The modified Boltzmann equation

& multi-component thermalization

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MTA Wigner RCP

ISS25 Prague 2013 – Understanding hot & dense QCD matter

Outline

- Few things about the Boltzmann equation
- Why do we need modifications?
- A toy model for modified BE
- Long-time behaviour in the case of two components

What is BE & why do we like it?

time evolution of an ensemble

$$\dot{f}_1 = \int_{234} \mathcal{W}_{1234}(f_3 f_4 - f_1 f_2)$$

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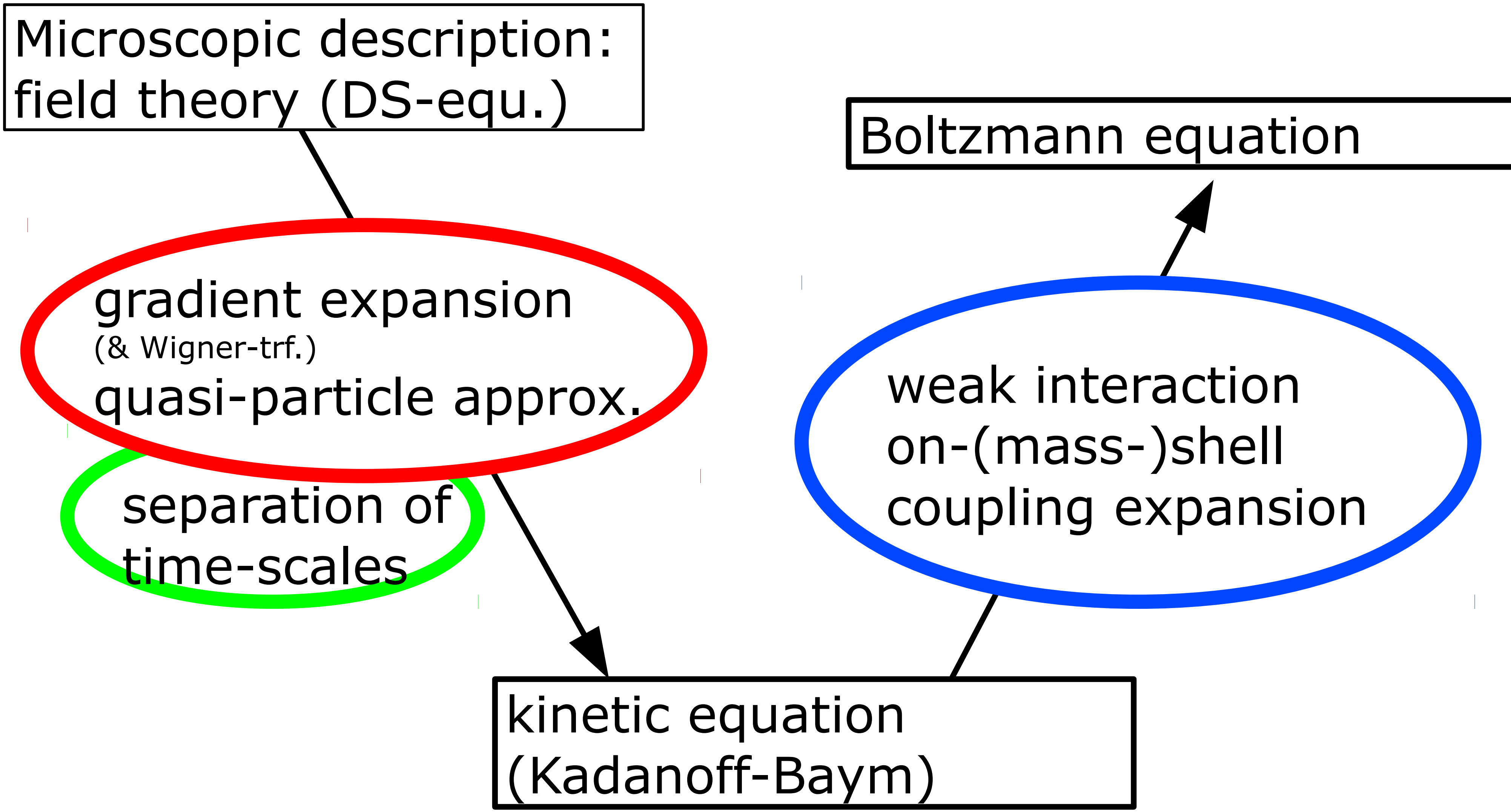
phenomenology

relatively easy to solve numerically

Where does a BE come from?



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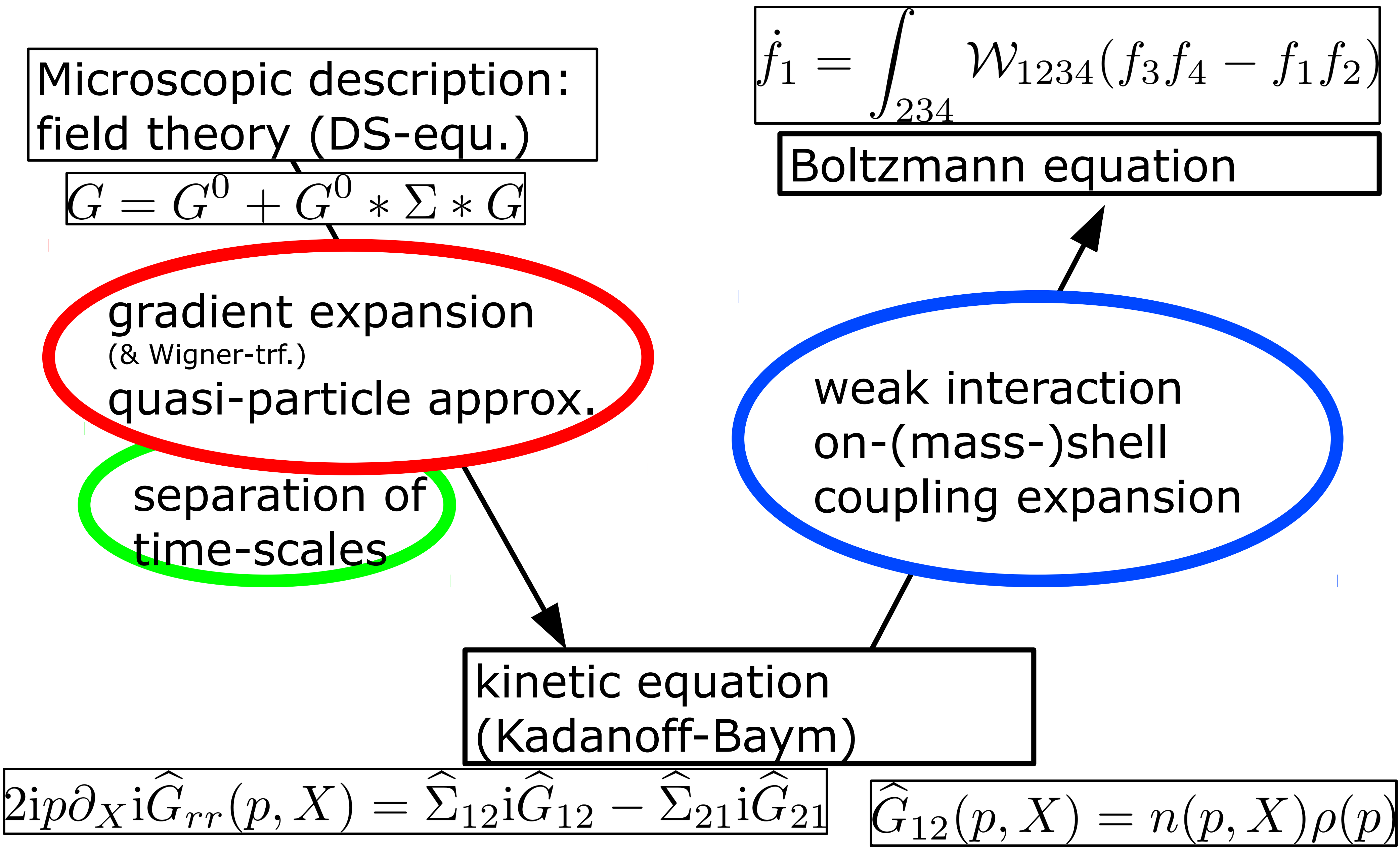

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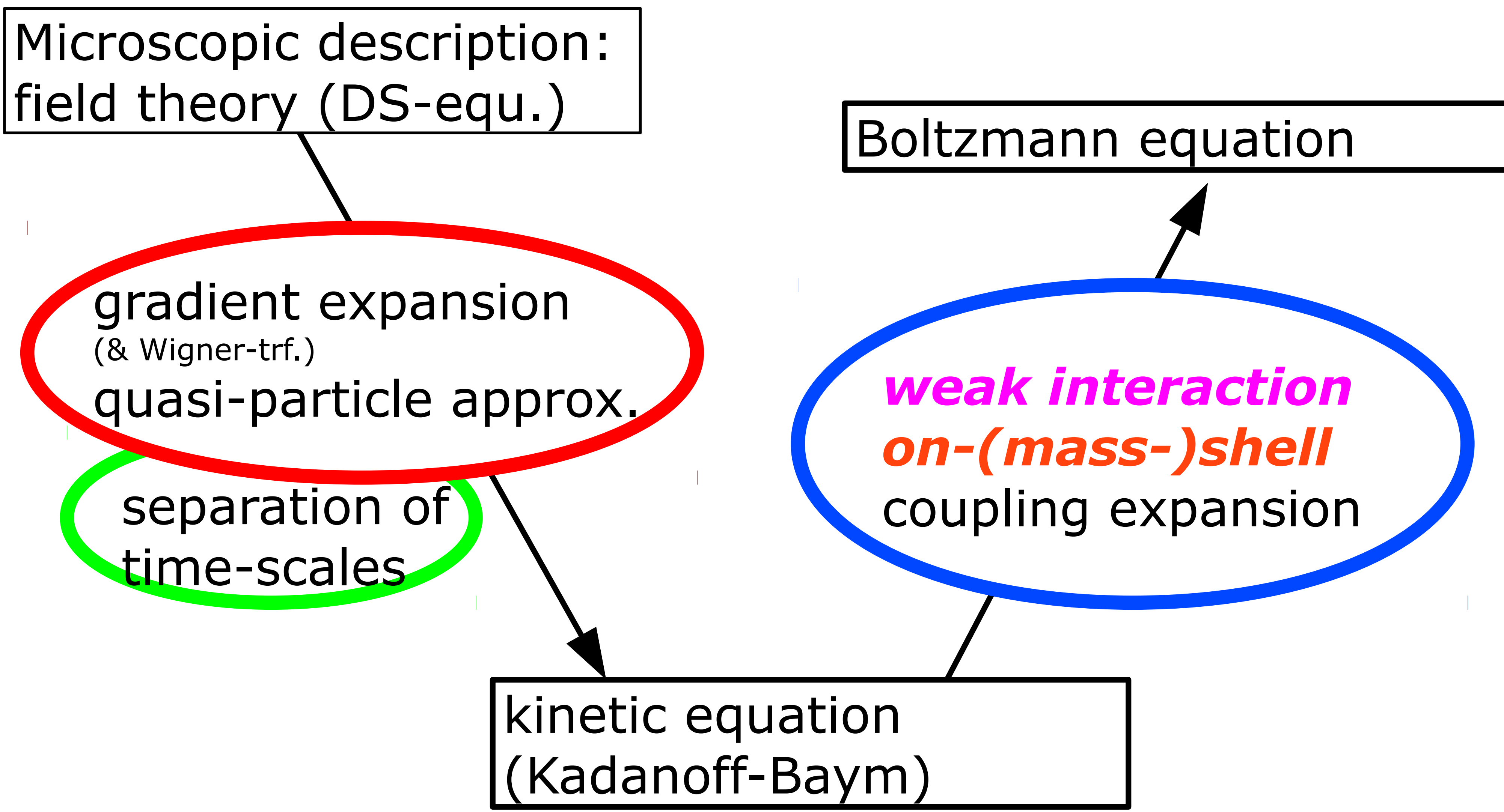
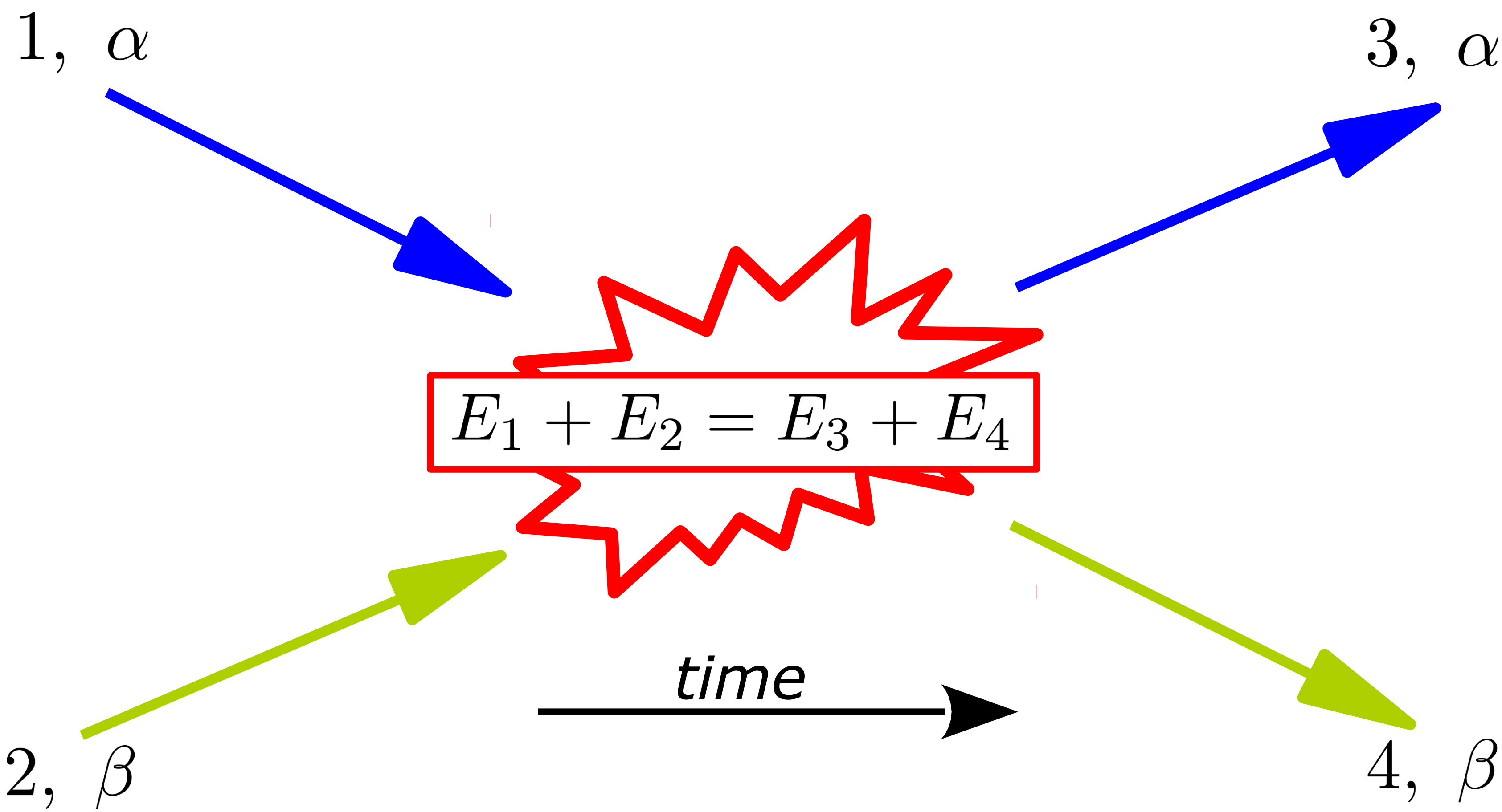


Modifying the Boltzmann equation



$$\dot{f}^{\alpha}(\mathbf{p}_1) = \sum_{\beta} \int_{234} \delta(\mathbf{p}_3 + \mathbf{p}_4 - \mathbf{p}_1 - \mathbf{p}_2)\delta(E_3 + E_4 - E_1 - E_2)\times$$
$$\times \mathcal{W}^{\alpha\beta}_{1234}(f^{\alpha}(\mathbf{p}_3)f^{\beta}(\mathbf{p}_4) - f^{\alpha}(\mathbf{p}_1)f^{\beta}(\mathbf{p}_2))$$

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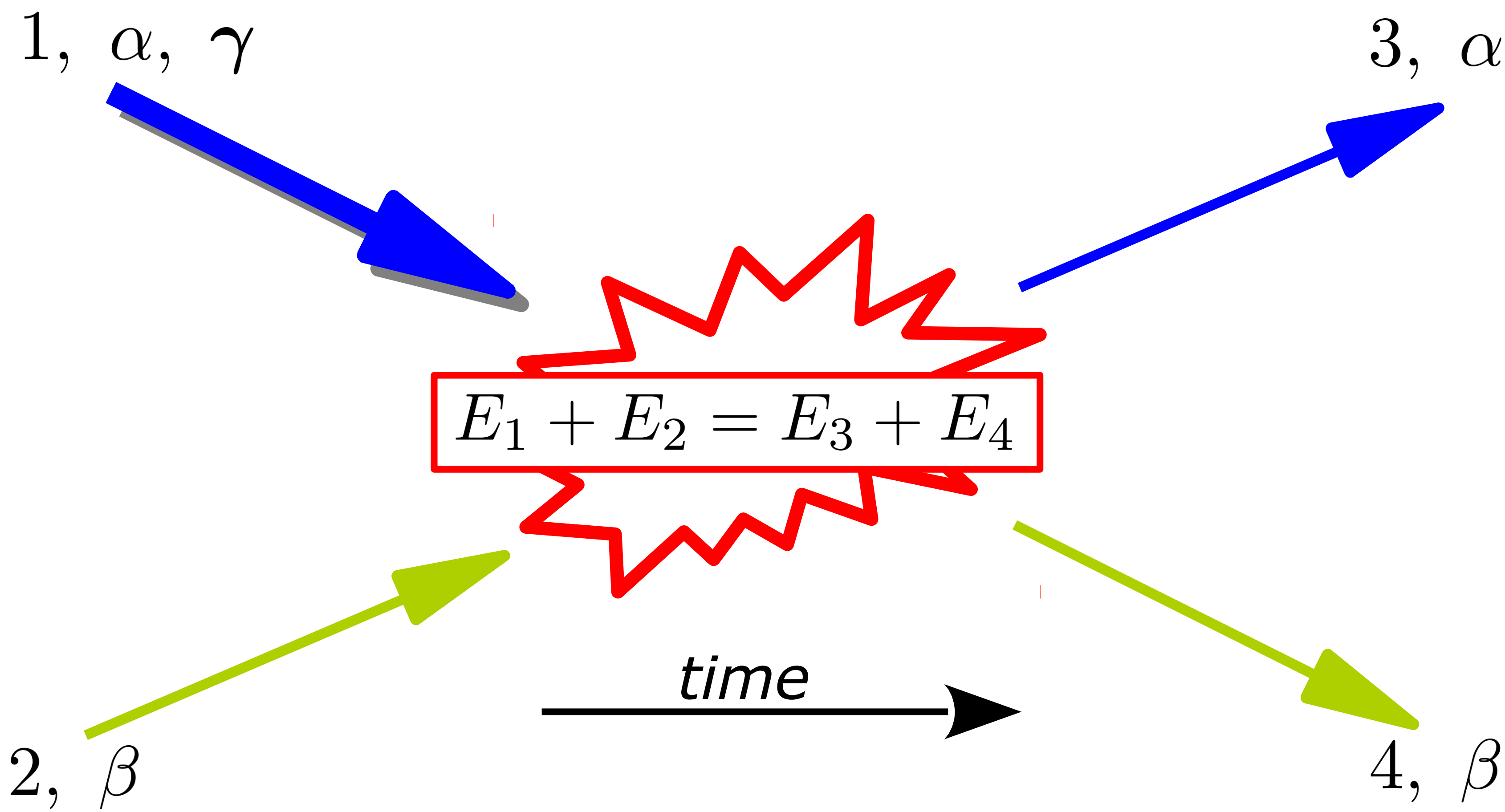

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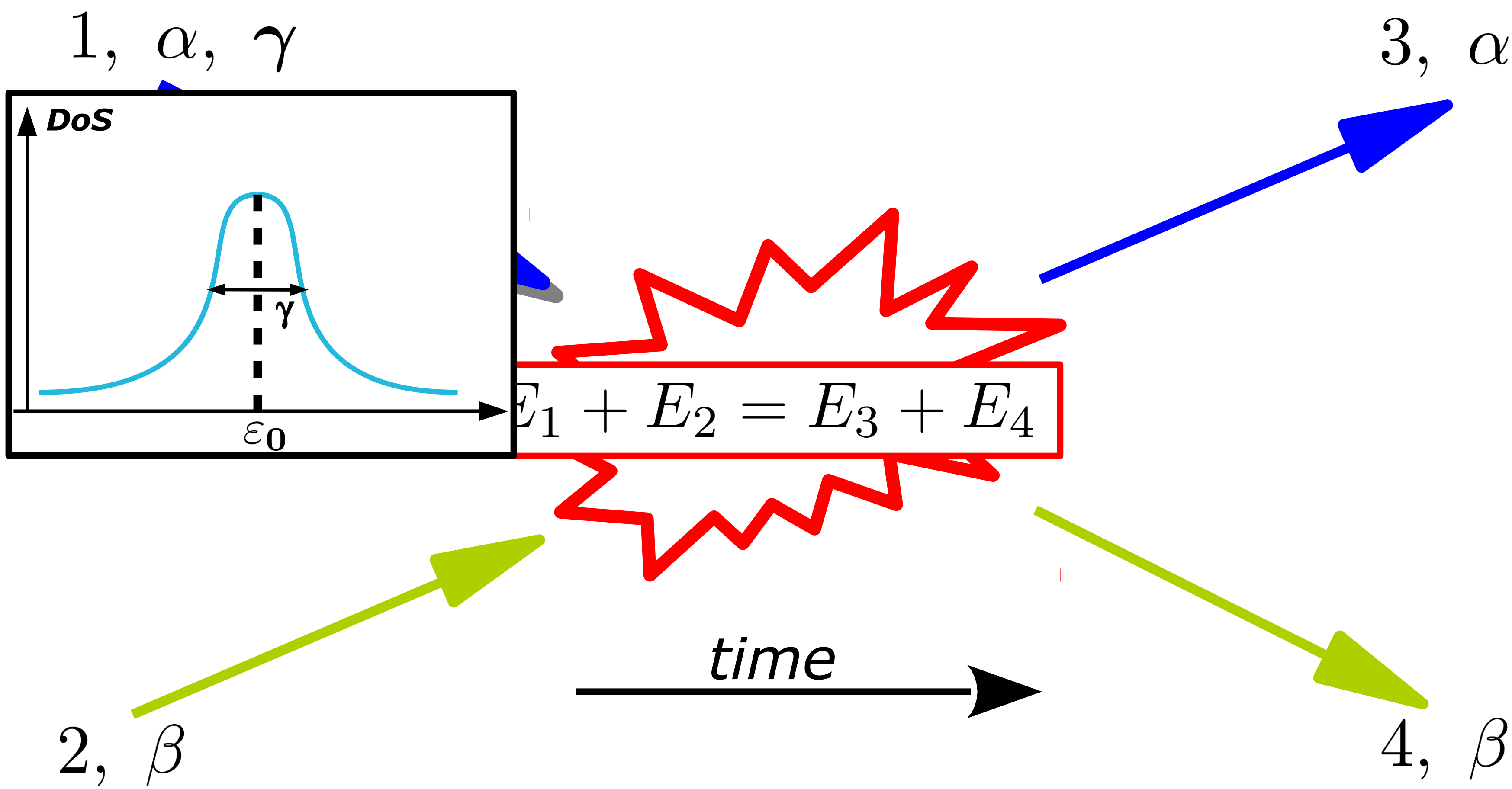

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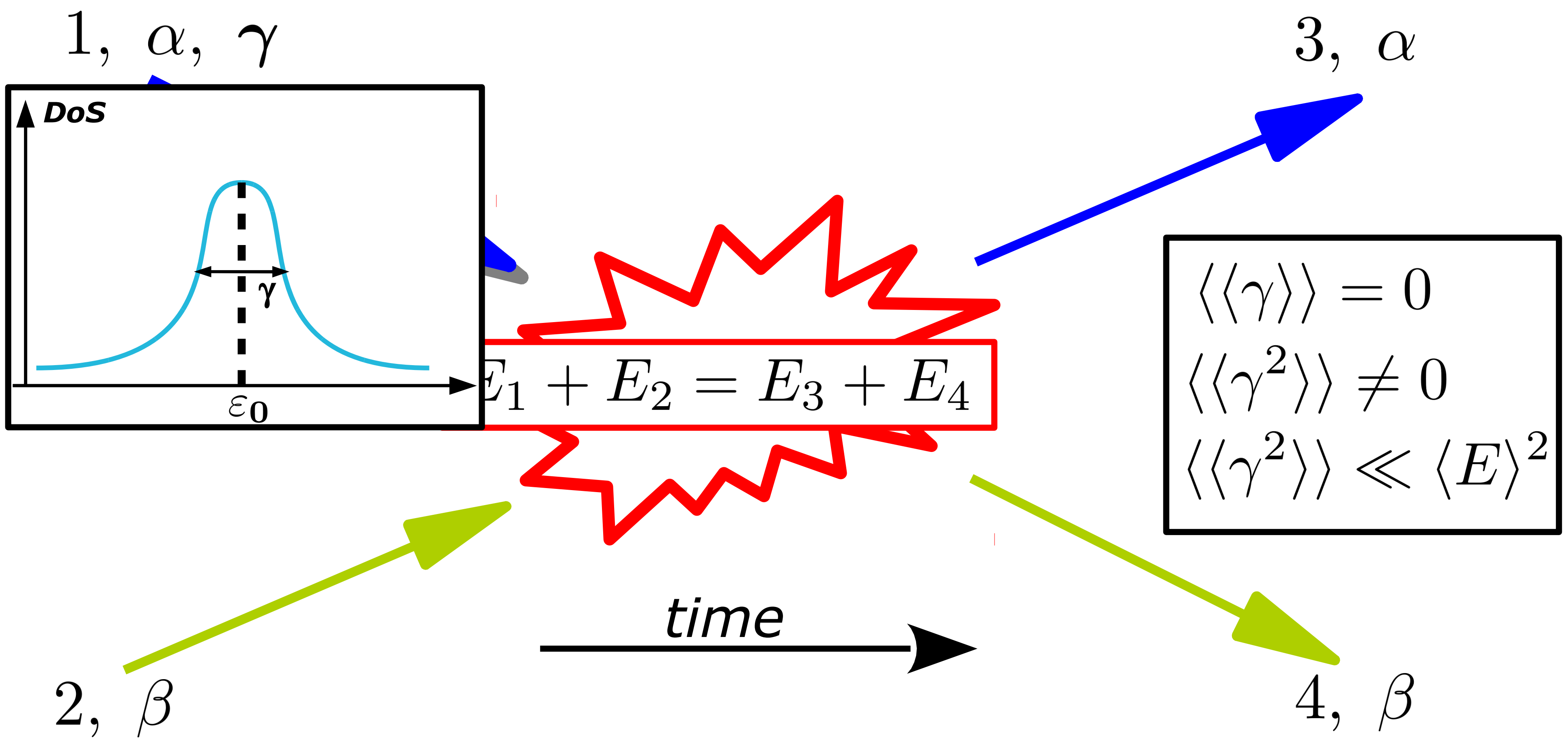

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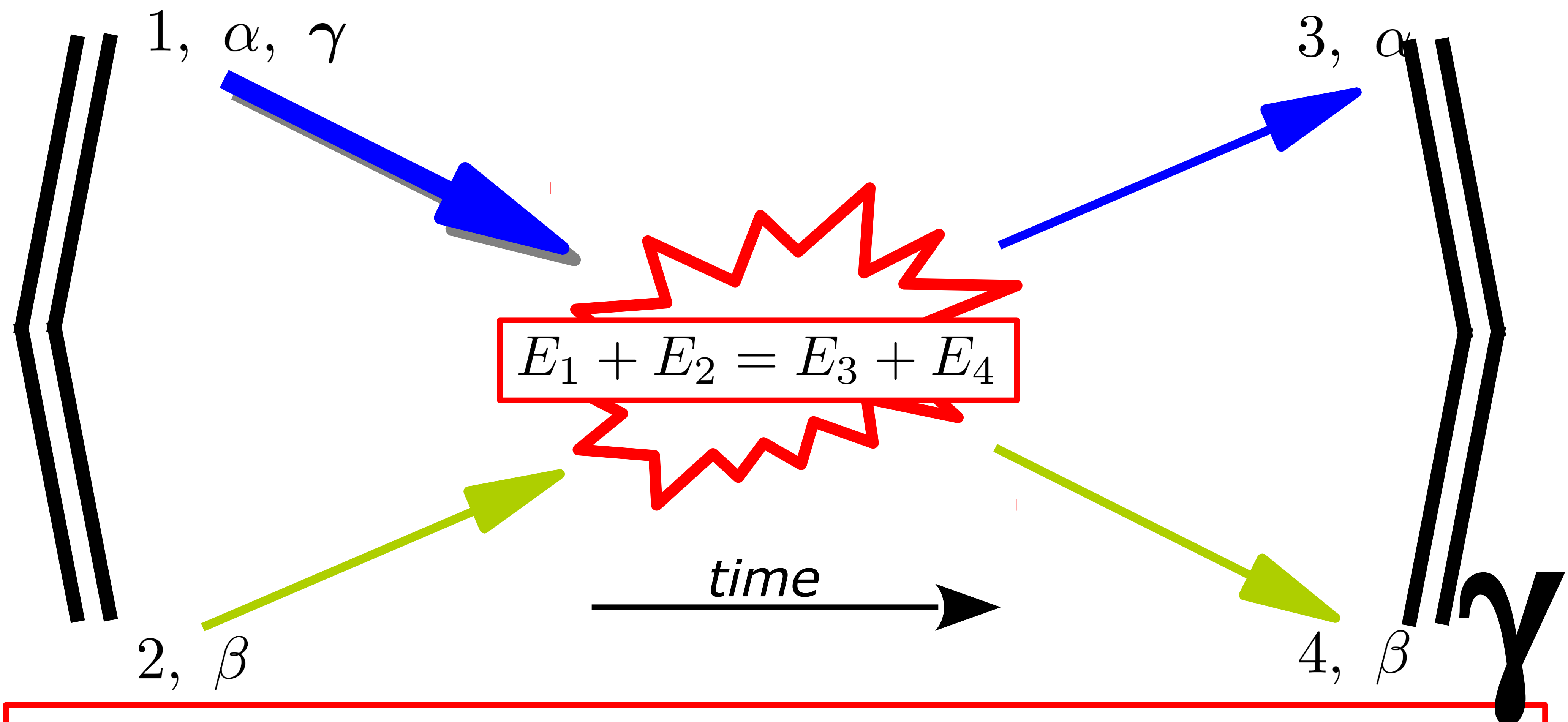

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$$\times \mathcal{W}^{\alpha\beta}_{1234}(f^{\alpha}(\mathbf{p}_3)f^{\beta}(\mathbf{p}_4) - f^{\alpha}(\mathbf{p}_1)f^{\beta}(\mathbf{p}_2))$$

Modification by averaging

$$\langle\langle \mathcal{C} \rangle\rangle = \int \mathrm{d}^3\mathbf{P} \int \mathrm{d}\widetilde{\omega} \int \mathrm{d}\omega \mathcal{K}_{1234}(\Omega - \omega) J^{\Delta}(\omega) \approx$$

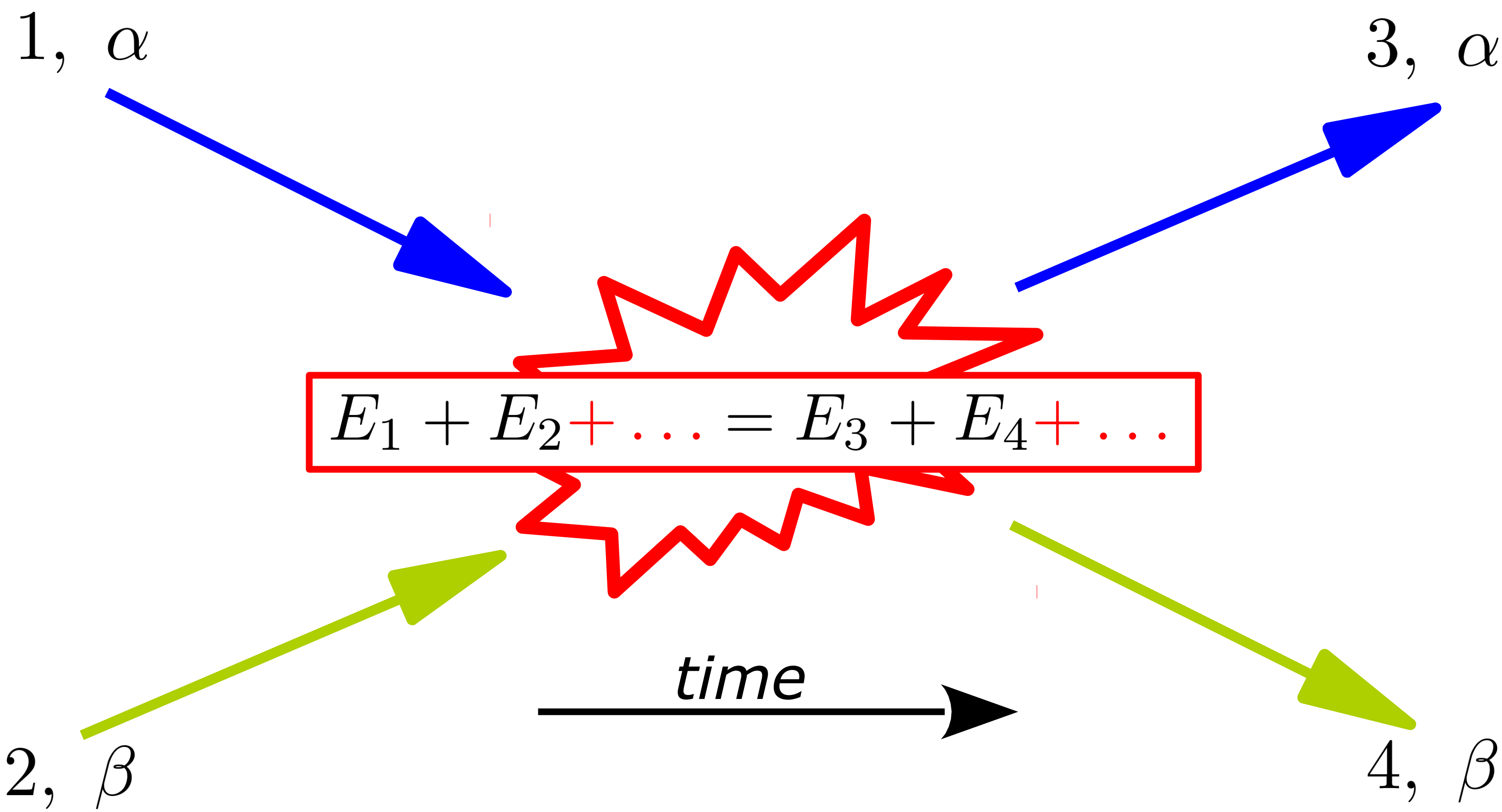
$$\mathcal{K}_{1234} = w_{1234}(f_3 f_4 - f_1 f_2)$$

$$J^{\Delta}(\omega) = \int_0^{\infty} \mathrm{d}\gamma g(\gamma) \rho_{\gamma}(\omega)$$

$$\approx \int \mathrm{d}^3\mathbf{P} \int \mathrm{d}\widetilde{\omega} \underbrace{\mathcal{K}_{1234}(\Omega - \Delta)}_{= \mathcal{K}_{1234}|_{E_4 = E_1 + \widetilde{\omega} - E_3 - \Delta}} + \mathcal{O}(\Delta^2) \approx$$

$$\approx \int_{234} \delta(E_1 + E_2 - E_3 - E_4 - \Delta) \mathcal{K}_{1234}$$

Modifying the Boltzmann equation



$$\dot{f}^{\alpha}(\mathbf{p}_1) = \sum_{\beta} \int_{234} \delta(\mathbf{p}_3 + \mathbf{p}_4 - \mathbf{p}_1 - \mathbf{p}_2)\delta(E_3 \oplus E_4 - E_1 \oplus E_2) \times \\ \times \mathcal{W}^{\alpha\beta}_{1234}(f^{\alpha}(\mathbf{p}_3) f^{\beta}(\mathbf{p}_4) - f^{\alpha}(\mathbf{p}_1) f^{\beta}(\mathbf{p}_2))$$

Examples for modified BEs

microscopic reasons:

dense electron gas

(H. Haug, C. Ell: Phys. Rev. B **46**, 2126; PRL **73**, 3439)

non-instantaneous collisions

(Špička et.al.: Phys. Rev. E **59**, 1219; Phys. Lett. A **240**, 160)

phenomenolgy:

power-law tailed spectra

(T. S. Biró et. al.: Eur. Phys. J. A **40**, 325)

noisy environment

(Rafelski et.al.: Lect. Notes Phys. **633**, 377, arXiv: physics/0204011v2)

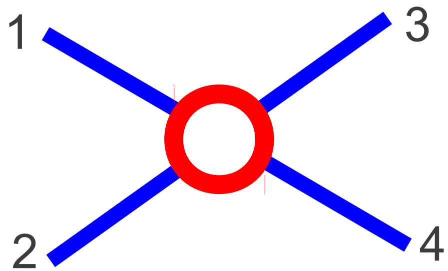
The case of two components

$$\dot{f}_1^A = \int_{234} \mathcal{W}_{1234}^{AA}(f_3^A f_4^A - f_1^A f_2^A) + \int_{234} \mathcal{W}_{1234}^{AB}(f_3^A f_4^B - f_1^A f_2^B)$$

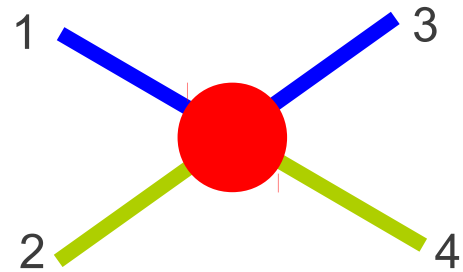
$$\dot{f}_1^B = \int_{234} \mathcal{W}_{1234}^{BB}(f_3^B f_4^B - f_1^B f_2^B) + \int_{234} \mathcal{W}_{1234}^{BA}(f_3^B f_4^A - f_1^B f_2^A)$$

The case of two components

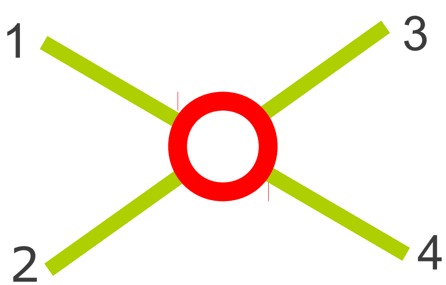
$\dot{f}_1^A = \int_{234}$



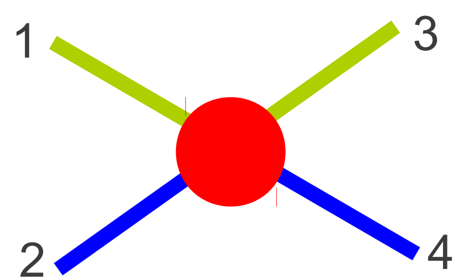
$+ \int_{234}$



$\dot{f}_1^B = \int_{234}$



$+ \int_{234}$



The case of two components

$$\dot{f}_1^A = \int_{234} \qquad + \int_{234}$$

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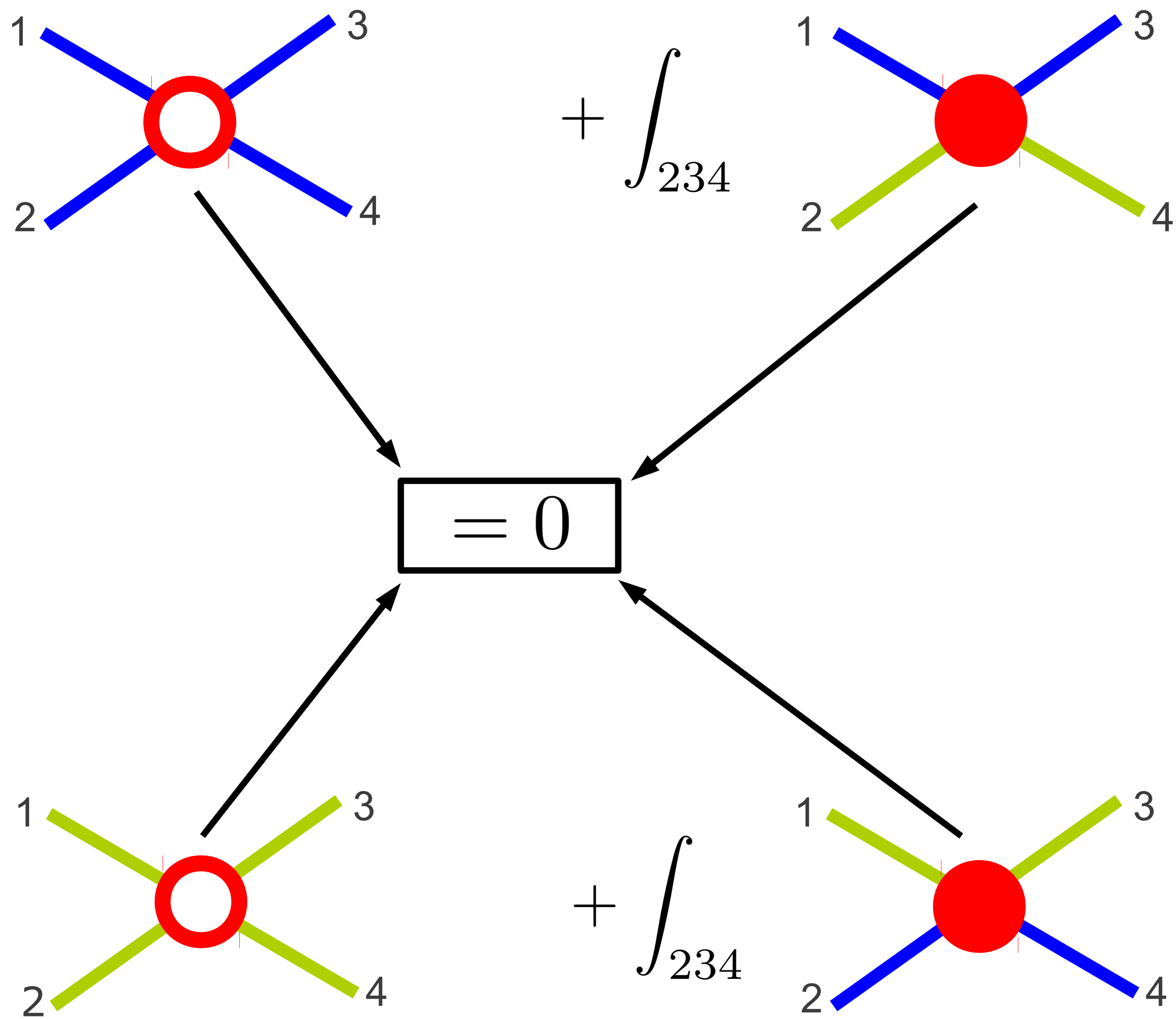

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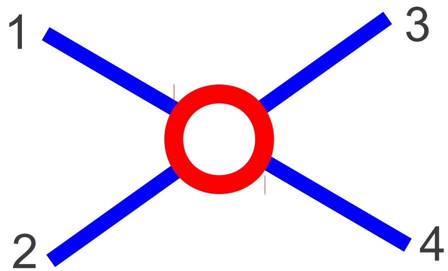
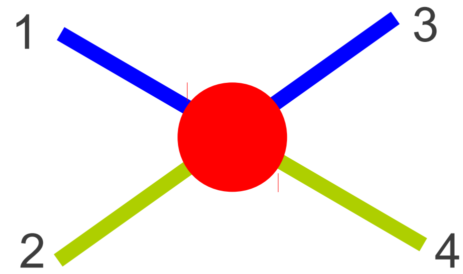
$$\dot{f}_1^A = \int_{234}$$ 

$$+ \int_{234}$$ 

$$f^A(E_3)f^A(E_1+E_2-E_3) = f^A(E_1)f^A(E_2)$$

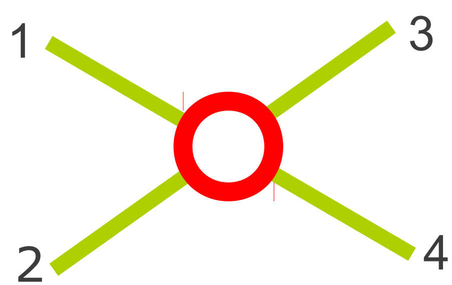
$$f^A(E_3)f^B(E_1 \oplus E_2 \ominus E_3) = f^A(E_1)f^B(E_2)$$

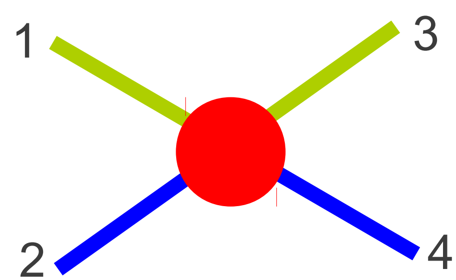
$$E_1 + E_2 = E_3 + E_4$$

$$E_1 \oplus E_2 = E_3 \oplus E_4$$

$$f^B(E_3)f^A(E_1 \oplus E_2 \ominus E_3) = f^B(E_1)f^A(E_2)$$

$$f^B(E_3)f^B(E_1+E_2-E_3) = f^B(E_1)f^B(E_2)$$

$$\dot{f}_1^B = \int_{234}$$ 

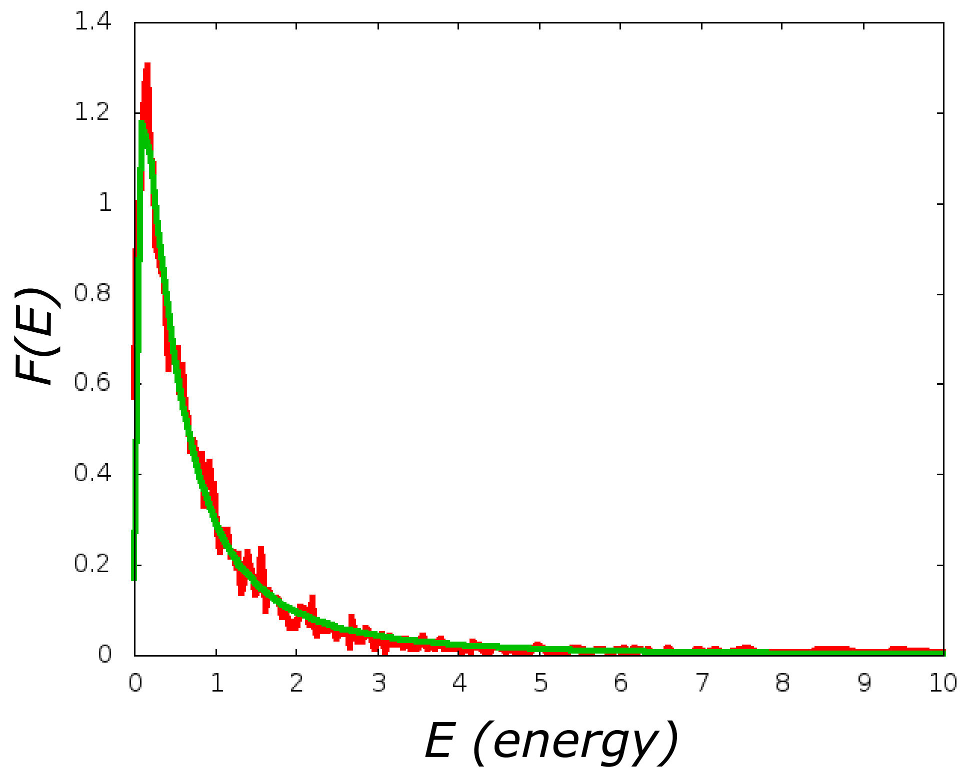
$$+ \int_{234}$$ 

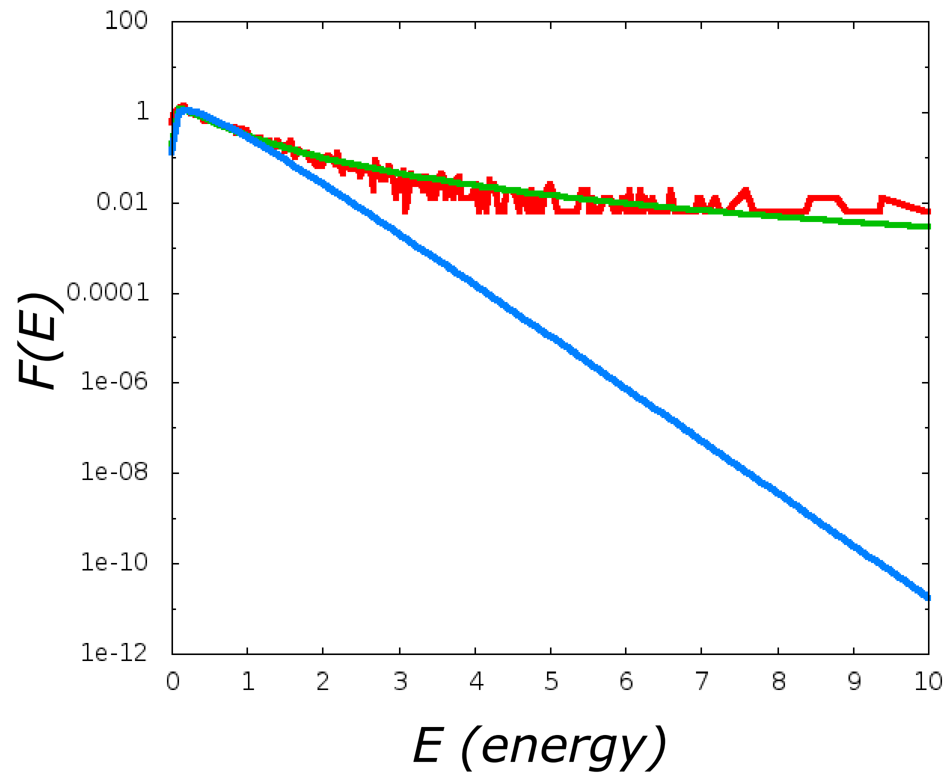
Long-time behaviour

- One-component
  - detailed balance: **YES** ✔
- Two-component
  - detailed balance: **NO** ✘
  - for long times: various scenarios
    - ➔ *warming or cooling system*
    - ➔ *saturation – in special cases*
    - ➔ *scaling solutions*

Long-time behaviour

- One-component
  - detailed balance: ***YES*** ✔





$$F_{eq}(E) \sim e^{-\beta L(E)} \sim (1 + AE)^{-\frac{B}{A}}$$

Two components – some details

simple modification $E \oplus_{\alpha\beta} E' = E + E' + A_{\alpha\beta} E E'$

modification scale fixed relative to avr. energy $A_{\alpha\beta} = \dfrac{a_{\alpha\beta}}{\langle E \rangle}$

case I.

$$a_{AA} = a_{BB} =: a_0 > 0$$

$$a_{AB} = a_{BA} =: a_1 > 0$$

$a_0, \ a_1$ are constant

case II.

$a_0 > 0 \qquad \Lambda = \Lambda_0 \langle E \rangle$

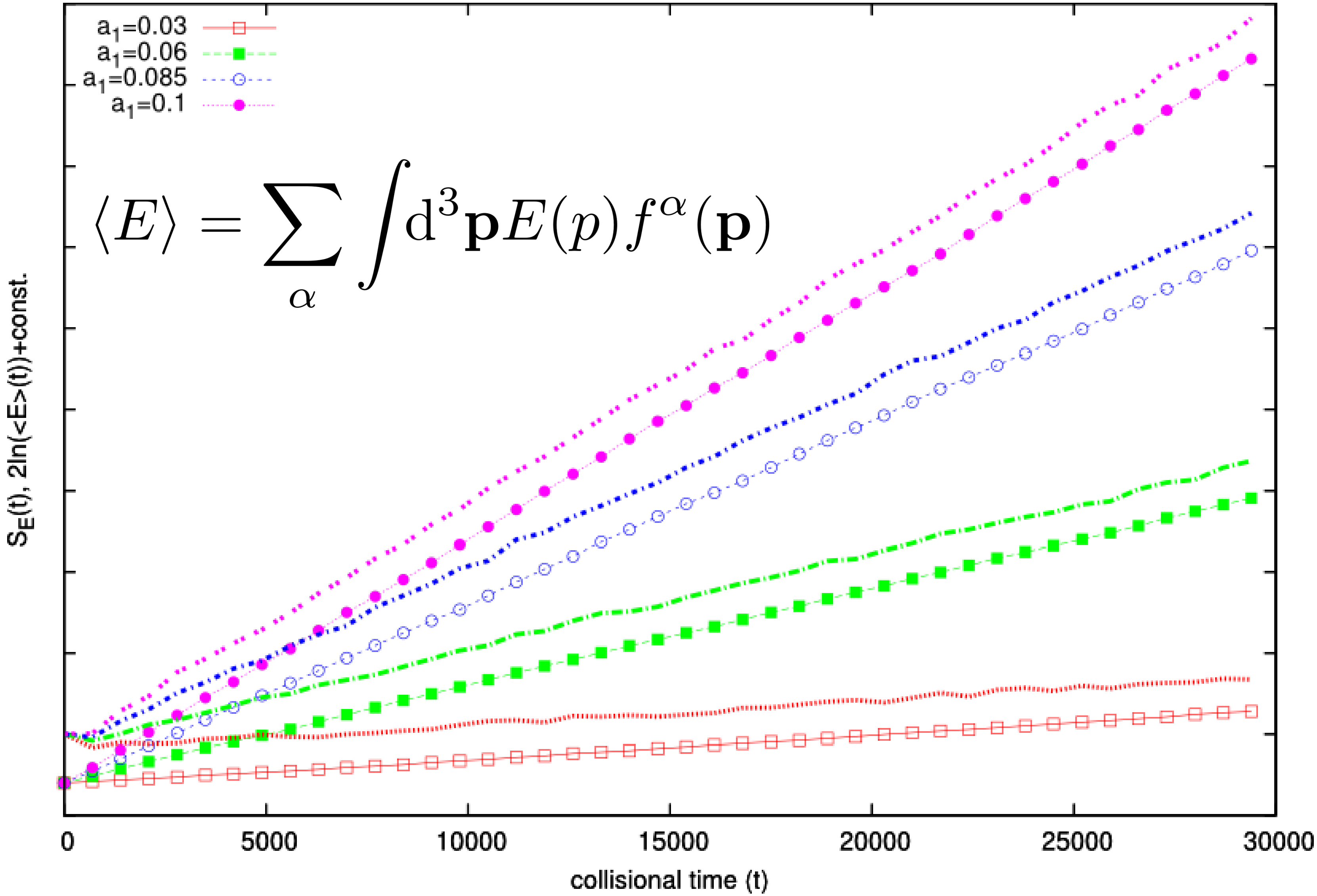
$$a_1 = \begin{cases} a_0, & E_1 + E_2 < \Lambda \\ b, & \Lambda < E_1 + E_2 \end{cases}$$

$a_0, \ b, \ \Lambda_0$ are constant

exact scaling: $\langle E \rangle \sim e^{\gamma t}$

Two components

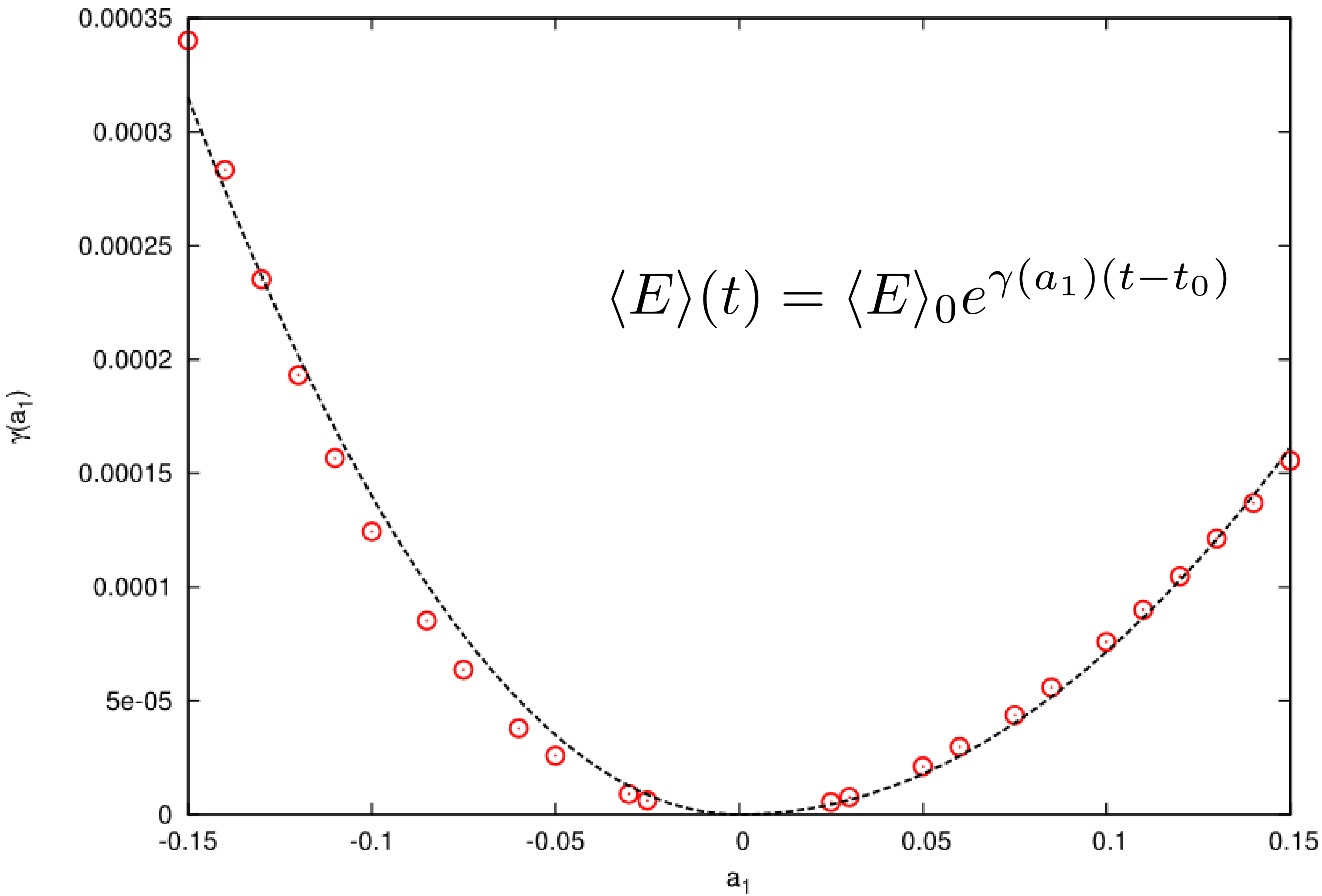
detailed balance ✗



Two components, case I.

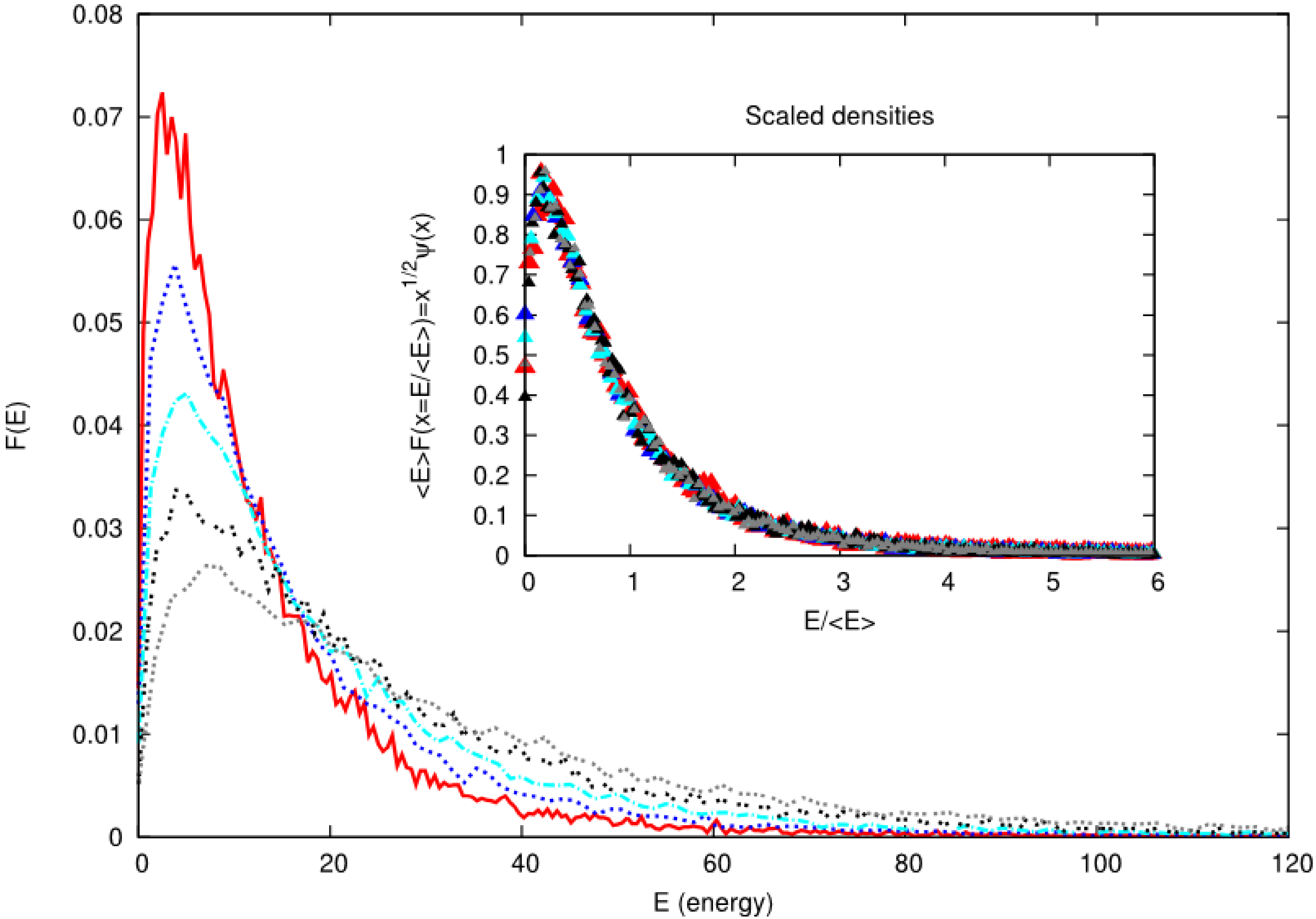
$a_0 = 0$

detailed balance ✗



Two components

detailed balance ✗



$$F^{\alpha}(E,t)$$

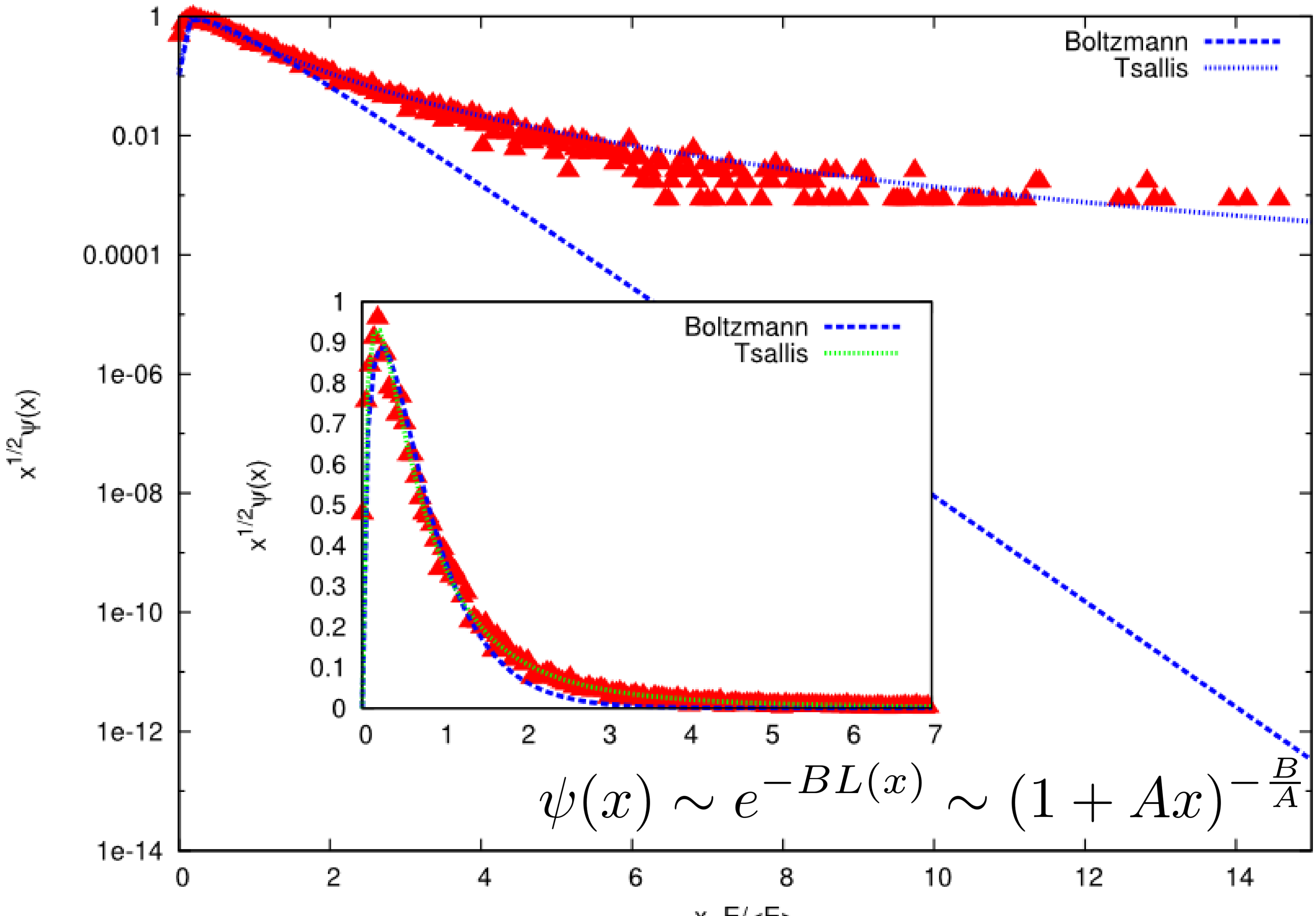
$$t \to \infty$$

$$\frac{1}{\langle E \rangle}\psi(E/\langle E \rangle)$$

(exact!)

Two components

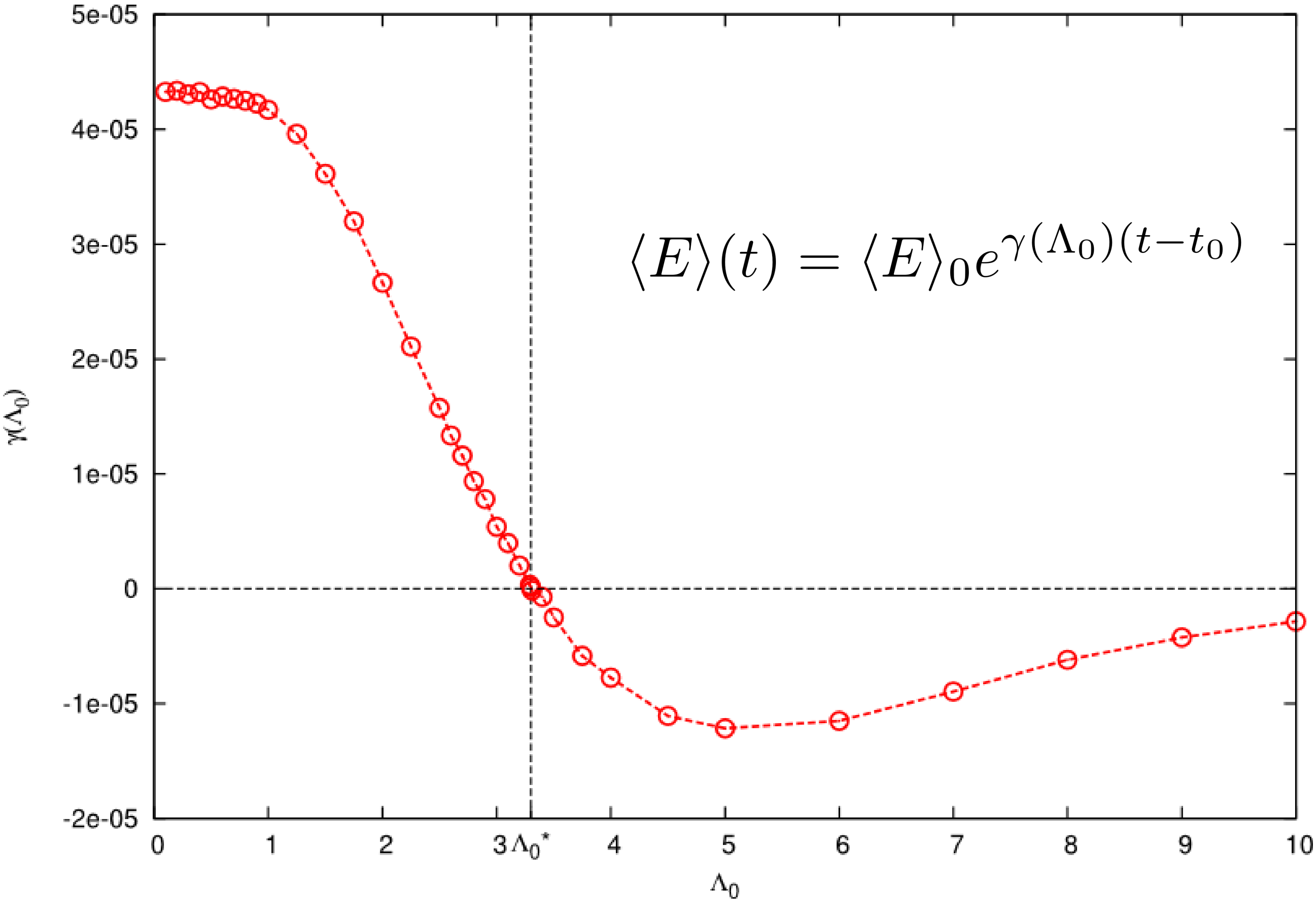
detailed balance ✘



$$\psi(x) \sim e^{-BL(x)} \sim (1+Ax)^{-\frac{B}{A}}$$

Two components, case II.

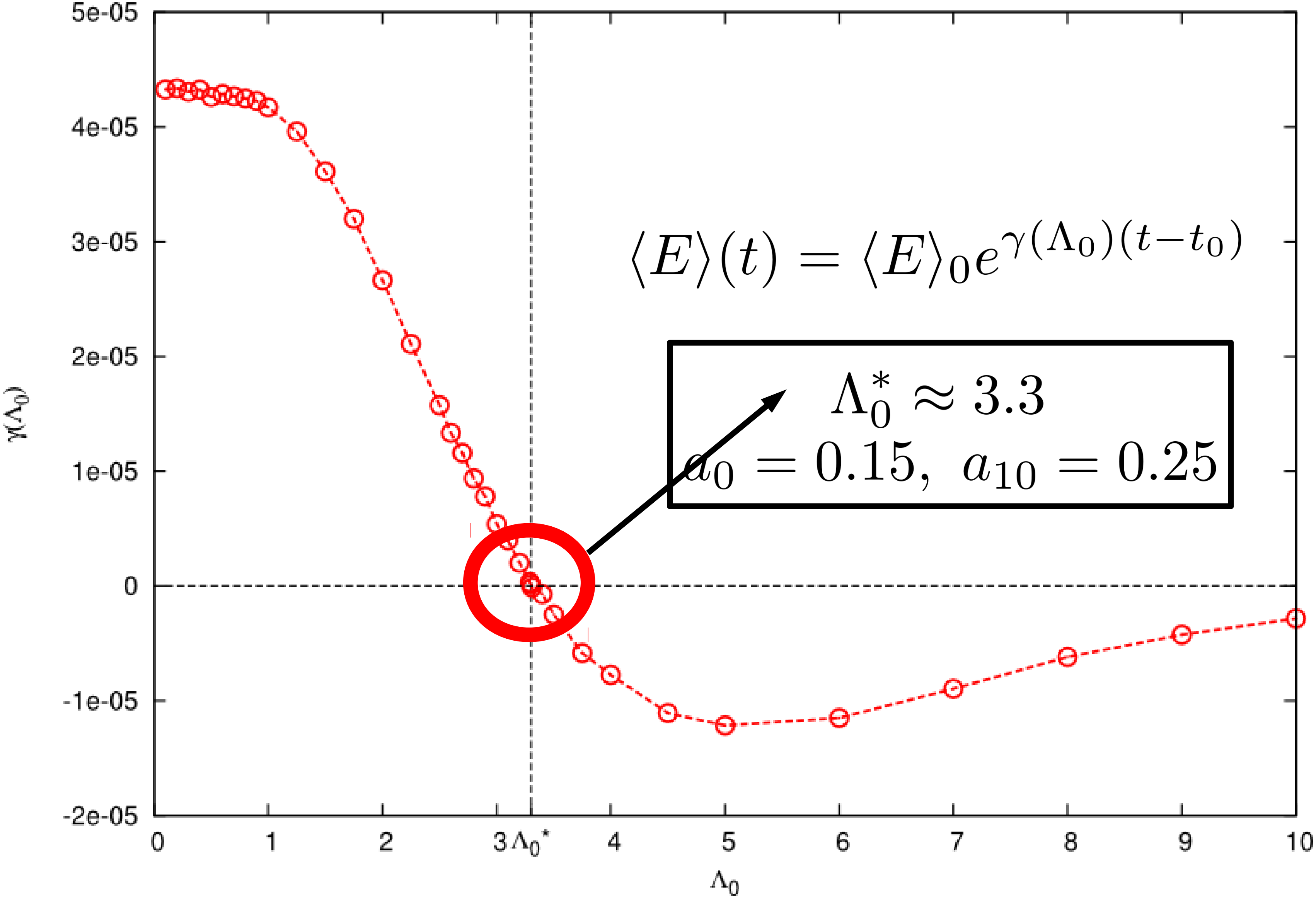
detailed balance ✗



$$\langle E\rangle(t) = \langle E\rangle_0 e^{\gamma(\Lambda_0)(t-t_0)}$$

Two components, case II.

detailed balance ✗



Summary

- modification from a phenomenological viewpoint
- multicomponent – several types of collisions
- no detailed balance in MC case
- asymptotic behaviour: driven or saturating

open system

dynamical threshold
for the modification

dynamical control is needed
for saturation

Thank you for your attention!

Questions?

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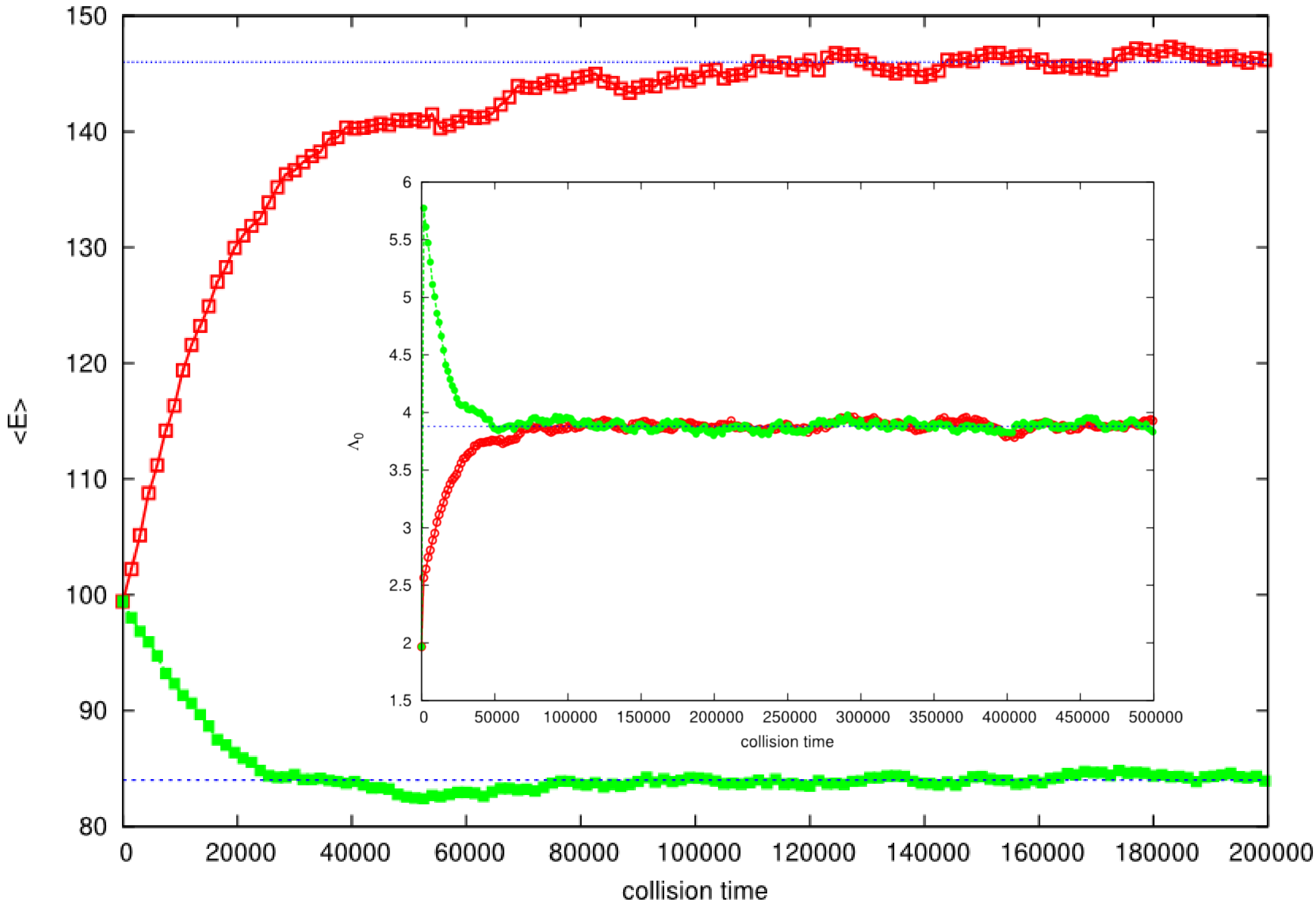
Questions?

Stay tuned!
arXiv: 1309.????

Backups – case II.: saturation



Backups – modification by averaging

$$\mathcal{I}_\gamma = \int_{234} \int \mathrm{d}\omega_2 \delta(E_1 + \omega_2 - E_3 - E_4) \rho_\gamma(\omega_2 - E_2) \mathcal{K}_{1234} =$$

$$= \int_{234} \rho_\gamma(E_3 + E_4 - E_1 - E_2) \mathcal{K}_{1234} = \int \mathrm{d}^3\mathbf{P} \int \mathrm{d}^2\sigma \rho_\gamma(\sigma) \mathcal{K}_{1234}(\sigma)$$

$$= \int \mathrm{d}^3\mathbf{P} \int \mathrm{d}\widetilde{\omega} \int \mathrm{d}\omega \rho_\gamma(\Omega(\widetilde{\omega}) - \omega) \mathcal{K}_{1234}(\omega) =$$

$$= \int \mathrm{d}^3\mathbf{P} \int \mathrm{d}\widetilde{\omega} \int \mathrm{d}\omega \rho_\gamma(\omega) \underbrace{\mathcal{K}_{1234}(\Omega - \omega)}_{= \mathcal{K}_{1234}|_{E_4 = E_1 + \widetilde{\omega} - \omega}} ,$$

Backups – modification by averaging

$$\mathcal{I} = \int\limits_0^\infty \mathrm{d}\gamma g(\gamma) \mathcal{I}_\gamma = \int \mathrm{d}^3\mathbf{P} \int \mathrm{d}\widetilde{\omega} \int \mathrm{d}\omega \mathcal{K}_{1234}(\Omega - \omega) J^\Delta(\omega) \approx$$

$$\mathcal{I}_\gamma = \int \mathrm{d}^3\mathbf{P} \int_{(\widetilde{\omega},\omega)=\sigma} \mathrm{d}^2\sigma \rho_\gamma(\omega) \mathcal{K}_{1234}(\Omega(\widetilde{\omega}) - \omega),$$

$$J^\Delta(\omega) = \int\limits_0^\infty \mathrm{d}\gamma g(\gamma) \rho_\gamma(\omega)$$

Backups – modification by averaging

$$\mathcal{I} = \int\limits_0^\infty \mathrm{d}\gamma g(\gamma) \mathcal{I}_\gamma = \int \mathrm{d}^3\mathbf{P} \int \mathrm{d}\widetilde{\omega} \int \mathrm{d}\omega \mathcal{K}_{1234}(\Omega - \omega) J^\Delta(\omega) \approx$$

$$\approx \int \mathrm{d}^3\mathbf{P} \int \mathrm{d}\widetilde{\omega} \Big\{ \mathcal{K}_{1234}(\Omega - \Delta) \int \mathrm{d}\omega J^\Delta(\omega) +$$

$$+ \frac{\partial}{\partial \omega} \mathcal{K}_{1234}(\Omega - \Delta) \int \mathrm{d}\omega (\omega - \Delta) J^\Delta(\omega) +$$

$$+ \frac{\partial^2}{\partial \omega^2} \mathcal{K}_{1234}(\Omega - \Delta) \int \mathrm{d}\omega (\omega - \Delta)^2 J^\Delta(\omega) +$$

$$+ \int \mathrm{d}\omega \mathcal{O}((\omega - \Delta)^3) J^\Delta(\omega) \Big\} \approx$$

Backups – modification by averaging

$$\mathcal{I} = \int_0^\infty \mathrm{d}\gamma g(\gamma) \mathcal{I}_\gamma = \int \mathrm{d}^3\mathbf{P} \int \mathrm{d}\widetilde{\omega} \int \mathrm{d}\omega \mathcal{K}_{1234}(\Omega - \omega) J^\Delta(\omega) \approx$$

$$\approx \int \mathrm{d}^3\mathbf{P} \int \mathrm{d}\widetilde{\omega} \underbrace{\mathcal{K}_{1234}(\Omega - \Delta)}_{= \mathcal{K}_{1234}|_{E_4 = E_1 + \widetilde{\omega} - E_3 - \Delta}} + \mathcal{O}(\Delta)^2 \approx$$

$$\approx \int_{234} \delta(E_1 + E_2 - E_3 - E_4 - \Delta) \mathcal{K}_{1234}$$

Backups – detailed balance

for $\alpha = \beta$

$$f^\alpha(E_1) f^\alpha(E_2) = f^\alpha(E_3) f^\alpha(E_4)$$

$$L^{\alpha\alpha}(E_1) + L^{\alpha\alpha}(E_2) = L^{\alpha\alpha}(E_3) + L^{\alpha\alpha}(E_4)$$

$$\Rightarrow \quad f^\alpha \sim e^{-\gamma_{\alpha\alpha} L^{\alpha\alpha}(E)}$$

$$f^\alpha(E_1) f^\beta(E_2) = f^\alpha(E_3) f^\beta(E_4)$$

$$\gamma_{\alpha\alpha} L^{\alpha\alpha}(E_1) + \gamma_{\beta\beta} L^{\beta\beta}(E_2) = \gamma_{\alpha\alpha} L^{\alpha\alpha}(E_3) + \gamma_{\beta\beta} L^{\beta\beta}(E_4)$$

AND for $\alpha \neq \beta$

$$L^{\alpha\beta}(E_1) + L^{\alpha\beta}(E_2) = L^{\alpha\beta}(E_3) + L^{\alpha\beta}(E_4)$$

Backups – model used in simulation

Constant rate function (no dep. on out-g. energies)

$$\dot{f}(p) =$$

$$= \int \mathrm{d}^3\mathbf{p}' \int \mathrm{d}^3\mathbf{q} \frac{1}{Z} \delta(K(p,p') - E(q) \oplus E(q^*))(f(q)f(q^*) - f(p)f(p'))$$

$$= \int \mathrm{d}^3\mathbf{p}' \int \mathrm{d}^3\mathbf{q} \frac{1}{Z} \delta(\dots) f(q) f(q^*) - f(p) \int \mathrm{d}^3\mathbf{p}' f(p') \underbrace{\int \mathrm{d}^3\mathbf{q} \frac{1}{Z} \delta(\dots)}_{:=1}$$

$$Z(p_1, p_2, \mathbf{p}_1 \cdot \mathbf{p}_2) = \int \mathrm{d}^3\mathbf{q} \delta(K - E(q) \oplus E(|\mathbf{P} - \mathbf{q}|))$$

$$q^* = |\mathbf{p} + \mathbf{p}' - \mathbf{q}|$$