Conference Applications of Mathematics 2013
in honor of the 70th birthday of Karel Segeth.
Jan Brandts, Sergey Korotov, Michal Křížek,
Jakub Šístek, and Tomáš Vejchodský (Eds.),
Institute of Mathematics AS CR, Prague 2013

FUZZY SETS AND SMALL SYSTEMS

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Abstract

Independently with [7] a corresponding fuzzy approach has been developed in [3–5] with applications in measure theory. One of the results the Egoroff theorem has been proved in an abstract form. In [1] a necessary and sufficient condition for holding the Egoroff theorem was presented in the case of a space with a monotone measure. By the help of [2] and [6] we prove a variant of the Egoroff theorem stated in [4].

1. Introduction

In [7] the notion of a fuzzy subset A of a space X has been defined as a mapping $A : X \to [0,1]$. Especially, if $A : X \to \{0,1\}$, then A can be identified with a classical set $B \subset X$ by the help of the equality $A = \chi_B$.

Almost at the same time the notion of a set of small measure has been characterized in [3–5] using a sequence $(\mathcal{N}_n)_{n=1}^{\infty}$ of subfamilies of a σ-algebra $\mathcal{S} \subset 2^X$ satisfying the following properties:

(i) $\emptyset \in \mathcal{N}_n, \mathcal{N}_{n+1} \subset \mathcal{N}_n$ for every $n \in \mathbb{N}$,

(ii) if $A \in \mathcal{N}_n, B \in \mathcal{S}$ and $B \subset A$, then $B \in \mathcal{N}_n$,

(iii) if $A, B, C \in \mathcal{N}_n$, then $A \cup B \cup C \in \mathcal{N}_{n-1}$,

(iv) if $A_i \supset A_{i+1}\,(i = 1, 2, \cdots)$ and $\bigcap_i A_i = \emptyset$, then to every $n \in \mathbb{N}$ there is i such that $A_i \in \mathcal{N}_n$.

The classical Egoroff theorem states that if a sequence $(f_n)_n$ of real measurable functions converges to a measurable function f almost everywhere, then it converges almost uniformly, i.e. $\forall \varepsilon > 0\ \exists A \in \mathcal{A}$ such that $\mu(A) < \varepsilon$ and $(f_n)_n$ converges uniformly to f on $X - A$.

Definition. We say that a sequence $(f_n)_n$ converges to f almost everywhere, if $\{x \in X; f_n(x)$ does not converge to $f(x)\} \in \mathcal{N}_n$ for every n. We say that $(f_n)_n$ converges to f almost uniformly, if for any $n \in \mathbb{N}$ there exists $A \in \mathcal{N}_n$ such that (f_n) converges uniformly to f on $X - A$.

2. Egoroff theorem

Theorem. Let $(\mathcal{N}_n)_n$ be a small system of subfamilies of a measurable space $(X, \mathcal{S})$. Let $(f_n)_n$ converges to f almost everywhere. Then $(f_n)_n$ converges to f almost uniformly.

Proof. First we use a result from [6]: If $(\mathcal{N}_n)_n$ satisfies (i)–(iv), then there exists a monotone continuous function $\mu : \mathcal{S} \to [0,1]$ such that

$$\mathcal{N}_n = \{A \in \mathcal{S}; \mu(A) < 3^{-n}\},$$

$n = 1, 2, 3, \ldots$ In [1] the following theorem has been proved: A monotone function $\mu : \mathcal{S} \to [0,1]$ satisfies the Egoroff theorem if and only if it satisfies the following condition (E):

For every double sequence $\left\{E_n^{(m)}\right\}$ of measurable sets which satisfies

$$E_n^{(m)} \searrow E^{(m)} \, (n \to \infty) \, , \quad \mu\left(\bigcup_{m=1}^{\infty} E^{(m)}\right) = 0$$

there exist increasing sequences $\{n_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$ of natural numbers such that

$$\lim_{k\to\infty} \mu\left(\bigcup_{i=k}^{\infty} E_{n_i}^{(m_i)}\right) = 0.$$

We are going to prove that the monotone continuous set function μ satisfies condition (E). Let $\left\{E_n^{(m)}\right\}$ is double sequence of measurable sets for which

$$E_n^{(m)} \searrow E^{(m)} \, (n \to \infty) \, , \quad \mu\left(\bigcup_{m=1}^{\infty} E^{(m)}\right) = 0.$$

From the monotonicity it follows that

$$0 = \mu(\emptyset) \leq \mu\big(E^{(m_0)}\big) \leq \mu\left(\bigcup_{m=1}^{\infty} E^{(m)}\right) = 0.$$

We have proven that $\mu\big(E^{(m)}\big) = 0$ for arbitrary m. From this it follows that there is a natural number n_1 for which

$$\mu\big(E_{n_1}^{(1)}\big) \leq \frac{1}{3}.$$

Similarly there is a number $n_2 > n_1$ for which

$$\mu\big(E_{n_2}^{(2)}\big) \leq \frac{1}{3^2},$$

etc. Putting $m_i = i$, we get

$$\mu\left(\bigcup_{i=k}^{\infty} E_{n_i}^{(m_i)}\right) \le \sum_{i=k}^{\infty} \frac{1}{3^i} = \frac{\frac{1}{3^k}}{1-\frac{1}{3}} = \frac{1}{2 \cdot 3^{k-1}}.$$

From this it follows that

$$\lim_{k\to\infty} \mu\left(\bigcup_{i=k}^{\infty} E_{n_i}^{(m_i)}\right) = 0.$$

□

3. Conclusion

We presented a new proof of the Egoroff theorem for small systems [4]. It follows from a representation theorem in [6] and the Egoroff theorem for monotone measures in [2].

Acknowledgements

This paper was supported by Grant VEGA 1/0621/11.

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