

The modified Boltzmann equation & multi-component thermalization

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Outline

- Few things about the Boltzmann equation
- Why do we need modifications?
- A toy model for modified BE
- Long-time behaviour in the case of two components

What is BE & why do we like it?

time evolution of an ensemble

$$\dot{f}_1 = \int_{234} \mathcal{W}_{1234}(f_3 f_4 - f_1 f_2)$$

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equilibration & thermalization

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(transport coefficients, rel. time approx.)

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easy to solve

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relatively easy to solve

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str.fwd. calculations around the equilibrium

(transport coefficients, rel. time approx.)

moment equations  hydrodynamics

phenomenology

relatively easy to solve **numerically**

Where does a BE come from?

Microscopic description:
field theory (DS-equ.)

Boltzmann equation

gradient expansion
(& Wigner-trf.)
quasi-particle approx.

separation of
time-scales

weak interaction
on-(mass-)shell
coupling expansion

kinetic equation
(Kadanoff-Baym)

Where does a BE come from?

Microscopic description:
field theory (DS-equ.)

$$G = G^0 + G^0 * \Sigma * G$$

gradient expansion
(& Wigner-trf.)

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separation of
time-scales

$$\dot{f}_1 = \int_{234} \mathcal{W}_{1234} (f_3 f_4 - f_1 f_2)$$

Boltzmann equation

kinetic equation
(Kadanoff-Baym)

$$2ip\partial_X i\hat{G}_{rr}(p, X) = \hat{\Sigma}_{12} i\hat{G}_{12} - \hat{\Sigma}_{21} i\hat{G}_{21}$$

$$\hat{G}_{12}(p, X) = n(p, X)\rho(p)$$

Where does a BE come from?

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Boltzmann equation

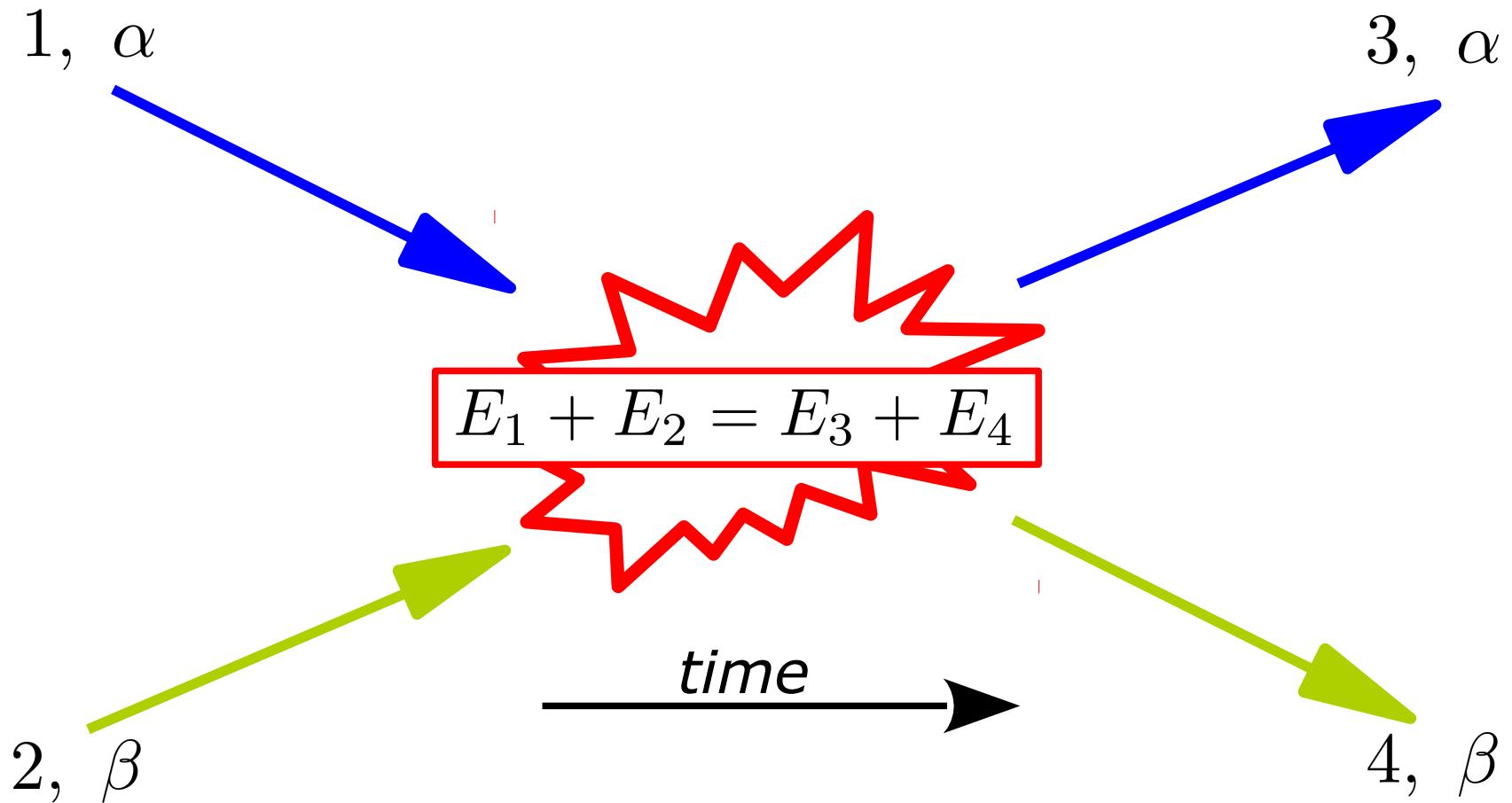
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***weak interaction
on-(mass-)shell***
coupling expansion

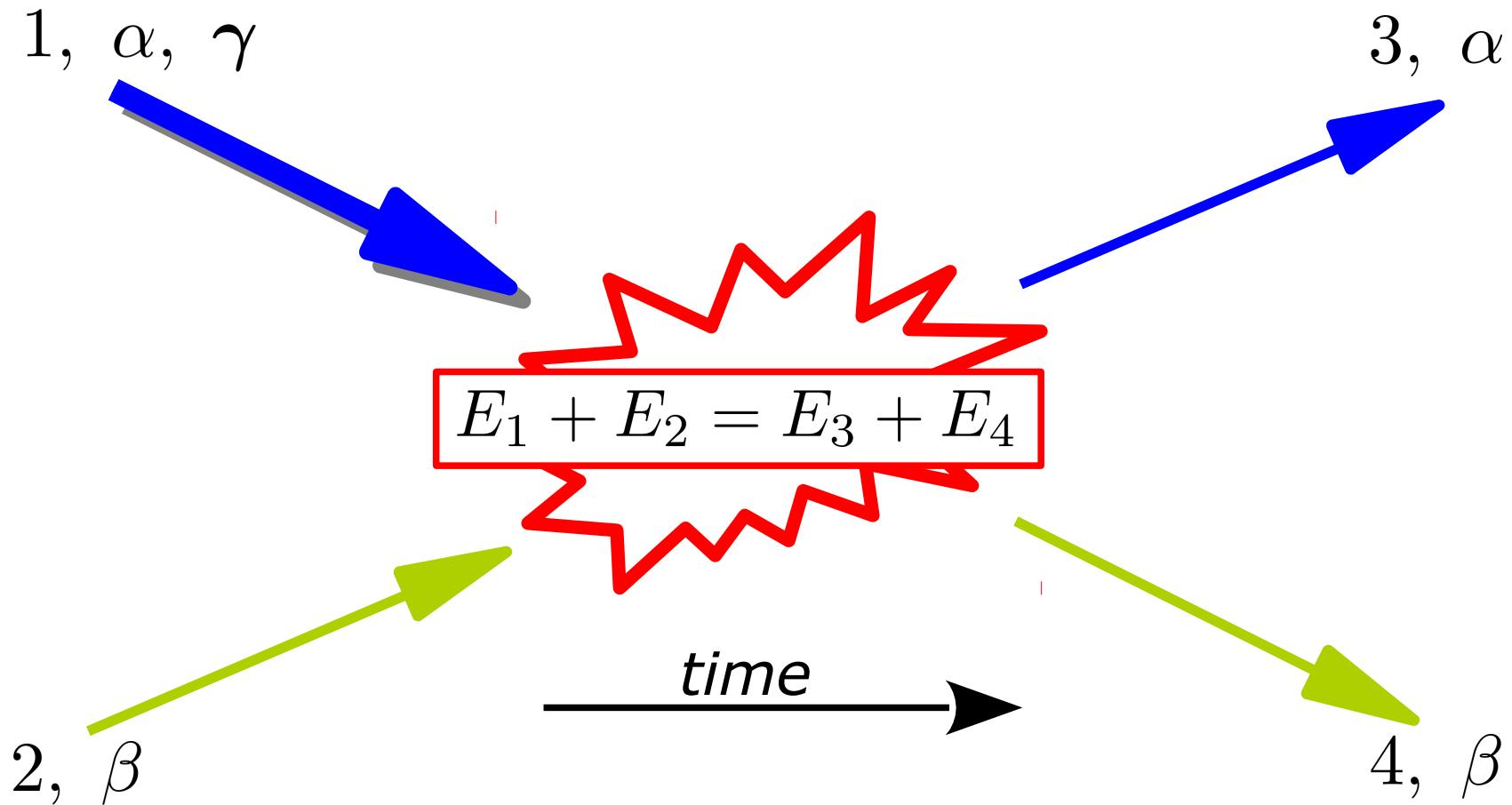
kinetic equation
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Modifying the Boltzmann equation



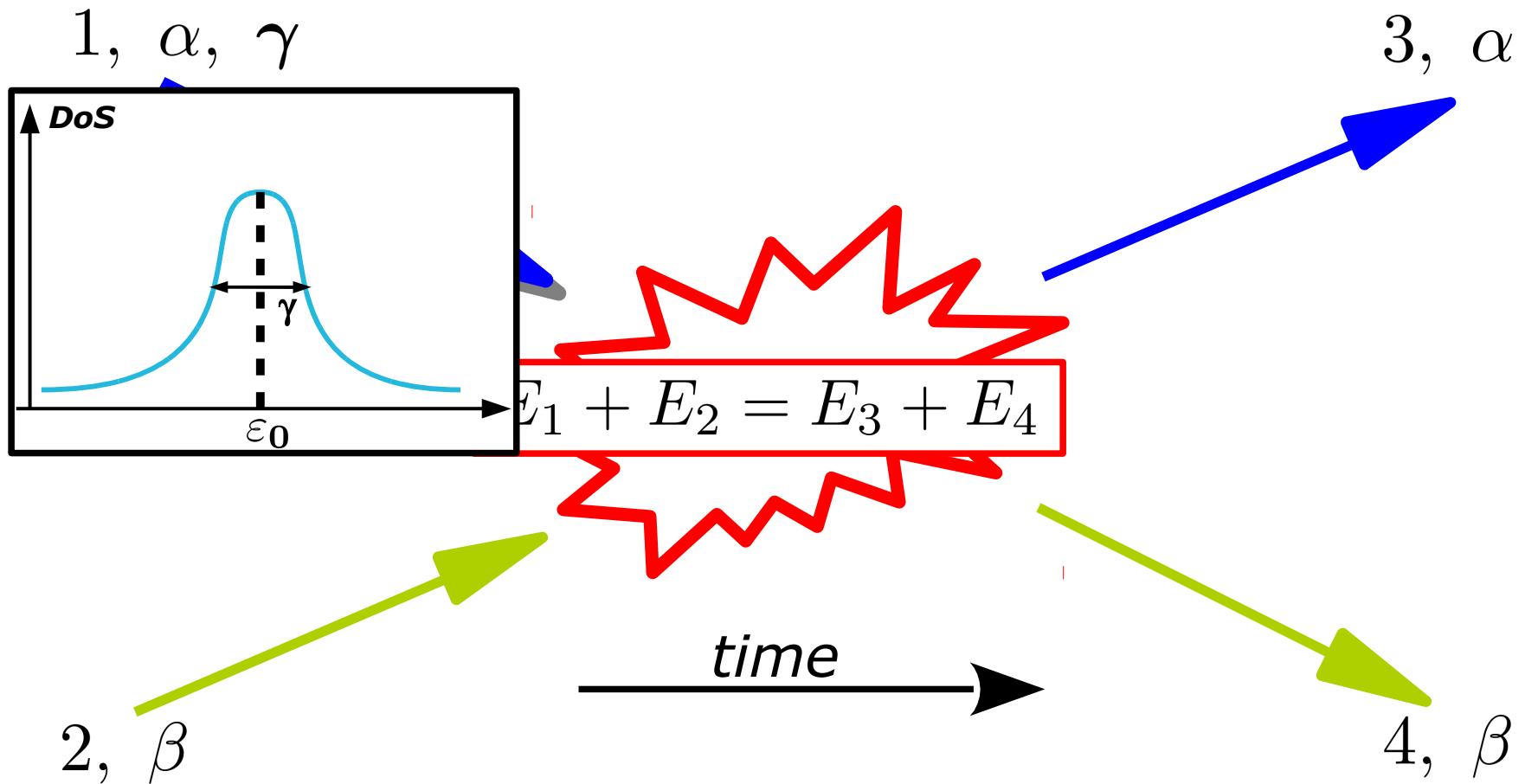
$$\dot{f}^\alpha(\mathbf{p}_1) = \sum_\beta \int_{234} \delta(\mathbf{p}_3 + \mathbf{p}_4 - \mathbf{p}_1 - \mathbf{p}_2) \delta(E_3 + E_4 - E_1 - E_2) \times \\ \times \mathcal{W}_{1234}^{\alpha\beta} (f^\alpha(\mathbf{p}_3) f^\beta(\mathbf{p}_4) - f^\alpha(\mathbf{p}_1) f^\beta(\mathbf{p}_2))$$

Modifying the Boltzmann equation



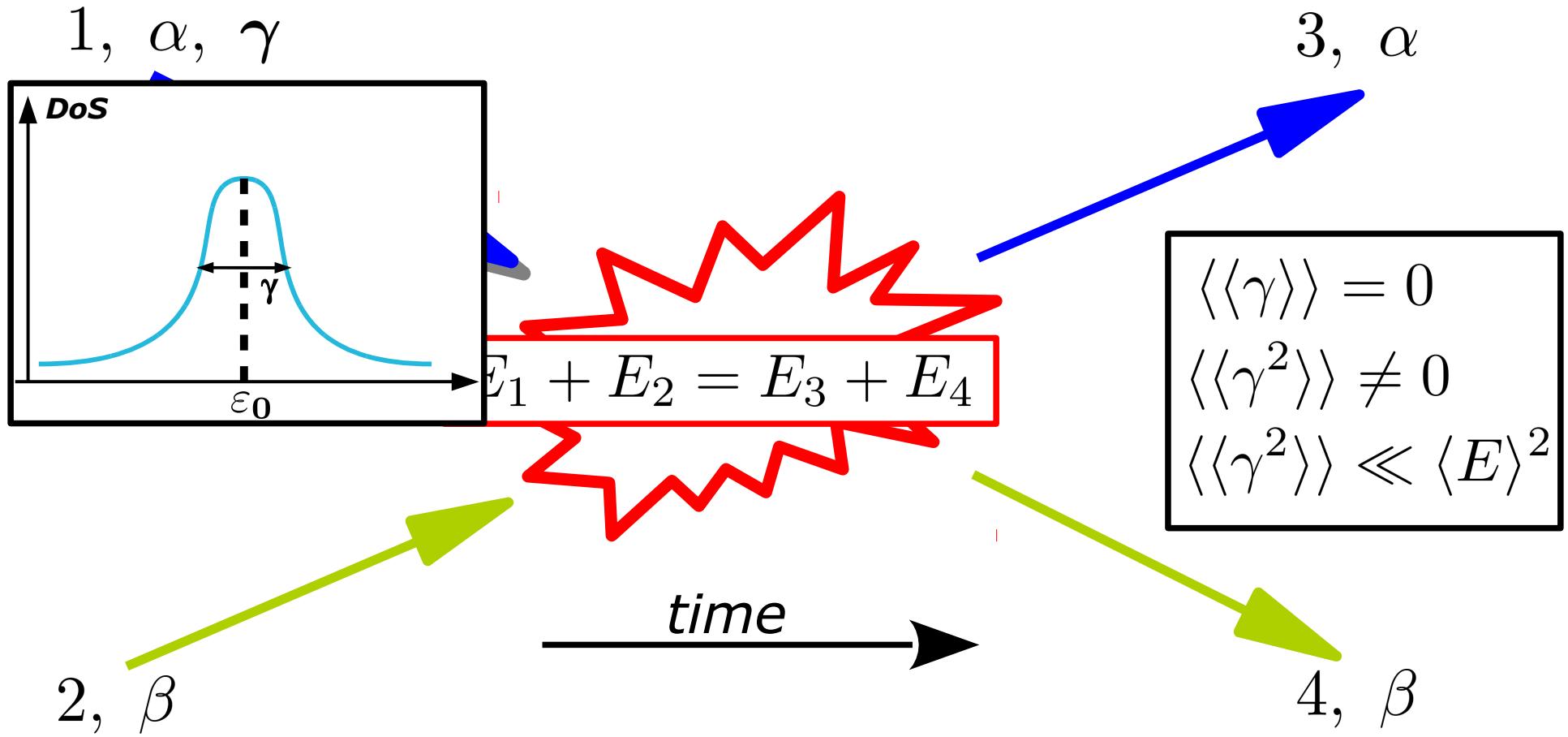
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Modifying the Boltzmann equation



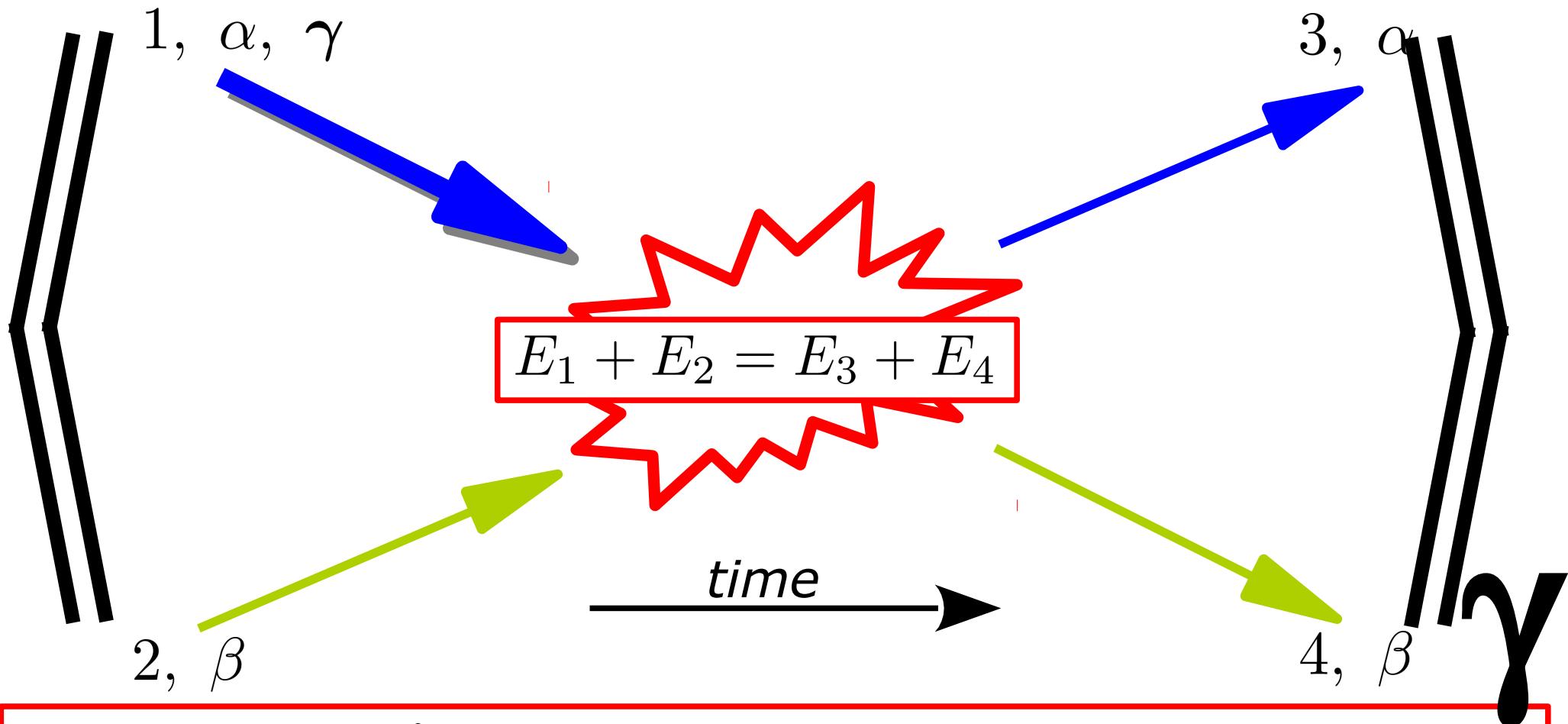
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Modifying the Boltzmann equation



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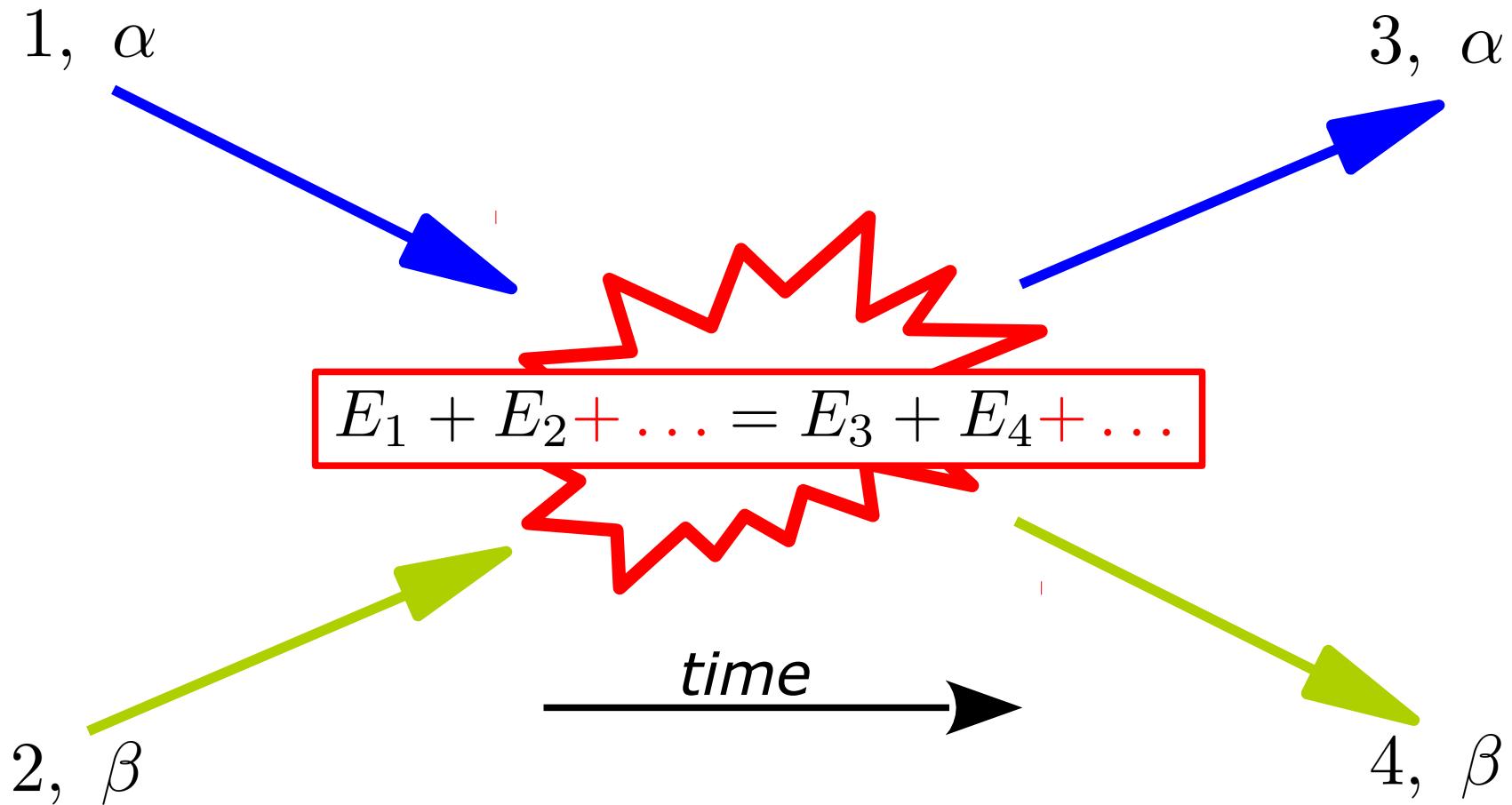
Modification by averaging

$$\langle\langle \mathcal{C} \rangle\rangle = \int d^3\mathbf{P} \int d\tilde{\omega} \int d\omega \mathcal{K}_{1234}(\Omega - \omega) J^\Delta(\omega) \approx$$

$\boxed{\mathcal{K}_{1234} = w_{1234}(f_3 f_4 - f_1 f_2)}$ $\boxed{J^\Delta(\omega) = \int_0^\infty d\gamma g(\gamma) \rho_\gamma(\omega)}$

$$\approx \int d^3\mathbf{P} \int d\tilde{\omega} \underbrace{\mathcal{K}_{1234}(\Omega - \Delta)}_{= \mathcal{K}_{1234}|_{E_4 = E_1 + \tilde{\omega} - E_3 - \Delta}} + \mathcal{O}(\Delta^2) \approx$$
$$\approx \int_{234} \delta(E_1 + E_2 - E_3 - E_4 - \Delta) \mathcal{K}_{1234}$$

Modifying the Boltzmann equation



$$\begin{aligned} \dot{f}^\alpha(\mathbf{p}_1) = & \sum_{\beta} \int_{234} \delta(\mathbf{p}_3 + \mathbf{p}_4 - \mathbf{p}_1 - \mathbf{p}_2) \delta(E_3 \oplus E_4 - E_1 \oplus E_2) \times \\ & \times \mathcal{W}_{1234}^{\alpha\beta} (f^\alpha(\mathbf{p}_3) f^\beta(\mathbf{p}_4) - f^\alpha(\mathbf{p}_1) f^\beta(\mathbf{p}_2)) \end{aligned}$$

Examples for modified BEs

microscopic reasons:

dense electron gas

(H. Haug, C. Ell: Phys. Rev. B **46**, 2126; PRL **73**, 3439)

non-instantaneous collisions

(Špička et.al.: Phys. Rev. E **59**, 1219; Phys. Lett. A **240**, 160)

phenomenology:

power-law tailed spectra

(T. S. Biró et. al.: Eur. Phys. J. A **40**, 325)

noisy environment

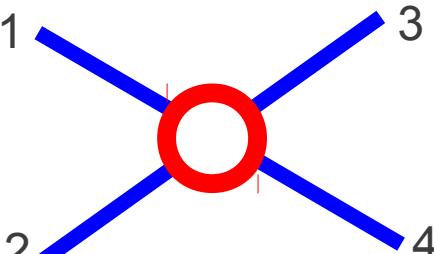
(Rafelski et.al.: Lect. Notes Phys. **633**, 377, arXiv: physics/0204011v2)

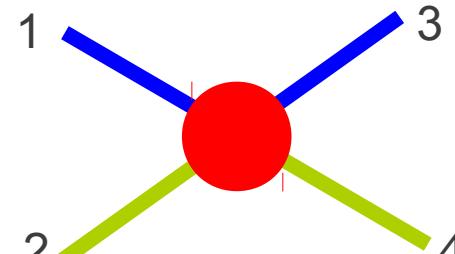
The case of two components

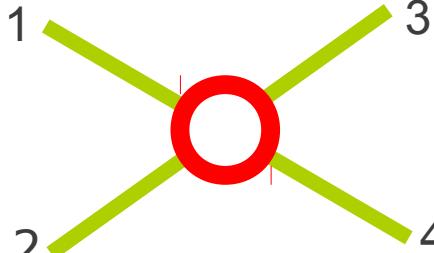
$$\dot{f}_1^A = \int_{234} \mathcal{W}_{1234}^{AA} (f_3^A f_4^A - f_1^A f_2^A) + \int_{234} \mathcal{W}_{1234}^{AB} (f_3^A f_4^B - f_1^A f_2^B)$$

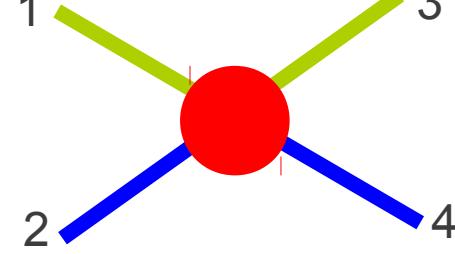
$$\dot{f}_1^B = \int_{234} \mathcal{W}_{1234}^{BB} (f_3^B f_4^B - f_1^B f_2^B) + \int_{234} \mathcal{W}_{1234}^{BA} (f_3^B f_4^A - f_1^B f_2^A)$$

The case of two components

$$\dot{f}_1^A = \int_{234} \text{Diagram A}$$


$$+ \int_{234} \text{Diagram B}$$


$$\dot{f}_1^B = \int_{234} \text{Diagram C}$$


$$+ \int_{234} \text{Diagram D}$$


The case of two components

$$\dot{f}_1^A = \int_{234} + \int_{234} + \int_{234} = 0$$
$$\dot{f}_1^B = \int_{234} + \int_{234}$$

The diagram illustrates the cancellation of terms in the Boltzmann equation for two components. It consists of four parts, each representing a term in the equation:

- The first part shows four blue lines meeting at a red circle, with a black arrow pointing to a central box labeled "0".
- The second part shows four green lines meeting at a red circle, with a black arrow pointing to the same central box labeled "0".
- The third part shows two blue lines and two green lines meeting at a red circle, with a black arrow pointing to the central box labeled "0".
- The fourth part shows two blue lines and two green lines meeting at a red circle, with a black arrow pointing to the central box labeled "0".

The central box labeled "0" indicates that the sum of all these terms is zero.

The case of two components

$$\dot{f}_1^A = \int_{234} \text{Diagram A}$$

$$+ \int_{234} \text{Diagram B}$$

$$f^A(E_3)f^A(E_1 + E_2 - E_3) = f^A(E_1)f^A(E_2)$$

$$E_1 + E_2 = E_3 + E_4$$

$$f^A(E_3)f^B(E_1 \oplus E_2 \ominus E_3) = f^A(E_1)f^B(E_2)$$

$$E_1 \oplus E_2 = E_3 \oplus E_4$$

$$f^B(E_3)f^A(E_1 \oplus E_2 \ominus E_3) = f^B(E_1)f^A(E_2)$$

$$f^B(E_3)f^B(E_1 + E_2 - E_3) = f^B(E_1)f^B(E_2)$$

$$\dot{f}_1^B = \int_{234} \text{Diagram C}$$

$$+ \int_{234} \text{Diagram D}$$

Long-time behaviour

- One-component

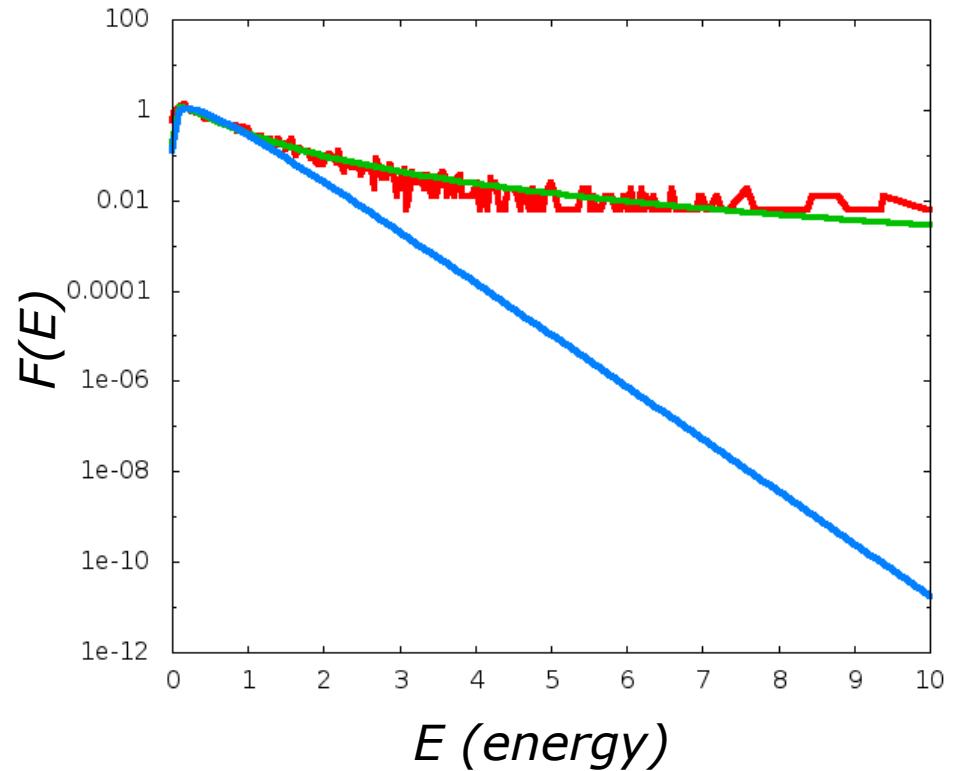
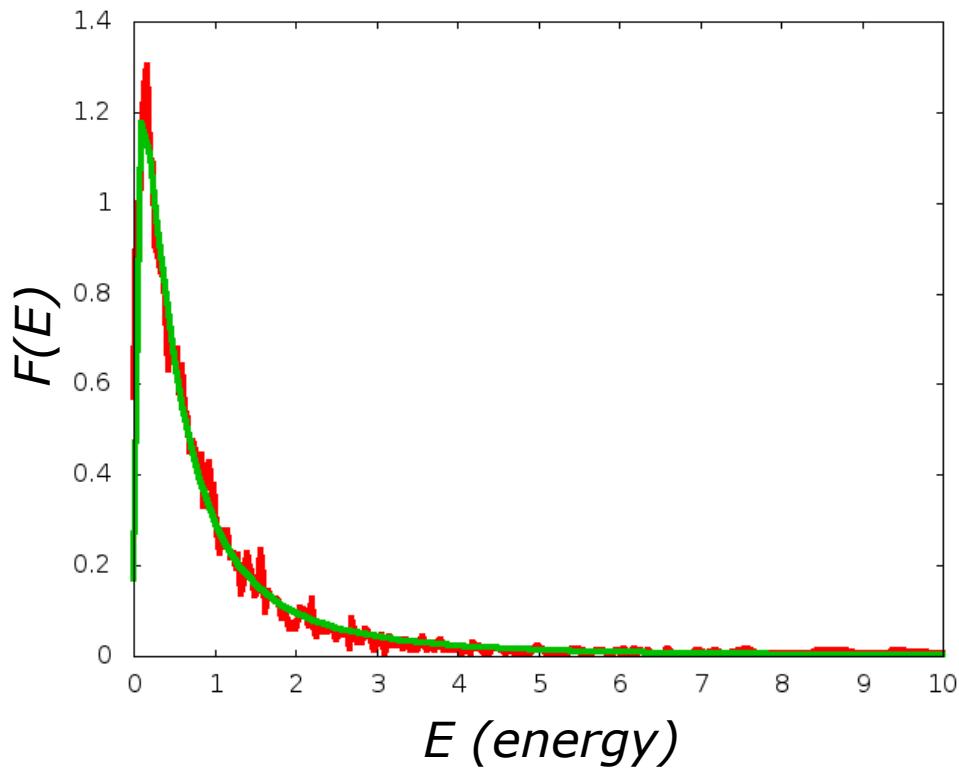
- detailed balance: **YES ✓**

- Two-component

- detailed balance: **NO ✗**
 - for long times: various scenarios
 - *warming or cooling system*
 - *saturation – in special cases*
 - *scaling solutions*

Long-time behaviour

- One-component
 - detailed balance: **YES** ✓



$$F_{eq}(E) \sim e^{-\beta L(E)} \sim (1 + AE)^{-\frac{B}{A}}$$

Two components – some details

simple modification $E \oplus_{\alpha\beta} E' = E + E' + A_{\alpha\beta}EE'$

modification scale fixed relative to avr. energy $A_{\alpha\beta} = \frac{a_{\alpha\beta}}{\langle E \rangle}$

case I.

$$a_{AA} = a_{BB} =: a_0 > 0$$

$$a_{AB} = a_{BA} =: a_1 > 0$$

a_0, a_1 are constant

case II.

$$a_0 > 0 \quad \Lambda = \Lambda_0 \langle E \rangle$$

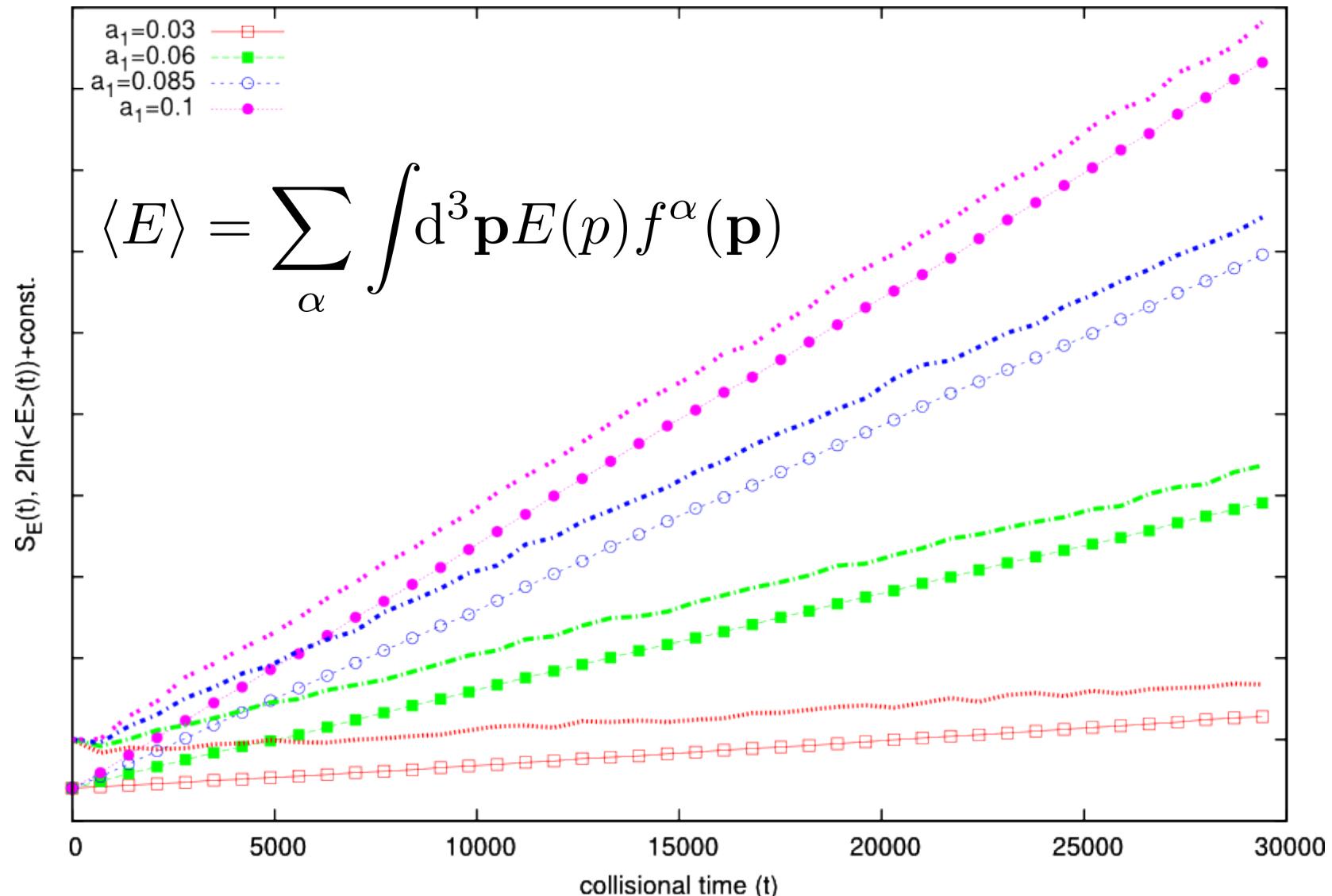
$$a_1 = \begin{cases} a_0, & E_1 + E_2 < \Lambda \\ b, & \Lambda < E_1 + E_2 \end{cases}$$

a_0, b, Λ_0 are constant

exact scaling: $\langle E \rangle \sim e^{\gamma t}$

Two components

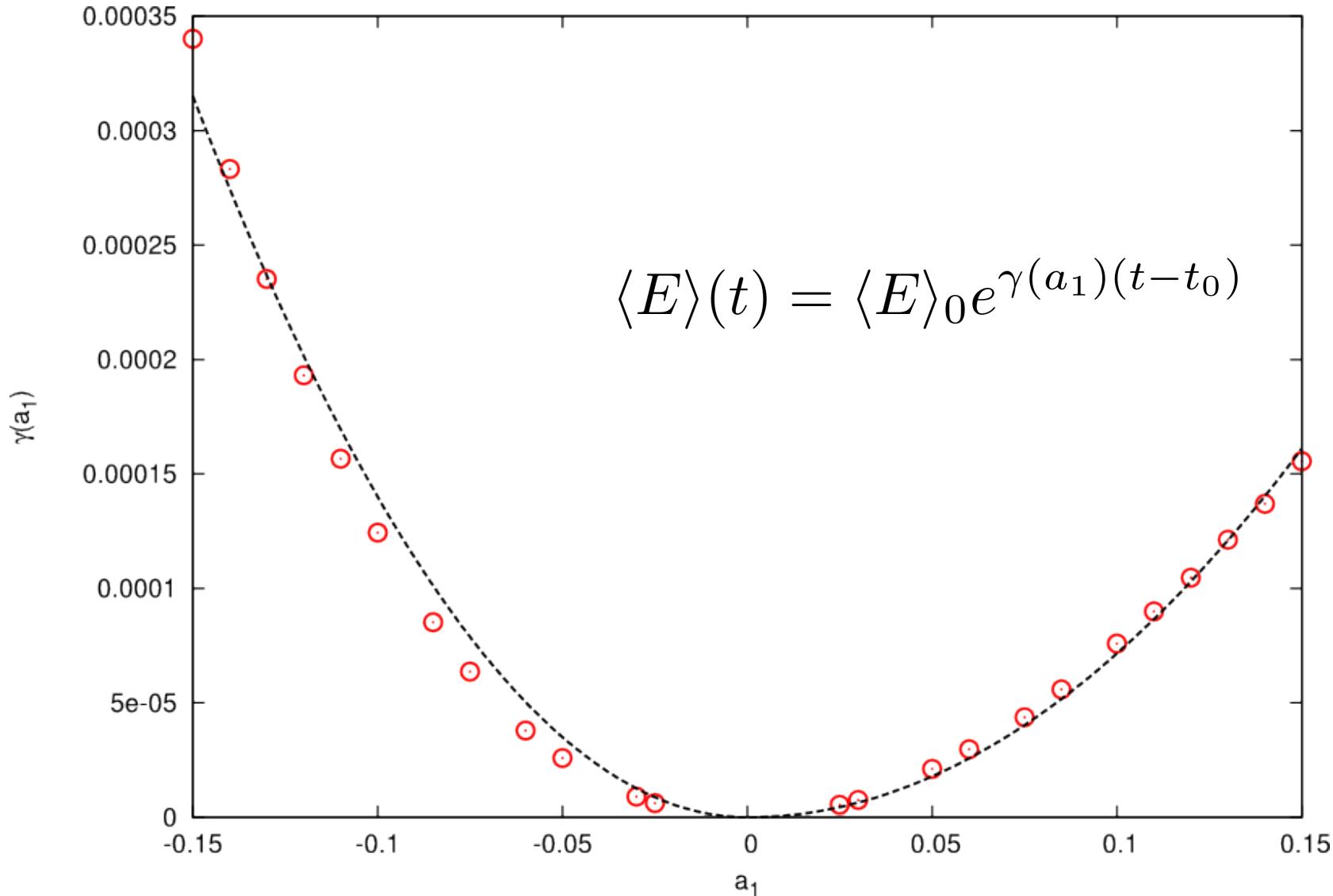
detailed balance **X**



Two components, case I.

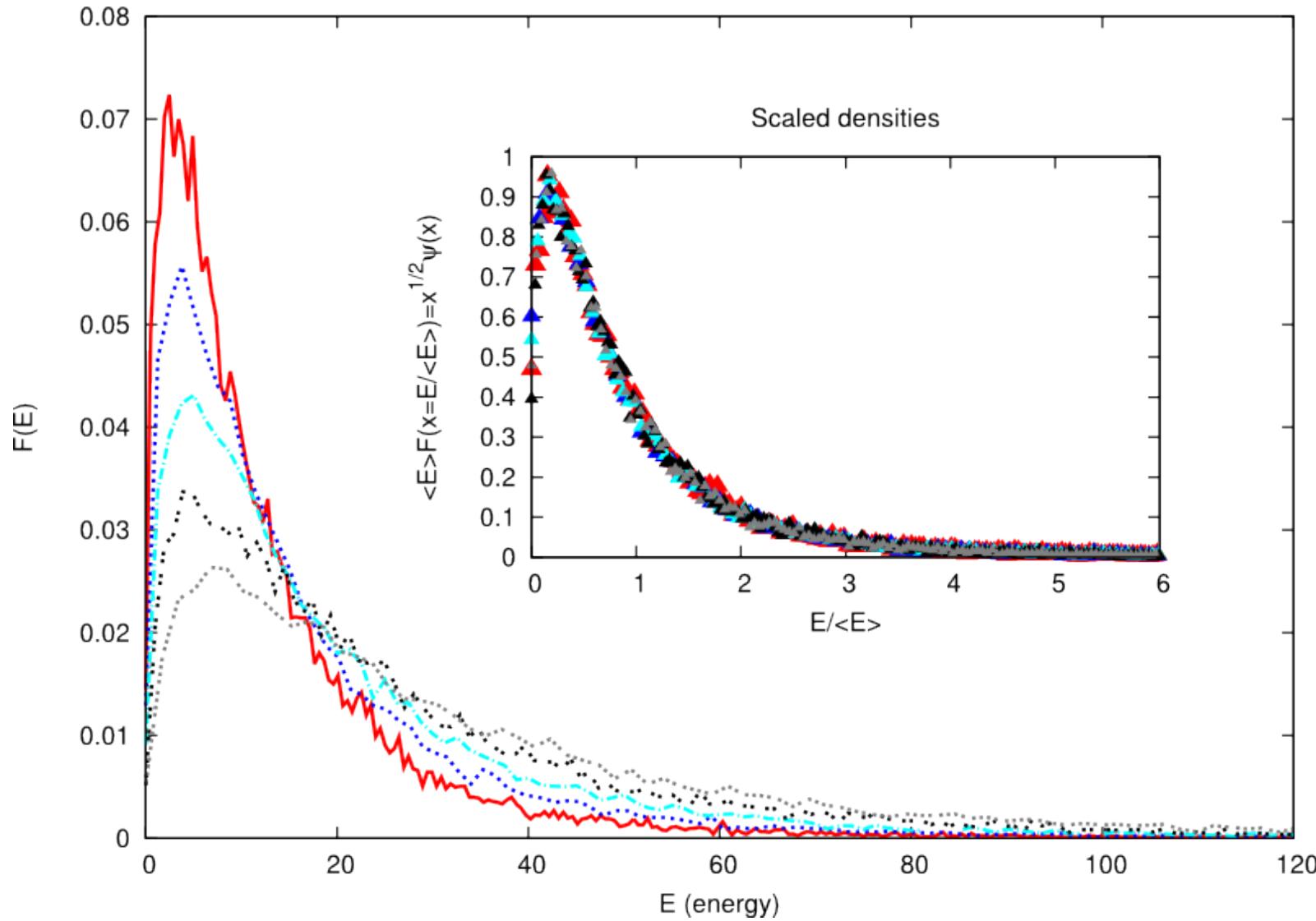
$$a_0 = 0$$

detailed balance X



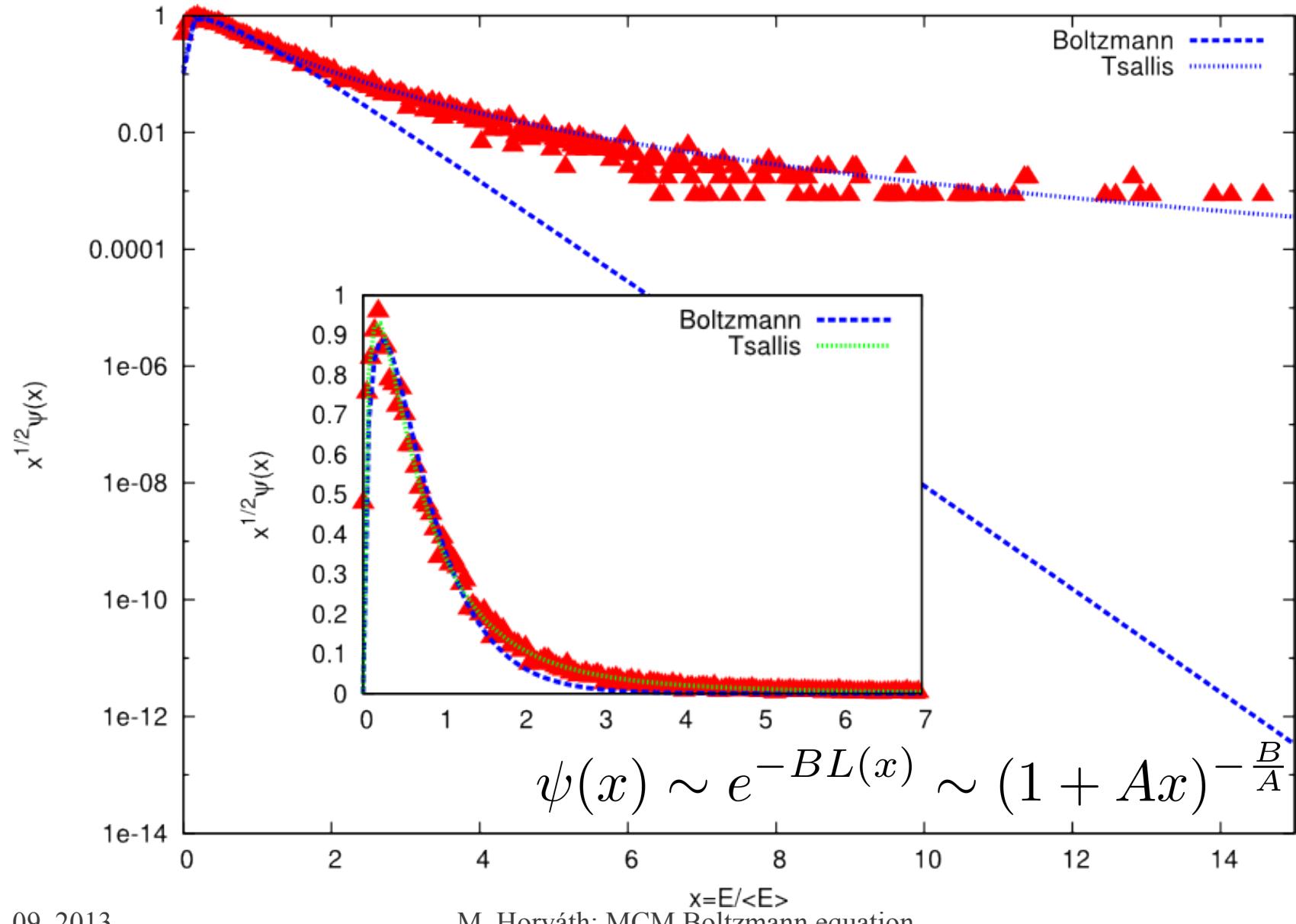
Two components

detailed balance **X**



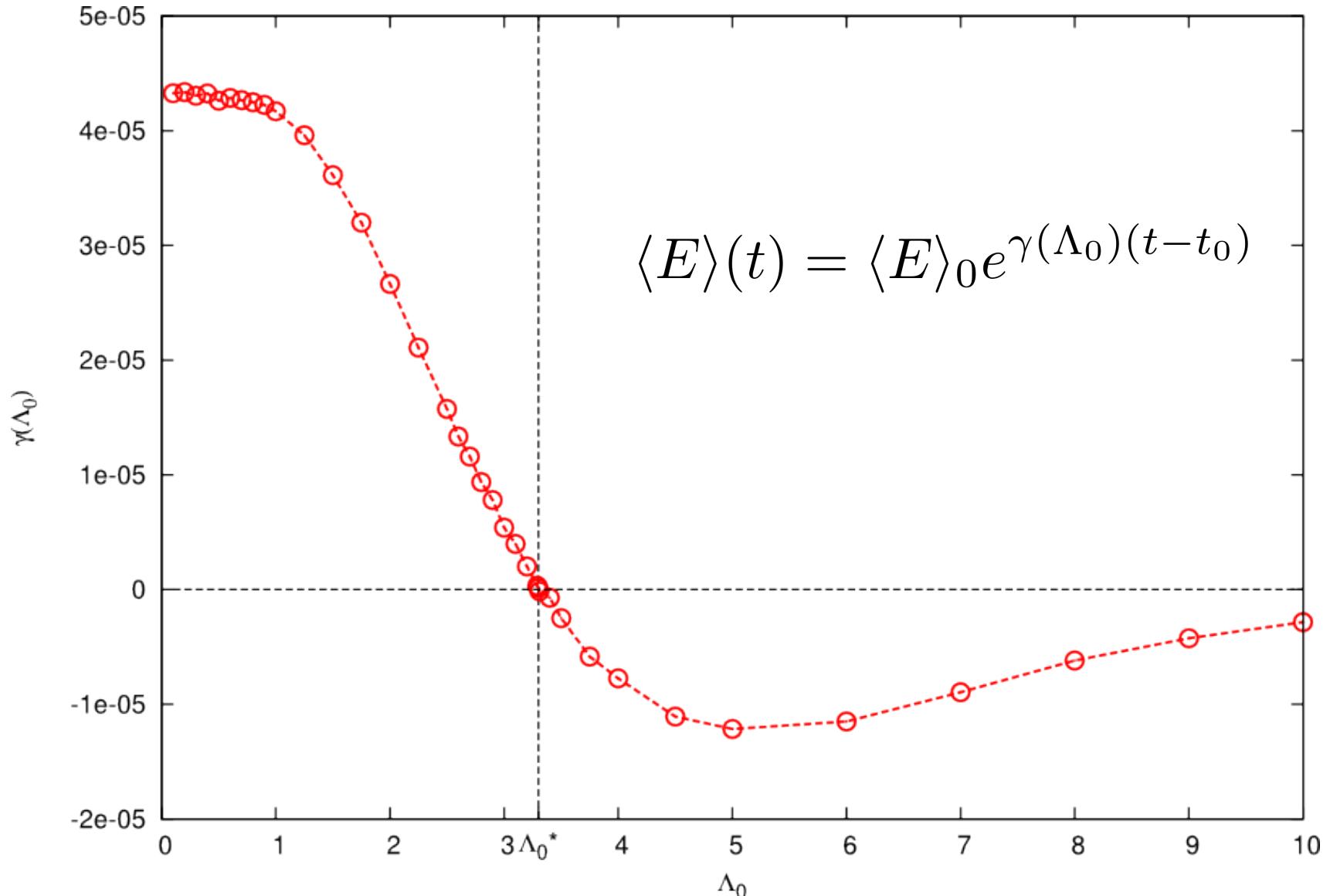
Two components

detailed balance **X**



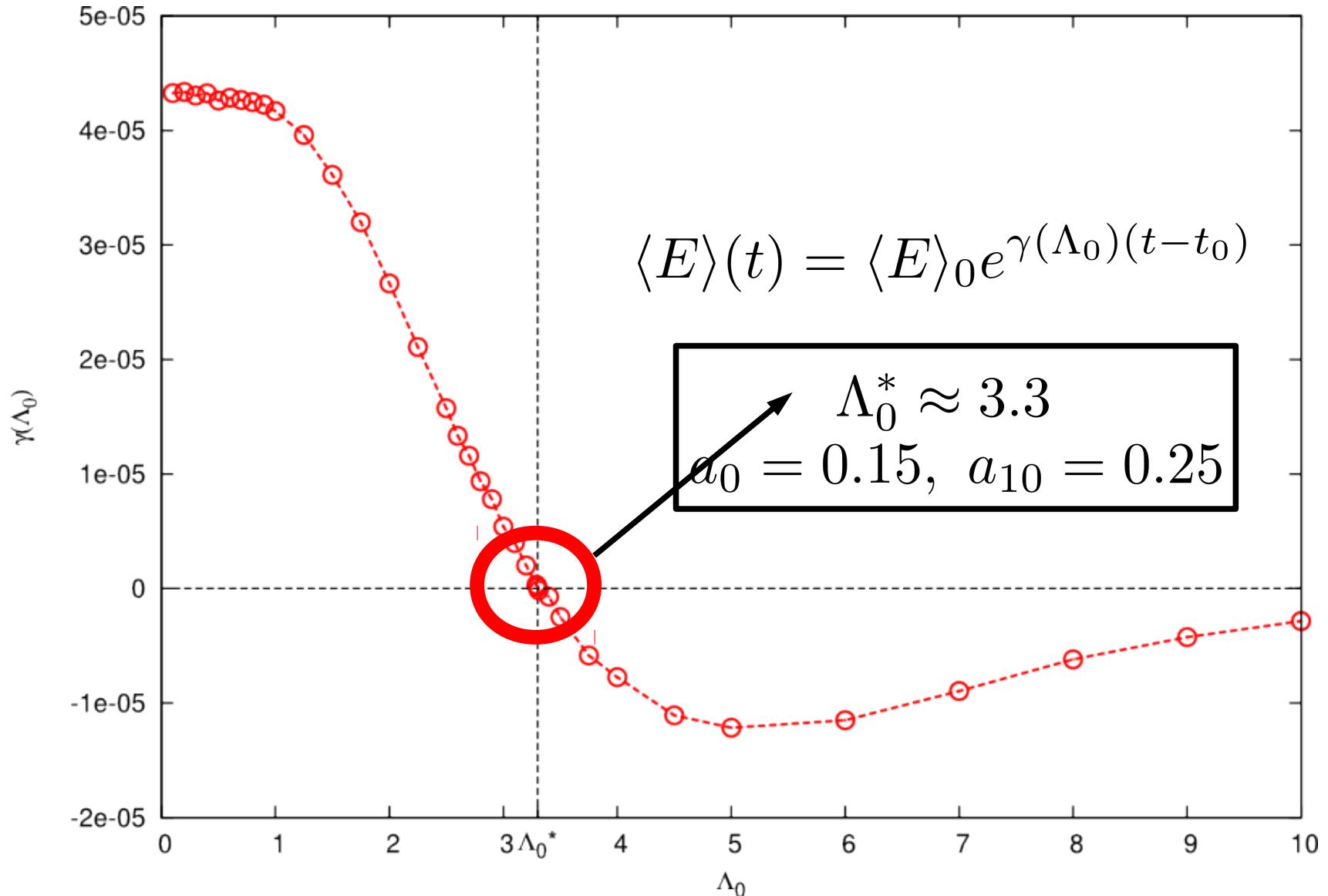
Two components, case II.

detailed balance **X**



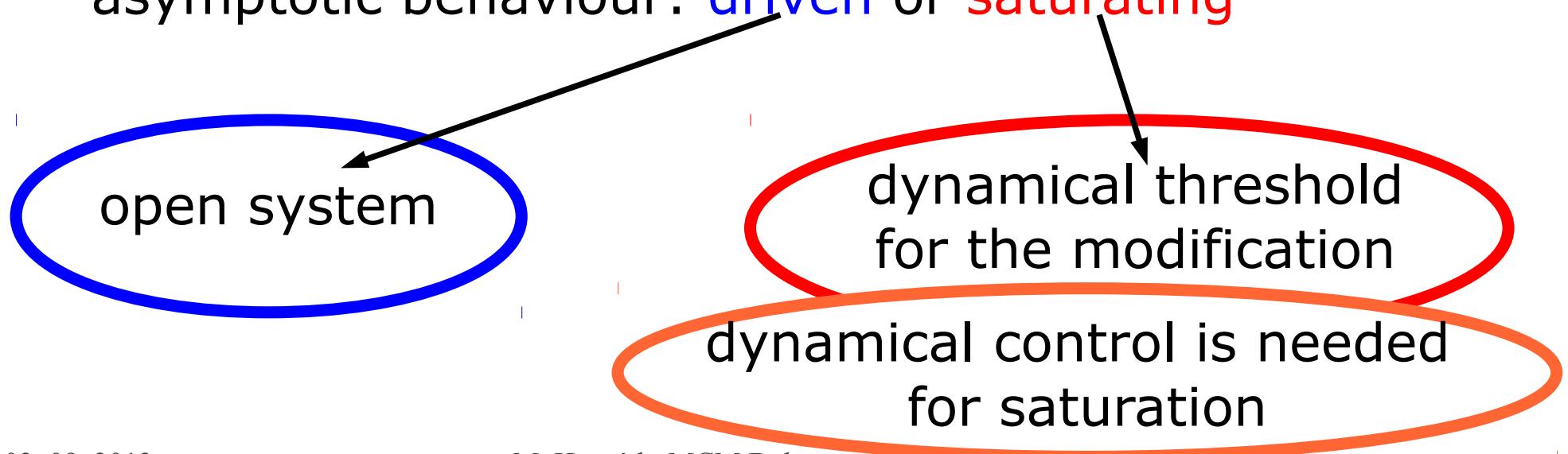
Two components, case II.

detailed balance X



Summary

- modification from a phenomenological viewpoint
- multicomponent – several types of collisions
- no detailed balance in MC case
- asymptotic behaviour: **driven** or **saturating**



Thank you for your attention!

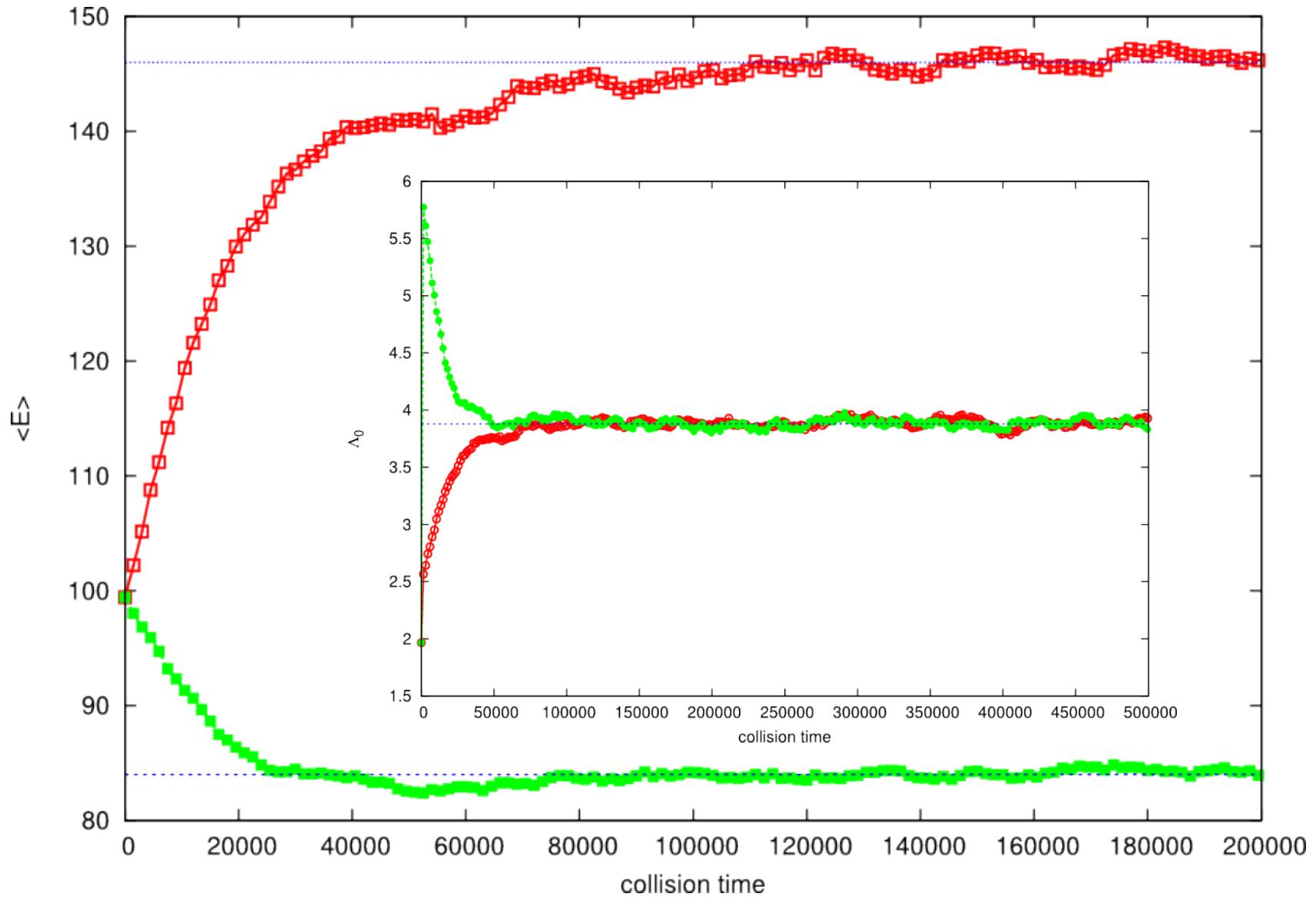
Questions?

Thank you for your attention!

Questions?

**Stay tuned!
arXiv: 1309.????**

Backups – case II.: saturation



Backups – modification by averaging

$$\begin{aligned}\mathcal{I}_\gamma &= \int_{234} \int d\omega_2 \delta(E_1 + \omega_2 - E_3 - E_4) \rho_\gamma(\omega_2 - E_2) \mathcal{K}_{1234} = \\ &= \int_{234} \rho_\gamma(E_3 + E_4 - E_1 - E_2) \mathcal{K}_{1234} = \int d^3 \mathbf{P} \int d^2 \sigma \rho_\gamma(\sigma) \mathcal{K}_{1234}(\sigma) \\ &= \int d^3 \mathbf{P} \int d\tilde{\omega} \int d\omega \rho_\gamma(\Omega(\tilde{\omega}) - \omega) \mathcal{K}_{1234}(\omega) = \\ &\quad = \int d^3 \mathbf{P} \int d\tilde{\omega} \int d\omega \rho_\gamma(\omega) \underbrace{\mathcal{K}_{1234}(\Omega - \omega)}_{= \mathcal{K}_{1234}|_{E_4 = E_1 + \tilde{\omega} - \omega}} ,\end{aligned}$$

Backups – modification by averaging

$$\mathcal{I} = \int_0^\infty d\gamma g(\gamma) \mathcal{I}_\gamma = \int d^3\mathbf{P} \int d\tilde{\omega} \int d\omega \mathcal{K}_{1234}(\Omega - \omega) J^\Delta(\omega) \approx$$

$$\mathcal{I}_\gamma = \int d^3\mathbf{P} \int_{(\tilde{\omega}, \omega) = \sigma} d^2\sigma \rho_\gamma(\omega) \mathcal{K}_{1234}(\Omega(\tilde{\omega}) - \omega),$$

$$J^\Delta(\omega) = \int_0^\infty d\gamma g(\gamma) \rho_\gamma(\omega)$$

Backups – modification by averaging

$$\begin{aligned} \mathcal{I} &= \int_0^\infty d\gamma g(\gamma) \mathcal{I}_\gamma = \int d^3 P \int d\tilde{\omega} \int d\omega \mathcal{K}_{1234}(\Omega - \omega) J^\Delta(\omega) \approx \\ &\approx \int d^3 P \int d\tilde{\omega} \left\{ \mathcal{K}_{1234}(\Omega - \Delta) \int d\omega J^\Delta(\omega) + \right. \\ &\quad + \frac{\partial}{\partial \omega} \mathcal{K}_{1234}(\Omega - \Delta) \int d\omega (\omega - \Delta) J^\Delta(\omega) + \\ &\quad + \frac{\partial^2}{\partial \omega^2} \mathcal{K}_{1234}(\Omega - \Delta) \int d\omega (\omega - \Delta)^2 J^\Delta(\omega) + \\ &\quad \left. + \int d\omega \mathcal{O}((\omega - \Delta)^3) J^\Delta(\omega) \right\} \approx \end{aligned}$$

Backups – modification by averaging

$$\begin{aligned}\mathcal{I} &= \int_0^\infty d\gamma g(\gamma) \mathcal{I}_\gamma = \int d^3\mathbf{P} \int d\tilde{\omega} \int d\omega \mathcal{K}_{1234}(\Omega - \omega) J^\Delta(\omega) \approx \\ &\approx \int d^3\mathbf{P} \int d\tilde{\omega} \underbrace{\mathcal{K}_{1234}(\Omega - \Delta)}_{= \mathcal{K}_{1234}|_{E_4 = E_1 + \tilde{\omega} - E_3 - \Delta}} + \mathcal{O}(\Delta)^2 \approx \\ &\approx \int_{234} \delta(E_1 + E_2 - E_3 - E_4 - \Delta) \mathcal{K}_{1234}\end{aligned}$$

Backups – detailed balance

for $\alpha = \beta$

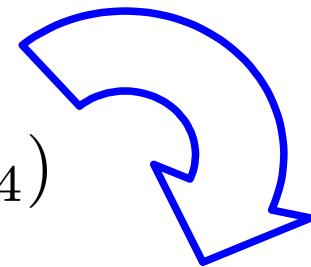
$$f^\alpha(E_1)f^\alpha(E_2) = f^\alpha(E_3)f^\alpha(E_4)$$

$$L^{\alpha\alpha}(E_1) + L^{\alpha\alpha}(E_2) = L^{\alpha\alpha}(E_3) + L^{\alpha\alpha}(E_4)$$



$$f^\alpha \sim e^{-\gamma_{\alpha\alpha} L^{\alpha\alpha}(E)}$$

$$f^\alpha(E_1)f^\beta(E_2) = f^\alpha(E_3)f^\beta(E_4)$$



$$\gamma_{\alpha\alpha} L^{\alpha\alpha}(E_1) + \gamma_{\beta\beta} L^{\beta\beta}(E_2) = \gamma_{\alpha\alpha} L^{\alpha\alpha}(E_3) + \gamma_{\beta\beta} L^{\beta\beta}(E_4)$$

AND for $\alpha \neq \beta$

$$L^{\alpha\beta}(E_1) + L^{\alpha\beta}(E_2) = L^{\alpha\beta}(E_3) + L^{\alpha\beta}(E_4)$$

Backups – model used in simulation

Constant rate function (no dep. on out-g. energies)

$$\dot{f}(p) =$$

$$= \int d^3 p' \int d^3 q \frac{1}{Z} \delta(K(p, p') - E(q) \oplus E(q^*)) (f(q)f(q^*) - f(p)f(p'))$$

$$= \int d^3 p' \int d^3 q \frac{1}{Z} \delta(\dots) f(q)f(q^*) - f(p) \underbrace{\int d^3 p' f(p') \int d^3 q \frac{1}{Z} \delta(\dots)}_{:=1}$$

$$Z(p_1, p_2, \mathbf{p}_1 \cdot \mathbf{p}_2) = \int d^3 q \delta(K - E(q) \oplus E(|\mathbf{P} - \mathbf{q}|))$$