Supremal Normal Sublanguages in Hierarchical Supervisory Control

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Abstract: In this paper, we study hierarchical supervisory control with partial observations. In particular, we are interested in preservation of supremal normal and supremal controllable and normal sublanguages from the abstracted system (high level) in the original (low level) system. Sufficient conditions are formulated under which the supremal normal or the supremal controllable and normal sublanguage computed at the high level (for the abstracted plant) is implementable at the low level, i.e., in the original plant.

Keywords: Discrete event system, supremal normal sublanguage, supremal controllable and normal sublanguage, hierarchical supervisory control, natural projection.

1. INTRODUCTION

In the study of large and complex discrete event plants there are two basic approaches to face the high (often prohibitive) computational complexity of their supervisor synthesis: *decentralized control* and *hierarchical control*. The decentralized (or distributed) approach can be seen as a horizontal abstraction, while the hierarchical approach can be seen as a vertical abstraction. Often both these perpendicular ways are combined in order to achieve a higher computational saving in the supervisory control synthesis, see Schmidt et al. (2008).

On one hand, there has been a considerable effort in applying the hierarchical control architecture to large discrete event systems (DES) modeled by automata with complete observations. Particularly, the sufficient conditions under which the high level synthesis of a nonblocking and optimal (the least restrictive) supervisor has a low level implementation are well known. These are called *observer condition*, and *output control consistency* (see Zhong and Wonham (1990)) or *local control consistency* (see Schmidt and Breindl (2008)). Mathematically, the key problem is when the supremal normal or supremal controllable and normal sublanguage of the high level specification commutes with the inverse abstraction (typically abstraction is given by a natural projection from all plant events, called *low level events*, to projected (*high level events*) that are used by the abstracted system).

On the other hand, however, there was almost no interest in applying the hierarchical control approach to partially observed automata, although it seems important to investigate the possibilities for the complexity reduction of the supervisor synthesis procedure known to be exponential for discrete event systems with partial observations. The basic supervisory control theorem under partial observations (see Cassandras and Lafortune (2008)) states that a specification language must be controllable, observable, and $L_m(G)$-closed in order to achieve the given specification as the language of the closed-loop system in a nonblocking manner, i.e., from all states reachable in the resulting automaton a marked state can be reached.

In the hierarchical control architecture the synthesis is done in the abstracted (high level) plant and the major problem is how to ensure that the high level solution is implementable at the low level, i.e., in the original plant. Of course, it is only implementable if observability and controllability are preserved in the original (low level) plant. Moreover, in the case when the (high level safety) specification language is not controllable or observable, an approximation (from below when considering safety specification) must be computed, i.e., a controllable and observable sublanguage. However, it is known that observability unlike controllability is not preserved under unions. Therefore, the supremal observable sublanguage does not always exist, and there are only maximal observable sublanguages, which are not unique in general. Fortunately, normality coincides with observability in the case when all controllable events are observable. This is a common assumption, which ensures that optimal solutions exist. It is therefore interesting to investigate when supremal normal sublanguages of high level specifications commute with the inverse projection (to high level events).

In this paper, the commutation, which is equivalent to the preservation of supremal normal and supremal controllable and normal sublanguages of the abstracted systems in the original system, is investigated. A simple algebraic formula for supremal normal and supremal controllable and normal sublanguages given in Brandt et al. (1990) is used in our results that specify preliminary sufficient conditions for this commutation in the case of prefix-closed specifications. Using another approach, inspired by a similar result for supremal controllable sublanguages from Feng (2007), our first result is extended to the general case, where the specification need not be a prefix-closed language. Note that our condition is also necessary in the sense that if it is not satisfied, then we can find a low level plant language and a high level specification language so that the supremal (controllable and) normal sublanguage computed in the abstracted (high level) system disagree with the supremal (controllable and) normal sublanguage computed in the original (low level) system, see Example 10.

The organization of this paper is as described below. In the next section preliminary fundamental results from supervisory control theory are recalled. Then, Section 3 presents the hierarchical control with partial observations, where the sufficient conditions for preservation of the high level supremal normal sublanguages at the low level are formulated in the case of prefix-closed languages. In Section 4, the sufficient conditions under which the supremal controllable and normal sublanguages of the high level are preserved by the inverse abstraction, i.e., they are implementable at the physical (low) level, are studied in the case of prefix-closed languages, and Section 5 then discusses the case of general (i.e., not necessarily prefix-closed) specifications.

2. PRELIMINARIES AND DEFINITIONS

The reader interested in further concepts and results concerning supervisory control is referred to Wonham (2009) or Cassandras and Lafortune (2008) for more details.

For an event alphabet (finite nonempty set) A, A^* denotes the free monoid generated by A. For $s_1, s_2 \in A^*$, $s_1 \leq s_2$ denotes that s_1 is a *prefix* of s_2, i.e., there exists $s \in A^*$ such that $s_2 = s_1 s$. The empty string is denoted by λ. A *language* L over the alphabet A is a subset $L \subseteq A^*$. The *prefix closure* of L is defined as $\overline{L} = \{s_1 \in A^* : s_1 \leq s$ for some $s \in L\}$. A language L is *prefix-closed* if $L = \overline{L}$.

Let $G = (S, A, f, s_0, Q_m)$ be a DES generator consisting of a finite *state set* S, an *event alphabet* A, a (partial) *transition function* $f : S \times A \to S$, the *initial state* $s_0 \in S$, and the set of *marked states* $Q_m \subseteq Q$. The standard notations for the prefix-closed *language* of G is $L(G) = \{w \in A^* : f(s_0, w) \in Q\}$, and for the *marked language* of G is $L_m(G) = \{w \in A^* : f(s_0, w) \in Q_m\}$. The DES is *nonblocking* if $\overline{L_m(G)} = L(G)$.

The *natural projection* $P : A^* \to A_0^*$, where $A_0 \subseteq A$, is a homomorphism defined so that $P(a) = \lambda$, for $a \in A - A_0$, and $P(a) = a$, for $a \in A_0$. The *inverse image* of P is denoted by $P^{-1} : A_0^* \to 2^{A^*}$.

In supervisory control theory with partial observations, one has $A = A_c \,\dot\cup\, A_{uc} = A_o \,\dot\cup\, A_{uo}$, where $\dot\cup$ denotes the disjoint union, to distinguish the sets A_c of *controllable* events and A_{uc} of *uncontrollable* events, A_o of *observable* events and A_{uo} of *unobservable* events. In hierarchical supervisory control, the abstracted (high level) plant is often simply the plant projected to the subset of high level events, denoted by A^{hi}. Thus, there is a similar decomposition $A = A^{hi} \,\dot\cup\, (A - A^{hi})$, and the associated projection (abstraction) is denoted by $Q : A^* \to (A^{hi})^*$.

Let L be a prefix-closed language and $K \subseteq L \subseteq A^*$. K is *normal* with respect to L and a natural projection P if, for all $s \in L$, $s \in \overline{K}$ if and only if $P(s) \in P(\overline{K})$, i.e.,

$$\overline{K} = P^{-1}(P(\overline{K})) \cap L\,.$$

In addition, let $\mathrm{N}(K, L, P)$ denote the set of all normal sublanguages of K with respect to L and P. Furthermore, K is *controllable* with respect to L and A_{uc} if

$$\overline{K} A_{uc} \cap L \subseteq \overline{K}\,.$$

Let $\mathrm{C}(K, L)$ denote the set of all controllable sublanguages of K with respect to L and A_{uc}. Let $\mathrm{CN}(K, L, P)$ denote the set of all controllable and normal sublanguages of K with respect to L, P, and A_{uc}. Note that A_{uc} is omitted from this notation because it will be clear from the context. In this paper, controllability is considered either at the low level and then with respect to A_{uc}, or at the high level and then with respect to $A_{uc} \cap A^{hi}$.

As it is known that a unique supremal normal (controllable) sublanguage exists, we can denote the supremal languages of $\mathrm{N}(K, L, P)$, $\mathrm{C}(K, L)$, and $\mathrm{CN}(K, L, P)$ by $\sup \mathrm{N}(K, L, P)$, $\sup \mathrm{C}(K, L)$, and $\sup \mathrm{CN}(K, L, P)$, respectively. Recall that there are algorithms for computation of these supremal languages. The reader is referred to Cassandras and Lafortune (2008); Komenda and van Schuppen (2005); Zad et al. (2005) for more details.

Let $K \subseteq L = \overline{L} \subseteq A^*$, where $A = A_c \,\dot\cup\, A_{uc} = A_o \,\dot\cup\, A_{uo}$. Let $P : A^* \to A_o^*$ be a natural projection. Then, K is *observable* with respect to L, A_o, and A_c if, for all $s \in \overline{K}$ and $c \in A_c$, $sc \in L$ and $sc \notin \overline{K}$ imply that

$$P^{-1}(P(s))\{c\} \cap \overline{K} = \emptyset\,.$$

There is also an algorithm for computation of controllable and observable sublanguages, see Takai and Ushio (2003).

In contrast to normal or controllable languages, no supremal observable sublanguage exists. However, normality implies observability as shown in Lin and Wonham (1988).

3. SUPREMAL NORMAL SUBLANGUAGES

This section presents sufficient conditions preserving the high level supremal normal sublanguages at the low level in the hierarchical control of discrete event systems with partial observations in the case of prefix-closed specifications.

First, recall the formula for supremal normal sublanguages of $K = \overline{K} \subseteq L = \overline{L} \subseteq A^*$ with respect to L and P shown in Brandt et al. (1990).

Fact 1. For any two prefix-closed languages $K \subseteq L \subseteq A^*$ and a natural projection $P : A^* \to A_o^*$,

$$\sup \mathrm{N}(K, L, P) = K - P^{-1}(P(L - K))A^*\,.$$

The following notation is implicitly used in this paper.

Definition 2. Let

$$Q : A^* \to (A^{hi})^* \qquad P : A^* \to A_o^*$$
$$Q^o : A_o^* \to (A_o \cap A^{hi})^* \qquad P^{hi} : (A^{hi})^* \to A_o^*$$

be natural projections such that the diagram in Fig. 1 commutes, i.e., $Q^o \circ P = P^{hi} \circ Q$.

These properties hold for natural projections.

Lemma 3. Let R be a natural projection and A, B be two languages. Then, $R^{-1}(A - B) = R^{-1}(A) - R^{-1}(B)$.

Lemma 4. Let $M \subseteq B^* \subseteq A^*$ and $R : A^* \to B^*$ be a natural projection. Then, M is prefix-closed if and only if $R^{-1}(M)$ is prefix-closed.

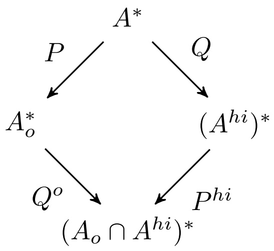

Fig. 1. Natural projections used in this paper.

The following theorem shows that the supremal normal sublanguage of the low level can be computed at the abstracted (high) level.

Theorem 5. Let $K \subseteq (A^{hi})^*$ and $L \subseteq A^*$ be prefix-closed languages. If $Q^{-1}(K) \subseteq L$ and $A_o \subseteq A^{hi}$, then

$$Q^{-1}(\sup \mathrm{N}(K, Q(L), P^{hi})) = \sup \mathrm{N}(Q^{-1}(K), L, P)\,.$$

Proof. First, $A_o \subseteq A^{hi}$ implies that Q^o is an identity. Then, by Fact 1, we have

$$\begin{aligned} &Q^{-1}(\sup \mathrm{N}(K, Q(L), P^{hi})) \\ &= Q^{-1}(K - P^{hi^{-1}} P^{hi}(Q(L) - K)(A^{hi})^*) \\ &= Q^{-1}(K) - Q^{-1} P^{hi^{-1}} P^{hi}(Q(L) - K) Q^{-1}((A^{hi})^*) \\ &= Q^{-1}(K) - Q^{-1} P^{hi^{-1}} P^{hi}(Q(L) - K) A^*\,. \end{aligned}$$

As $P^{-1} = Q^{-1} P^{hi^{-1}}$ and $P^{hi} = P^{hi} Q Q^{-1} = P Q^{-1}$, we also have

$$\begin{aligned} &Q^{-1}(K) - Q^{-1} P^{hi^{-1}} P^{hi}(Q(L) - K) A^* \\ &\quad = Q^{-1}(K) - P^{-1} P Q^{-1}(Q(L) - K) A^* \\ &\quad = Q^{-1}(K) - P^{-1} P (Q^{-1} Q(L) - Q^{-1}(K)) A^*\,. \end{aligned}$$

Now, as $L \subseteq Q^{-1} Q(L)$,

$$P^{-1} P(Q^{-1} Q(L) - Q^{-1}(K)) \supseteq P^{-1} P(L - Q^{-1}(K))\,.$$

On the other hand, let $u \in Q^{-1} Q(L) - Q^{-1}(K)$ be such that $u \notin L$. As $Q(u) \in Q(L)$, there exists $w \in L$ such that $Q(w) = Q(u)$ and $w \notin Q^{-1}(K)$; otherwise, $w \in Q^{-1}(K)$ implies that $Q(u) = Q(w) \in K$, hence $u \in Q^{-1}(K)$ which is a contradiction. However, $P^{-1}P(u) = P^{-1} P^{hi} Q(u) = P^{-1} P^{hi} Q(w) = P^{-1} P(w)$. Therefore,

$$\begin{aligned} &Q^{-1}(K) - P^{-1} P(Q^{-1} Q(L) - Q^{-1}(K)) A^* \\ &\qquad = Q^{-1}(K) - P^{-1} P(L - Q^{-1}(K)) A^* \\ &\qquad = \sup \mathrm{N}(Q^{-1}(K), L, P)\,, \end{aligned}$$

which completes the proof. □

The following result is an immediate consequence of the previous theorem.

Corollary 6. Let $K \subseteq (A^{hi})^*$ and $L \subseteq A^*$ be prefix-closed languages. Let $Q^{-1}(K) \subseteq L$ and $A_o \subseteq A^{hi}$. If K is normal with respect to $Q(L)$ and P^{hi}, then $Q^{-1}(K)$ is normal with respect to L and P.

On the other hand, the following example demonstrates that the condition $A_o \subseteq A^{hi}$ is (in some sense) necessary.

Example 7. Let $A = \{a, b, c\}$, where $A_o = \{a, c\}$, $A_{uo} = \{b\}$, and $A^{hi} = \{b, c\}$. Let

$$L = \{abc\} \cup a^*ba^* \cup a^*ca^* \cup a^* \quad \text{and} \quad K = \{\lambda, b, c\}\,.$$

Using Fact 1, we have

- $Q^{-1}(K) = a^*ba^* \cup a^*ca^* \cup a^* \subseteq L$,
- $L - Q^{-1}(K) = \{abc\}$, and
- $P^{-1}P(abc) = P^{-1}(ac) = b^*ab^*cb^*$,

which implies $c \in Q^{-1}(K) - P^{-1}P(L - Q^{-1}(K))A^*$, i.e., $c \in \sup \mathrm{N}(Q^{-1}(K), L, P)$. On the other hand, however,

- $Q(L) - K = \{\lambda, b, c, bc\} - \{\lambda, b, c\} = \{bc\}$,
- $P^{hi}(bc) = c$, and $P^{hi^{-1}}(c) = b^*cb^*$.

Thus, we have $c \notin Q^{-1}(K - P^{hi^{-1}} P^{hi}(Q(L) - K)A^{hi^*}) = Q^{-1}(\{\lambda, b\}) = a^* \cup a^*ba^* = Q^{-1}(\sup \mathrm{N}(K, Q(L), P^{hi}))$. ◇

Theorem 5 and Example 7 imply the following result.

Corollary 8. The following equivalence holds.

$$Q^{-1}(\sup \mathrm{N}(K, Q(L), P^{hi})) = \sup \mathrm{N}(Q^{-1}(K), L, P) \quad (1)$$

for all prefix-closed languages K and L such that $K \subseteq (A^{hi})^*$ and $Q^{-1}(K) \subseteq L \subseteq A^*$ if and only if $A_o \subseteq A^{hi}$.

Remark that the necessity part of Corollary 8 only holds in a restricted sense, i.e., there might exist languages K and L such that $K \subseteq (A^{hi})^*$, $Q^{-1}(K) \subseteq L \subseteq A^*$, and $A_o \not\subseteq A^{hi}$ satisfying equality (1). However, if the equality holds for all such languages K and L, then necessarily $A_o \subseteq A^{hi}$.

Note that we could study a condition called output normal consistency as an analogy to output control consistency discussed in Schmidt and Breindl (2008) or Feng (2007).

Definition 9. The natural projection $Q : A^* \to (A^{hi})^*$ is *output normal consistent* (ONC) for $L \subseteq A^*$ if for every $s \in \overline{L}$ of the form

$$s = \sigma_1 \ldots \sigma_k \quad \text{or} \quad s = s'\sigma_0\sigma_1 \ldots \sigma_k, \quad k \geq 1\,,$$

where $s' \in A^*$, $\sigma_0, \sigma_k \in A^{hi}$ and $\sigma_i \in A - A^{hi}$, for $i = 1, \ldots, k-1$, if $\sigma_k \in A_{uo}$, then $\sigma_i \in A_{uo}$, for $i = 1, \ldots, k-1$.

However, if we assume that there exists $a \in A_o - A^{hi}$, then if there also exists $b \in A^{hi} \cap A_{uo}$ such that $ubv \in K$, for some $u, v \in (A^{hi})^*$, then $uabv \in Q^{-1}(K) \subseteq L$. Thus, Q is not ONC for L.

The assumption that $ubv \in K$ can be justified so that if no such string containing an unobservable high level event b exists in K, i.e., all strings of K are composed of observable events only, we have that $P^{hi}(K) = K$ and, clearly, K is observable with respect to $Q(L)$ and A^{hi}. In this case, however, our approach using supremal normal sublanguages is not needed because the high level supervisor can be synthesized using K itself.

This observation motivates the weakening of our assumption from $Q^{-1}(K) \subseteq L$ to $K \subseteq Q(L)$. Nevertheless, the following example shows that the assumption $A_o \subseteq A^{hi}$ is still required.

Example 10. Let $A = \{a, b, c\}$, where $A_o = \{a, c\}$, $A_{uo} = \{b\}$, and $A^{hi} = \{b, c\}$. Let

$$L = \{\lambda, a, b, c, ba, bac\} \quad \text{and} \quad K = \{\lambda, b, c\} \subseteq Q(L)\,.$$

Note that $A_o \not\subseteq A^{hi}$. It is not hard to see that Q is ONC for L because there are no low level observable symbols preceding b, the only uncontrollable high level event, in any word of L. Furthermore, using Fact 1, we get the following.

- $Q^{-1}(K) \cap L = a^*ba^* \cup a^*ca^* \cup a^* \cap L = \{\lambda, a, b, c, ba\}$,
- $L - Q^{-1}(K) \cap L = \{bac\}$, and

- $P^{-1}P(bac) = P^{-1}(ac) = b^*ab^*cb^*$.

Thus, we have $c \in Q^{-1}(K) \cap L - P^{-1}P(L - Q^{-1}(K))A^* = \{\lambda, a, b, c, ba\} = \sup \mathrm{N}(Q^{-1}(K) \cap L, L, P)$. On the other hand, however,

- $Q(L) - K = \{\lambda, b, c, bc\} - \{\lambda, b, c\} = \{bc\}$,
- $P^{hi}(bc) = c$, and $P^{hi^{-1}}(c) = b^*cb^*$.

Thus, we obtain that $c \notin Q^{-1}(\sup \mathrm{N}(K, Q(L), P^{hi})) \cap L = Q^{-1}(K - P^{hi^{-1}}P^{hi}(Q(L) - K)A^{hi^*}) \cap L = Q^{-1}(\{\lambda, b\}) \cap L = \{\lambda, a, b, ba\}$. $\diamond$

Considering $A_o \subseteq A^{hi}$, the following result is an analogous result to Theorem 5 above.

Theorem 11. For any two prefix-closed languages $L \subseteq A^*$ and $K \subseteq Q(L)$. If $A_o \subseteq A^{hi}$, then

$$Q^{-1}(\sup \mathrm{N}(K, Q(L), P^{hi})) \cap L = \sup \mathrm{N}(Q^{-1}(K) \cap L, L, P)\,.$$

Proof. Again, by Fact 1, we have

$$\begin{aligned} &Q^{-1}(\sup \mathrm{N}(K, Q(L), P^{hi})) \cap L \\ &= Q^{-1}(K - P^{hi^{-1}}P^{hi}(Q(L) - K)(A^{hi})^*) \cap L \\ &= \left(Q^{-1}K - Q^{-1}P^{hi^{-1}}P^{hi}(Q(L) - K)Q^{-1}A^{hi^*}\right) \cap L \\ &= Q^{-1}(K) \cap L - Q^{-1}P^{hi^{-1}}P^{hi}(Q(L) - K)A^*\,. \end{aligned}$$

The last equality follows from the fact that $(X - Y) \cap Z = X \cap Z - Y$. In addition, as $P^{-1} = Q^{-1}P^{hi^{-1}}$ and $P^{hi} = P^{hi}QQ^{-1} = PQ^{-1}$, we also have

$$\begin{aligned} &Q^{-1}(K) \cap L - Q^{-1}P^{hi^{-1}}P^{hi}(Q(L) - K)A^* \\ &\quad = Q^{-1}(K) \cap L - P^{-1}PQ^{-1}(Q(L) - K)A^* \\ &\quad = Q^{-1}(K) \cap L - P^{-1}P(Q^{-1}Q(L) - Q^{-1}(K))A^*\,. \end{aligned}$$

Analogously as above, we can prove that

$$P^{-1}P(Q^{-1}Q(L) - Q^{-1}(K)) = P^{-1}P(L - Q^{-1}(K))\,.$$

Therefore,

$$\begin{aligned} &Q^{-1}(K) \cap L - P^{-1}P(Q^{-1}Q(L) - Q^{-1}(K))A^* \\ &\qquad = Q^{-1}(K) \cap L - P^{-1}P(L - Q^{-1}(K))A^* \\ &\qquad = Q^{-1}(K) \cap L - P^{-1}P(L - Q^{-1}(K) \cap L)A^* \\ &\qquad = \sup \mathrm{N}(Q^{-1}(K) \cap L, L, P)\,, \end{aligned}$$

which completes the proof. $\square$

From Theorem 11 and Example 10, we immediately have the following consequence similar to Corollary 8.

Corollary 12. The following equivalence holds.

$$Q^{-1}(\sup \mathrm{N}(K, Q(L), P^{hi})) \cap L = \sup \mathrm{N}(Q^{-1}(K) \cap L, L, P)$$

for all two prefix-closed languages $L \subseteq A^*$ and $K \subseteq Q(L)$ if and only if $A_o \subseteq A^{hi}$.

4. SUPREMAL CONTROLLABLE AND NORMAL SUBLANGUAGES

In what follows, the result of the previous section is generalized to the case where preservation of both controllability and normality is considered in the supremal sublanguage between the computations at high and low levels.

By Brandt et al. (1990), we have the following formula for supremal controllable and normal sublanguages.

Fact 13. For any two prefix-closed languages $B \subseteq M \subseteq A^*$ and a natural projection $R : A^* \to A_o^*$,

$$\begin{aligned} &\sup \mathrm{CN}(B, M, R) \\ &\qquad = M \cap R^{-1}(\sup \mathrm{C}(R(\sup \mathrm{N}(B, M, R)), R(M)))\,. \end{aligned}$$

Similarly as in Brandt et al. (1990) the set of uncontrollable events is not listed to simplify the formulas. In fact, $\sup \mathrm{CN}(B, M, R)$ stands for the supremal controllable and normal sublanguages of B with respect to M, a natural projection R, and the uncontrollable event set A_{uc}.

To prove the main result concerning prefix-closed languages, we need the following lemma.

Lemma 14. Let $Q : A^* \to (A^{hi})^*$ be as defined above, and let $Y \subseteq A^*$ and $X \subseteq Q(Y)$ be two languages. Then,

$$Q(Q^{-1}(X) \cap Y) = X\,.$$

Proof. Let $x \in Q(Q^{-1}(X) \cap Y)$, then there exists $y \in Q^{-1}(X) \cap Y$ such that $Q(y) = x$. As $y \in Q^{-1}(X)$, we have that $x = Q(y) \in X$. On the other hand, let $x \in X$. As $X \subseteq Q(Y)$, there exists $y \in Y$ such that $Q(y) = x$. Thus, we have that $y \in Q^{-1}(X) \cap Y$, which implies that $x \in Q(Q^{-1}(X) \cap Y)$. $\square$

Now, we can prove the following theorem, which says that the supremal controllable and normal sublanguage of a prefix-closed language of the low level can be computed at the high level.

Theorem 15. For any two prefix-closed languages $L \subseteq A^*$ and $K \subseteq Q(L)$. If $A_o \subseteq A^{hi}$, then

$$Q^{-1}(\sup \mathrm{CN}(K, Q(L), P^{hi})) \cap L = \sup \mathrm{CN}(Q^{-1}(K) \cap L, L, P)\,.$$

Proof. First we make clear that the controllability of the sublanguage on the left hand side is meant with respect to uncontrollable event set $P^{hi}(A_{uc}) = A^{hi} \cap A_{uc}$, while the controllability of the sublanguage on the right hand side is meant with respect to A_{uc}. As $A_o \subseteq A^{hi}$, we have that $P = P^{hi}Q$.

Now, note that by Theorem 11, we have

$$Q^{-1}(\sup \mathrm{N}(K, Q(L), P^{hi})) \cap L = \sup \mathrm{N}(Q^{-1}(K) \cap L, L, P)\,.$$

As $\sup \mathrm{N}(K, Q(L), P^{hi}) \subseteq K \subseteq Q(L)$, we obtain using Lemma 14 that

$$\begin{aligned} &Q(\sup \mathrm{N}(Q^{-1}(K) \cap L, L, P)) \\ &\qquad = Q(Q^{-1}(\sup \mathrm{N}(K, Q(L), P^{hi})) \cap L) \\ &\qquad = \sup \mathrm{N}(K, Q(L), P^{hi})\,. \end{aligned}$$

By Fact 13, we have

$$\begin{aligned} &Q^{-1}(\sup \mathrm{CN}(K, Q(L), P^{hi})) \cap L \\ &\quad = L \cap Q^{-1}Q(L) \cap \\ &\qquad P^{hi^{-1}} \sup \mathrm{C}(P^{hi} \sup \mathrm{N}(K, Q(L), P^{hi}), P^{hi}Q(L))\,. \end{aligned}$$

Note that the controllability is with respect to $A_{uc} \cap A^{hi} \cap A_o = P^{hi}(Q(A_{uc})) = P(A_{uc})$. Therefore,

$= Q^{-1}Q(L) \cap L \cap$
$Q^{-1}{P^{hi}}^{-1}\sup \mathrm{C}(P^{hi}\sup \mathrm{N}(K, Q(L), P^{hi}), P(L))$
$= L \cap P^{-1}\sup \mathrm{C}(P^{hi}\sup \mathrm{N}(K, Q(L), P^{hi}), P(L))$
$= L \cap P^{-1}\sup \mathrm{C}(P^{hi}Q \sup \mathrm{N}(Q^{-1}(K) \cap L, L, P), P(L))$
$= L \cap P^{-1}\sup \mathrm{C}(P \sup \mathrm{N}(Q^{-1}(K) \cap L, L, P), P(L))$
$= \sup \mathrm{CN}(Q^{-1}(K) \cap L, L, P)\,,$

which proves the theorem. □

Note that we do not have to limit ourselves to the situation where $K \subseteq Q(L)$. Assume that a given specification language K' is such that $K' \nsubseteq Q(L)$. Then, however, it is of interest to consider only the corresponding part of the specification, i.e., the sublanguage $K := K' \cap Q(L)$. As it holds that $K \subseteq Q(L)$ and we have that $Q^{-1}(K) \cap L = Q^{-1}(K' \cap Q(L)) \cap L = Q^{-1}(K') \cap Q^{-1}Q(L) \cap L = Q^{-1}(K') \cap L$, the previous theorem gives the following consequence.

Corollary 16. For any prefix-closed languages $K' \subseteq (A^{hi})^*$ and $L \subseteq A^*$. If $A_o \subseteq A^{hi}$, then

$$Q^{-1}(\sup \mathrm{CN}(K' \cap Q(L), Q(L), P^{hi})) \cap L = \sup \mathrm{CN}(Q^{-1}(K') \cap L, L, P)\,.$$

5. NON-PREFIX-CLOSED SPECIFICATIONS

Since the formulas for supremal normal and supremal controllable and normal sublanguages from Brandt et al. (1990) are applicable only to prefix-closed languages, to show analogous results for non-prefix-closed specifications we extend the result for supremal controllable sublanguages shown in Feng (2007).

We recall the notions of output control consistency and observer property of abstractions (natural projections).

Definition 17. The natural projection $Q : A^* \to (A^{hi})^*$ is *output control consistent* (OCC) for $L \subseteq A^*$ if for every $s \in \overline{L}$ of the form

$$s = \sigma_1 \dots \sigma_k \quad \text{or} \quad s = s'\sigma_0\sigma_1 \dots \sigma_k, \quad k \geq 1\,,$$

where $s' \in A^*$, $\sigma_0, \sigma_k \in A^{hi}$ and $\sigma_i \in A - A^{hi}$, for $i = 1, \dots, k-1$, if $\sigma_k \in A_{uc}$, then $\sigma_i \in A_{uc}$, for $i = 1, \dots, k-1$.

Wong and Wonham (1996) defined the notion of observer which we use in this section.

Definition 18. The natural projection $Q : A^* \to (A^{hi})^*$ is an *L-observer* for $L \subseteq A^*$ if, for all $t \in Q(L)$ and $s \in \overline{L}$, $Q(s) \leq t$ implies that there exists $u \in A^*$ such that $su \in L$ and $Q(su) = t$.

First we recall Theorem 4.2 from Feng (2007) concerning the case of supremal controllable sublanguages.

Theorem 19. Let G be a plant over A, where $L = L(G)$ and $L_m = L_m(G)$ are nonblocking plant languages, i.e., $L = \overline{L_m}$. Let $Q : A^* \to (A^{hi})^*$ be an L_m-observer and OCC for L. Then, for any specification $K \subseteq (A^{hi})^*$,

$$Q^{-1}(\sup \mathrm{C}(K \cap Q(L_m), Q(L))) \cap L_m = \sup \mathrm{C}(Q^{-1}(K) \cap L_m, L)\,.$$

Now, we can prove the following theorem.

Theorem 20. Let G be a plant over A, where $L = L(G)$ and $L_m = L_m(G)$ are nonblocking plant languages. Let $Q : A^* \to (A^{hi})^*$ be an L_m-observer and OCC for L. For any specification $K \subseteq (A^{hi})^*$, if $A_0 \subseteq A^{hi}$, then

$$Q^{-1}(\sup \mathrm{CN}(K \cap Q(L_m), Q(L), P^{hi})) \cap L_m = \sup \mathrm{CN}(Q^{-1}(K) \cap L_m, L, P)\,.$$

Proof. Let us denote $N = \sup \mathrm{CN}(Q^{-1}(K) \cap L_m, L, P)$ and $N_{hi} = \sup \mathrm{CN}(K \cap Q(L_m), Q(L), P^{hi})$.

We first prove that $N \subseteq Q^{-1}(N_{hi}) \cap L_m$, i.e., we need to show that $Q(N) \subseteq N_{hi}$ and $N \subseteq L_m$. However, as $N \subseteq Q^{-1}(K) \cap L_m$, the latter inclusion is immediate. In order to show that $Q(N) \subseteq N_{hi}$, we prove (i) that $Q(N) \subseteq K \cap Q(L_m)$, (ii) that $Q(N)$ is controllable with respect to $Q(L)$ and $A_{uc} \cap A^{hi}$, and (iii) that $Q(N)$ is normal with respect to $Q(L)$ and P^{hi}.

To show (i), note that from $N \subseteq Q^{-1}(K) \cap L_m$ we immediately have that $Q(N) \subseteq Q(Q^{-1}(K) \cap L_m) \subseteq K \cap Q(L_m)$. Item (ii) is proven in Theorem 4.2 in Feng (2007), more precisely it is shown in Lemma 4.3 therein. Note that in this part of the proof, the observer and OCC properties are used in Lemma 4.3 of Feng (2007). Finally, to show (iii), we know that N is normal with respect to L and P, i.e., $\overline{N} = P^{-1}P(\overline{N}) \cap L$. Applying Q on this equation we obtain $Q(\overline{N}) = Q(P^{-1}P(\overline{N}) \cap L) = Q(P(\overline{N}) \| L)$, where the last equality is by definition of the synchronous product (see Cassandras and Lafortune (2008)). Using Proposition 4.3 in Feng (2007) along with the assumption that $A_o \subseteq A^{hi}$ means that projection below distributes with the synchronous product. Moreover, since $A_o \subseteq A^{hi}$, Q^o is an identity in the commutative diagram of Fig. 1, i.e., $P = P^{hi} \circ Q$. Hence, we have that

$$\begin{aligned} Q(\overline{N}) = Q(P(\overline{N}) \| L) = Q^oP(\overline{N}) \| Q(L) = P(\overline{N}) \| Q(L) \\ = (P^{hi})^{-1}P(\overline{N}) \cap Q(L) \\ = (P^{hi})^{-1}P^{hi}Q(\overline{N}) \cap Q(L)\,, \end{aligned}$$

where the equalities are in turn by the previous equation, Proposition 4.3 in Feng (2007), Q^o is an identity, definition of the synchronous product, and $P = P^{hi} \circ Q$. Thus, $Q(N)$ is normal with respect to $Q(L)$ and P^{hi}.

In order to prove the other inclusion, $Q^{-1}(N_{hi}) \cap L_m \subseteq N$, we show (1) that $Q^{-1}(N_{hi}) \cap L_m \subseteq Q^{-1}(K) \cap L_m$, which immediately follows from the following: $Q^{-1}(N_{hi}) \cap L_m \subseteq Q^{-1}(K \cap Q(L_m)) \cap L_m = Q^{-1}(K) \cap Q^{-1}Q(L_m) \cap L_m = Q^{-1}(K) \cap L_m$; and (2) that (i) $Q^{-1}(N_{hi}) \cap L_m$ is controllable with respect to L and A_{uc}, and (ii) $Q^{-1}(N_{hi}) \cap L_m$ is normal with respect to L and P. Again, (i) has been proven in Theorem 4.2 in Feng (2007). In fact, the proof of controllability of $Q^{-1}(N_{hi}) \cap L_m$ is based on two auxiliary results: first it is shown that $\overline{Q^{-1}(N_{hi})}$ and L_m are non-conflicting, i.e., $\overline{Q^{-1}(N_{hi}) \cap L_m} = Q^{-1}(\overline{N_{hi}}) \cap L$, and then Proposition 4.6 of Feng (2007) gives (i). The equality $\overline{Q^{-1}(N_{hi}) \cap L_m} = Q^{-1}(\overline{N_{hi}}) \cap L$ that we need below in the proof of (ii) is actually shown using Theorem 4.1 therein that uses the assumption that Q is L_m-observer. Now, to prove (ii), we know that $\overline{N_{hi}} = (P^{hi})^{-1}P^{hi}(\overline{N_{hi}}) \cap Q(L)$. Then,

$$\begin{aligned}Q^{-1}(\overline{N_{hi}}) \cap L &= Q^{-1}((P^{hi})^{-1}P^{hi}(\overline{N_{hi}}) \cap Q(L)) \cap L\\ &= Q^{-1}(P^{hi})^{-1}P^{hi}(\overline{N_{hi}}) \cap Q^{-1}Q(L) \cap L\\ &= Q^{-1}(P^{hi})^{-1}P^{hi}(\overline{N_{hi}}) \cap L\\ &= P^{-1}P^{hi}(\overline{N_{hi}}) \cap L\\ &= P^{-1}PQ^{-1}(\overline{N_{hi}}) \cap L\,.\end{aligned}$$

Then, as $\overline{Q^{-1}(N_{hi}) \cap L_m} = Q^{-1}(\overline{N_{hi}}) \cap L$,

$$\begin{aligned}&P^{-1}P(\overline{Q^{-1}(N_{hi}) \cap L_m}) \cap L\\ &\quad= P^{-1}P(Q^{-1}(\overline{N_{hi}}) \cap L) \cap L\\ &\quad\subseteq P^{-1}PQ^{-1}(\overline{N_{hi}}) \cap P^{-1}P(L) \cap L\\ &\quad= P^{-1}PQ^{-1}(\overline{N_{hi}}) \cap L\\ &\quad= Q^{-1}(\overline{N_{hi}}) \cap L = \overline{Q^{-1}(N_{hi}) \cap L_m}\,,\end{aligned}$$

where the last two equalities follow from the previous equations. Since the opposite inclusion of normality holds always true, this means that $Q^{-1}(N_{hi}) \cap L_m$ is normal with respect to L and P, which was to be shown. □

In the previous proof, the OCC property has been required only to show the controllability of the language and it plays no role for normality. On the other hand, however, the property of Q being an L_m-observer has been required also to show normality. Thus, we have the following corollary concerning supremal normal sublanguages.

Corollary 21. Let G be a plant over A, where $L = L(G)$ and $L_m = L_m(G)$ are nonblocking plant languages. Let $Q : A^* \to (A^{hi})^*$ be an L_m-observer. For any specification $K \subseteq (A^{hi})^*$, if $A_0 \subseteq A^{hi}$, then

$$\begin{aligned}&Q^{-1}(\sup \mathrm{N}(K \cap Q(L_m), Q(L), P^{hi})) \cap L_m\\ &\qquad= \sup \mathrm{N}(Q^{-1}(K) \cap L_m, L, P)\,.\end{aligned}$$

6. CONCLUSION

In this paper, hierarchical supervisory control of partially observed discrete event systems has been studied. First, the sufficient conditions for preservations of high level supremal normal sublanguages at the low level have been formulated. Then, sufficient conditions for preservations of high level supremal controllable and normal sublanguages at the low level have been proposed, which ensure that the optimal high level supervisor with partial observations is implementable in the original plant (low level).

Among the problems opened for a future investigation we mention that it is important to combine both hierarchical and decentralized approaches in the settings of partially observed distributed plants in order to achieve a higher degree of computational savings in synthesizing supervisory controllers for large systems.

In this paper, we have also considered only natural projections. However, in the future investigation, we will address general reporter maps instead.

Finally, the assumption that $A_o \subseteq A^{hi}$ is very restrictive. Thus, it is of interest to find some less restrictive sufficient conditions. We would like to address this issue in a future work.

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