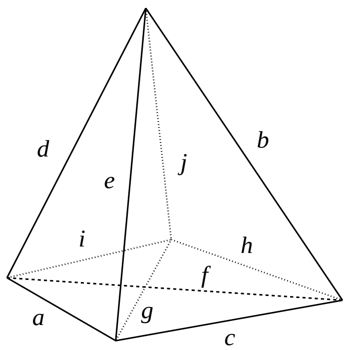

Figure 4: Schematic illustration of a 4-simplex and notation of its edges.

Now we show that there is no 4-simplex whose five facets are all the Sommerville tetrahedra T_1. Suppose to the contrary that such 4-simplex S exists. Denote its 10 edges by $a, b, c, d, e, f, g, h, i, j$ as indicated in Figure 4. Let one of its facets be the Sommerville tetrahedron T_1. Without lost of generality we may assume that its edges satisfy $|a| = |b| = |c| = |d| = \sqrt{3}$ and $|e| = |f| = 2$. Since e is opposite to h and i; and f is opposite to g and j, we get

$$|g| = |h| = |i| = |j| = 2.$$

However, this relation does not allow that all five facets are the Sommerville tetrahedra T_1, since the edges g, h, i, j contain a common point and thus their pairs are not opposite. This is a contradiction. ■

Theorem 4 *The red refinement of an acute simplex for $n > 2$ never yields subsimplices that would be all mutually congruent.*

Proof: Assume, on the contrary, that there exists an acute simplex whose red refinement produces mutually congruent subsimplicies, which should be then, obviously, acute as the exterior subsimplices are always similar to the father simplex. As the red refinement of the simplex implies by induction the red refinement of all its lower-dimensional facets (cf. Remark 2), any of its three-dimensional facets would be partitioned as in Figure 3. But then some nonacute angles between lower-dimensional faces appear, since the inner edge g is surrounded by four tetrahedra. This contradicts by Theorem 2 to the assumption that all subsimplicies are acute. ∎

Remark 6 In fact, from the above proof we observe even a stronger result than the one stated in Theorem 4. The red refinement of n-simplex never produces only acute subsimplices for $n > 2$.

Acknowledgements

The first author was supported by Grant MTM2011–24766 of the MICINN, Spain. The second author was supported by the Institutional Research Plan RVO 67985840. Both authors are thankful to Jan Brandts for fruitful discussions.

References

[1] Bank, R. E., Sherman, A. H., and Weiser, A.: Refinement algorithms and data structures for regular local mesh refinement. In: R. Stepleman (Ed.), *Scientific Computing*, pp. 3–17. IMACS/North Holland, Amsterdam, 1983.

[2] Bey, J.: Simplicial grid refinement: on Freudenthal's algorithm and the optimal number of congruence classes. Numer. Math. **85** (2000), 1–29.

[3] Brandts, J., Korotov, S., and Křížek, M.: The strengthened Cauchy-Bunyakowski-Schwarz inequality for n-simplicial linear finite elements. In: Z. Li et al. (Eds.), *Proc. Third Internat. Conf. Numer. Anal. Appl. (NAA–2004), Rousse, Bulgaria*, pp. 203–210. Springer, 2005.

[4] Brandts, J., Korotov, S., and Křížek, M.: Simplicial finite elements in higher dimensions. Appl. Math. **52** (2007), 251–265.

[5] Brandts, J., Korotov, S., and Křížek, M.: The discrete maximum principle for linear simplicial finite element approximations of a reaction-diffusion problem. Linear Algebra Appl. **429** (2008), 2344–2357.

[6] Ciarlet, P. G.: *The finite element method for elliptic problems.* North-Holland, Amsterdam, 1978.

[7] Debrunner, H. E.: Tiling Euclidean d-space with congruent simplexes. Ann. N. Y. Acad. Sci. **440** (1985), 230–261.

[8] Fiedler, M.: Über qualitative Winkeleigenschaften der Simplexe. Czechoslovak Math. J. **7** (1957), 463–476.

[9] Freudenthal, H.: Simplizialzerlegungen von bescharaenkter. Flachheit. Ann. of Math. in Sci. and Engrg. **43** (1942), 580–582.

[10] Hertel, E.: Self-similar simplices. Beiträge Algebra Geom. **41** (2000), 589–595.

[11] Karátson, J. and Korotov, S.: Discrete maximum principles for finite element solutions of nonlinear elliptic problems with mixed boundary conditions. Numer. Math. **99** (2005), 669–698.

[12] Korotov, S. and Křížek, M.: Acute type refinements of tetrahedral partitions of polyhedral domains. SIAM J. Numer. Anal. **39** (2001), 724–733.

[13] Křížek, M.: An equilibrium finite element method in three-dimensional elasticity. Apl. Mat. **27** (1982), 46–75.

[14] Křížek, M.: Superconvergence phenomena on three-dimensional meshes. Internat. J. Numer. Anal. Model. **2** (2005), 43–56.

[15] Křížek, M. and Strouboulis, T.: How to generate local refinements of unstructured tetrahedral meshes satisfying a regularity ball condition. Numer. Methods Partial Differential Equations **13** (1997), 201–214.

[16] Křížek, M. and Lin, Q.: On diagonal dominance of stiffness matrices in 3D. East-West J. Numer. Math. **3** (1995), 59–69.

[17] Liu, A. and Joe, B.: Quality local refinement of tetrahedral meshes based on 8-subtetrahedron subdivision. Math. Comp. **65** (1996), 1183–1200.

[18] Matoušek, J. and Safernová, Z.: On the nonexistence of k-reptile tetrahedra. Discrete Comput. Geom. **46** (2011), 599–609.

[19] Milnor, J.: Hilbert's problem 18: On crystalographic groups, fundamental domains, and on sphere packing. In: F. E. Browder (Ed.), *Mathematical Developments Arising from Hilbert Problems, Part 2*, pp. 491–506. AMS Providence, Rhode Island, 1976.

[20] Moore, D. and Warren, J.: Adaptive simplicial mesh quadtrees. Houston J. Math. **21** (1995), 525–540.

[21] Sommerville, D. M. Y.: Space-filling tetrahedra in Euclidean space. Proc. Edinb. Math. Soc. **41** (1923), 49–57.

[22] Więsław, W. (Ed.): *Problemy Hilberta* (in Polish). Inst. Hist. Nauk PAN, Warsaw, 1997.

[23] Zhang, S.: Successive subdivisions of tetrahedra and multigrid methods on tetrahedral meshes. Houston J. Math. **21** (1995), 541–556.