Blending margins: The modal logic $\mathbf{K}$ has nullary unification type

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Abstract

We investigate properties of the formula $p \to \Box p$ in the basic modal logic $\mathbf{K}$. We show that $\mathbf{K}$ satisfies an infinitary weaker variant of the rule of margins $\varphi \to \Box\varphi \,/\, \varphi, \neg\varphi$, and as a consequence, we obtain various negative results about admissibility and unification in $\mathbf{K}$. We describe a complete set of unifiers (i.e., substitutions making the formula provable) of $p \to \Box p$, and use it to establish that $\mathbf{K}$ has the worst possible unification type: nullary. In well-behaved transitive modal logics, admissibility and unification can be analyzed in terms of projective formulas, introduced by Ghilardi; in particular, projective formulas coincide for these logics with formulas that are admissibly saturated (i.e., derive all their multiple-conclusion admissible consequences) or exact (i.e., axiomatize a theory of a substitution). In contrast, we show that in $\mathbf{K}$, the formula $p \to \Box p$ is admissibly saturated, but neither projective nor exact. All our results for $\mathbf{K}$ also apply to the basic description logic $\mathcal{ALC}$.

Key words: modal logic, description logic, unification type, admissible rules, rule of margins.

1 Introduction

Equational unification studies the problem of making terms equivalent modulo an equational theory by means of a substitution. It has been thoroughly investigated for basic algebraic theories, such as the theory of commutative semigroups, see Baader and Snyder [5] for an overview. If L is a propositional logic algebraizable with respect to a class of algebras V, unification modulo the equational theory of V can be stated purely in terms of propositional logic: an L-unifier of a set of formulas Γ is a substitution which turns all formulas from Γ into L-tautologies.

In the realm of modal logics, the seminal results of Ghilardi [11] show that unification is at most finitary, decidable, and generally well-behaved for a representative class of transitive modal logics, including e.g. $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{GL}$, $\mathbf{Grz}$. Unification was also studied for fragments of description logics, which have applications in ontology generation and maintenance; see

Baader and Ghilardi [1]. In particular, the description logics treated in [4, 3] can be thought of as the $\{\wedge, \Box\}$ and $\{\wedge, \Diamond\}$ fragments of multimodal **K**.

Unification in propositional logics is closely connected to admissibility of inference rules: a multiple-conclusion rule $\Gamma \mathbin{/} \Delta$ is L-admissible if every L-unifier of Γ also unifies some formula from Δ. Rybakov [17] proved that admissibility is decidable for a class of transitive modal logics (similar to the one mentioned above) and provided characterizations of their admissible rules. Some of these results can be alternatively obtained using Ghilardi's approach (cf. also [13]). It is also possible to treat intuitionistic and intermediate logics in parallel with the transitive modal case [17, 10, 12].

In contrast to these results, not much is known about unification and admissibility in nontransitive modal logics with a complete set of Boolean connectives. In particular, one of the main open problems in the area is decidability of unification or admissibility in the basic modal logic **K**. (Wolter and Zakharyaschev [21] have shown that unifiability is undecidable in the bimodal extension of **K** with the universal modality and in some description logics, but it is wide open whether one can extend these results to **K** itself.)

In this note we present some negative properties of unification and admissibility in **K**. The main result is that unification in **K** is nullary (i.e., of the worst possible type). In terms of description logic, unification in $\mathcal{ALC}$ is nullary, even if we consider formulas with only one role and one concept name. We also show that there exists a formula (namely, $p \to \Box p$) which is admissibly saturated in the sense of [14], but it is not projective (or even exact). In contrast, the results of Ghilardi [11] imply that in well-behaved transitive modal logics such as **K4**, projective, exact, and admissibly saturated formulas coincide, and indeed this is an important precondition which makes possible the characterization of admissibility in terms of projective approximations. Thus, admissible rules of **K** cannot be directly analyzed in a similar way.

Our results are based on a classification of unifiers of the formula $p \to \Box p$. The main ingredient is establishing that **K** admits a weaker version of the so-called rule of margins

$$\varphi \to \Box\varphi \mathbin{/} \varphi, \neg\varphi$$

(meaning that whenever a formula of the form $\varphi \to \Box\varphi$ is valid, one of the formulas φ, $\neg\varphi$ is also valid). The rule of margins was investigated by Williamson [18, 19, 20] in the context of epistemic logic. (The rule is supposed to express the ubiquity of vagueness. We read $\Box$ as "clearly". Since all our learning processes have a certain margin of error, the only way we can know for sure that φ is clearly true whenever it is true is that we know in fact whether φ is true or false.) The rule of margins is admissible e.g. in the logics **KD**, **KT**, **KDB**, and **KTB**, but not in **K**. However, we will show that **K** satisfies a variant of the rule whose conclusion is that either φ holds, or it is almost contradictory in the sense of implying $\Box^n\bot$ for some $n \in \omega$. We remark that the rule of margins was also used in connection with unification by Dzik [9].

2 Preliminaries

We refer the reader to [8, 6, 5] for background on modal logic and unification. We review below the needed definitions to fix the notation, and some relevant basic facts.

We work with formulas in the propositional modal language using propositional variables p_n for $n < \omega$ (we will often write just p for p_0), Boolean connectives (including the nullary connectives $\bot, \top$), and the unary modal connective $\Box$. We will use lower-case Greek letters $\varphi, \psi, \dots$ to denote formulas, and upper-case Greek letters $\Gamma, \Delta, \dots$ for finite sets of formulas. We define $\Diamond\varphi$, $\Box^n\varphi$, $\Box^{<n}\varphi$, and $\Diamond^n\varphi$ as shorthands for $\neg\Box\neg\varphi$, $\underbrace{\Box\cdots\Box}_{n \text{ boxes}}\varphi$, $\bigwedge_{i=0}^{n-1}\Box^i\varphi$, and $\neg\Box^n\neg\varphi$, respectively. (As a special case, $\Box^0\varphi = \varphi$ and $\Box^{<0}\varphi = \top$.) The *modal degree* $\mathrm{md}(\varphi)$ of a formula φ is defined so that $\mathrm{md}(p_i) = 0$, $\mathrm{md}(\circ(\varphi_0, \dots, \varphi_{k-1})) = \max_{i<k}\mathrm{md}(\varphi_i)$ for a k-ary Boolean connective $\circ$, and $\mathrm{md}(\Box\varphi) = 1 + \mathrm{md}(\varphi)$.

We use $\vdash$ to denote the *global consequence relation* of **K**. That is, $\Gamma \vdash \varphi$ iff there exists a sequence of formulas $\varphi_0, \dots, \varphi_n$ such that $\varphi_n = \varphi$, and each φ_i is an element of Γ, a classical propositional tautology, an instance of the axiom

$$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta),$$

or it is derived from some of the formulas φ_j with $j < i$ by an instance of necessitation $\alpha \,/\, \Box\alpha$ or modus ponens $\alpha, (\alpha \to \beta) \,/\, \beta$.

A *Kripke model* is a triple $\langle F, R, \vDash\rangle$, where the *accessibility relation* R is a binary relation on a set F, and the *valuation* $\vDash$ is a relation between elements of F and formulas, written as $F, x \vDash \varphi$, which commutes with propositional connectives and satisfies

$$F, x \vDash \Box\varphi \quad \text{iff} \quad \forall y \in F\,(x \mathrel{R} y \Rightarrow F, y \vDash \varphi).$$

If there is no danger of confusion, we will denote the model $\langle F, R, \vDash\rangle$ by just F. We write $F \vDash \varphi$ if $F, x \vDash \varphi$ for every $x \in F$, and $F \vDash \Gamma$ if $F \vDash \varphi$ for every $\varphi \in \Gamma$. The *strong completeness theorem* for **K** [8, Thms. 3.55, 10.5] states

Fact 2.1 *$\Gamma \vdash \varphi$ iff $F \vDash \Gamma$ implies $F \vDash \varphi$ for every model $\langle F, R, \vDash\rangle$.*

We write $R(x) = \{y : x \mathrel{R} y\}$. Let

$$R^n = \{\langle x_0, x_n\rangle \in F^2 : \exists x_1, \dots, x_{n-1} \in F\, \forall i < n\, x_i \mathrel{R} x_{i+1}\}$$

be the n-fold composition of R (where the case $n = 0$ is understood to mean $R^0 = \{\langle x, x\rangle : x \in F\}$), and $R^{\le n} = \bigcup_{i \le n} R^i$. We say that x is a *root* of F if $F = \bigcup_{n\in\omega} R^n(x)$.

Fact 2.2 ([8, Cor. 3.29], cf. [6, Thm. 2.34]) *If $\nvdash \varphi$, then there exists a model $\langle F, R, \vDash\rangle$ based on a finite irreflexive intransitive tree with root x such that $F, x \nvDash \varphi$.*

(That is, R is the edge relation of a directed tree with edges oriented away from x and no self-loops.)

A model $\langle F', R', \vDash'\rangle$ is the restriction of $\langle F, R, \vDash\rangle$ to F', denoted as $\langle F, R, \vDash\rangle \restriction F'$, if $F' \subseteq F$, $R' = R \cap F'^2$, and $F, x \vDash p_i$ iff $F', x \vDash p_i$ for every $x \in F'$ and p_i.

Fact 2.3 ([8, Prop. 3.2], [6, L. 2.33]) *If $n \ge \mathrm{md}(\varphi)$, $x \in F \cap G$, and $\langle F, R, \vDash\rangle \restriction R^{\le n}(x) = \langle G, S, \vDash\rangle \restriction S^{\le n}(x)$, then $F, x \vDash \varphi$ iff $G, x \vDash \varphi$.*

A *p-morphism* between models $\langle F, R, \vDash\rangle$ and $\langle G, S, \vDash\rangle$ is a function $f\colon F \to G$ such that

(i) $x\ R\ y$ implies $f(x)\ S\ f(y)$,

(ii) if $f(x)\ S\ z$, there exists $y \in F$ such that $x\ R\ y$ and $f(y) = z$,

(iii) $F, x \vDash p_i$ iff $G, f(x) \vDash p_i$ for every variable p_i.

Fact 2.4 ([8, Thm. 3.15], [6, Prop. 2.14]) *If $f\colon F \to G$ is a p-morphism, then $F, x \vDash \varphi$ iff $G, f(x) \vDash \varphi$ for every formula φ.*

A *substitution* is a mapping from formulas to formulas which commutes with all connectives. A *unifier* of a finite set of formulas Γ is a substitution σ such that $\vdash \sigma(\varphi)$ for all $\varphi \in \Gamma$. In logics with a well-behaved conjunction connective such as **K**, unifiers of Γ are the same as unifiers of the single formula $\bigwedge \Gamma$, hence we will mostly restrict the discussion below to plain formulas instead of sets in order to simplify the notation.

Let $U(\varphi)$ be the set of all unifiers of φ. The *composition* of substitutions σ, τ is the substitution $\sigma \circ \tau$ such that $(\sigma \circ \tau)(\varphi) = \sigma(\tau(\varphi))$. Let $\sigma \equiv \tau$ if $\vdash \sigma(p_i) \leftrightarrow \tau(p_i)$ for every i. A substitution τ is *more general* than σ, written as $\sigma \preceq \tau$, if there exists a substitution υ such that $\sigma \equiv \upsilon \circ \tau$. We warn the reader that $\preceq$ is often written in the opposite direction in literature on unification theory. We write $\sigma \approx \tau$ if $\sigma \preceq \tau$ and $\tau \preceq \sigma$, and $\sigma \prec \tau$ if $\sigma \preceq \tau$ but $\tau \not\preceq \sigma$. Note that $\preceq$ is a preorder, and $\approx$ is the induced equivalence relation. A *complete set of unifiers of* φ is a cofinal subset C of $\langle U(\varphi), \preceq\rangle$ (i.e., a set of unifiers of φ such that every unifier of φ is less general than some element of C). If $\{\sigma\}$ is a complete set of unifiers of φ, then σ is a *most general unifier (mgu)* of φ.

If $\langle P, \leq\rangle$ is a nonempty poset, let M be the set of its maximal elements (i.e., $x \in P$ such that $x < y$ for no $y \in P$). If every element of P is below an element of M, we say that $\langle P, \leq\rangle$ is of

- type 1 (unitary), if $|M| = 1$,
- type ω (finitary), if M is finite and $|M| > 1$,
- type ∞ (infinitary), if M is infinite.

Otherwise, it is of type 0 (nullary).

The *unification type* of φ is the type of the quotient poset $\langle U(\varphi), \preceq\rangle/{\approx}$. Note that φ is of unitary type iff it has an mgu, and it is of at most finitary type (i.e., 1 or ω) iff it has a finite complete set of unifiers. The unification type of a logic (that is, for us, of **K**) is the maximal type of a unifiable formula φ, where we order the unification types as $1 < \omega < \infty < 0$.

In unification theory, it is more customary to define the equivalence of unifiers $\sigma, \tau \in U(\varphi)$ (and derived notions such as $\preceq$ and unification types) so that $\sigma \equiv \tau$ iff $\vdash \sigma(p_i) \leftrightarrow \tau(p_i)$ for variables p_i *that occur in* φ, whereas we demanded this for all variables. Our results hold equally well under the restricted definition, and in fact, the proofs could be slightly simplified in this case (we could replace conditions (ii), (iii) in Lemma 3.5 with just $\sigma \equiv \sigma_\top$). The latter is one reason for our choice of the definition: in order to make the results most general, we carry out the proofs for the most complicated case. We also find it convenient to have an absolute notion of equivalence of substitutions, independent of which formula they are considered to be unifiers of. Our results are robust under further variations of the definition, for example we could consider substitutions with domain consisting of formulas using only variables occurring in φ, and target consisting of formulas using variables from a fixed finite set (which could be the same as the domain).

A *multiple-conclusion rule* is an expression $\Gamma \,/\, \Delta$, where Γ, Δ are finite sets of formulas. A rule $\Gamma \,/\, \Delta$ is *derivable* if $\Gamma \vdash \psi$ for some $\psi \in \Delta$. A rule $\Gamma \,/\, \Delta$ is *admissible*, written as $\Gamma \mathrel{|\!\!\sim} \Delta$, if every unifier of Γ also unifies some $\psi \in \Delta$. Note that all derivable rules are admissible, but not vice versa. A formula φ is *admissibly saturated* [14], if every admissible rule of the form $\varphi \,/\, \Delta$ is derivable. φ is *exact* [15] if there exists a substitution σ such that

$$\varphi \vdash \psi \quad \text{iff} \quad \vdash \sigma(\psi)$$

for every formula ψ. φ is *projective* [10] if it has a unifier σ (called a *projective unifier*) such that

$$\varphi \vdash p_i \leftrightarrow \sigma(p_i)$$

for every p_i. This implies that $\varphi \vdash \psi \leftrightarrow \sigma(\psi)$ for every ψ, and that σ is an mgu of φ: if $\tau \in U(\varphi)$, we have $\tau \equiv \tau \circ \sigma$.

Fact 2.5 *Let φ be a formula.*

(i) *If φ is projective, it is exact.*

(ii) *If φ is exact, it is admissibly saturated.*

Proof: (i): On the one hand, σ is a unifier of φ. On the other hand, if $\vdash \sigma(\psi)$, then $\varphi \vdash \psi \leftrightarrow \sigma(\psi)$ implies $\varphi \vdash \psi$.

(ii): If $\varphi \mathrel{|\!\!\sim} \Delta$, then $\vdash \sigma(\psi)$ for some $\psi \in \Delta$ as σ is a unifier of φ, hence $\varphi \vdash \psi$ by exactness. $\square$

A *projective approximation* of φ [10] is a finite set Π of projective formulas such that $\varphi \mathrel{|\!\!\sim} \Pi$, and $\pi \vdash \varphi$ for every $\pi \in \Pi$. More generally, an *admissibly saturated approximation* [14] is a set with properties as above, except that its elements are only required to be admissibly saturated instead of projective. If Π is an admissibly saturated approximation of $\bigwedge \Gamma$, it is easy to see ([14, Obs. 3.7]) that

$$\Gamma \mathrel{|\!\!\sim} \Delta \quad \text{iff} \quad \forall \pi \in \Pi \, \exists \psi \in \Delta \, \pi \vdash \psi. \tag{1}$$

If Π is a projective approximation of φ, then the set of projective unifiers of elements of Π is a finite complete set of unifiers of φ. This does not hold for admissibly saturated approximations in general.

The definition immediately implies that if Π is any admissibly saturated approximation of an admissibly saturated formula φ, then there is a formula $\pi \in \Pi$ interderivable with φ (i.e., $\varphi \vdash \pi$ and $\pi \vdash \varphi$). In particular, if an admissibly saturated formula has a projective approximation, it must be projective itself, hence we have:

Fact 2.6 *The following are equivalent.*

(i) *Every φ has a projective approximation.*

(ii) *Every φ has an admissibly saturated approximation, and every admissibly saturated formula is projective.*

Projective formulas and approximations are the backbone of Ghilardi's analysis [11] of unification and admissibility in transitive modal logics such as **K4**, **S4**, or **GL**. He shows that in these logics, every formula has a projective approximation, which implies that unification is at most finitary, and gives a description of admissibility by means of (1). By Facts 2.5 and 2.6, the same property also implies that admissibly saturated, exact, and projective formulas coincide.

For an example exhibiting different behaviour, in Łukasiewicz logic every formula has an admissibly saturated approximation, and exact formulas coincide with admissibly saturated formulas, but the logic has nullary unification type, and some exact formulas are not projective [14, 16, 7].

3 Results

As all of our results concern properties of the formula $p \to \Box p$, our first task is to describe a complete set of unifiers of this formula. Without further ado, this set will consist of the following substitutions.

Definition 3.1 For any $n \in \omega$, we introduce the substitutions

$$\sigma_n(p) = \Box^{<n} p \wedge \Box^n \bot,$$
$$\sigma_\top(p) = \top,$$

where $\sigma_\alpha(q) = q$ for every variable $q \neq p$ and $\alpha \in \omega_+ := \omega \cup \{\top\}$.

Lemma 3.2 *σ_α is a unifier of $p \to \Box p$ for every $\alpha \in \omega_+$.*

Proof: Using the principle $\varphi \to \psi \vdash \Box^n \varphi \to \Box^n \psi$, and distributivity of $\Box$ over $\wedge$, we have

$$\vdash \Box^{<n} p \wedge \Box^n \bot \to \Box^{\leq n} p \to \Box\Box^{<n} p,$$
$$\vdash \Box^n \bot \to \Box^{n+1} \bot,$$

whence

$$\vdash \Box^{<n} p \wedge \Box^n \bot \to \Box\Box^{<n} p \wedge \Box\Box^n \bot \to \Box(\Box^{<n} p \wedge \Box^n \bot).$$

Clearly, $\vdash \top \to \Box\top$. $\Box$

We start with simple criteria for recognizing that a given unifier of $p \to \Box p$ is below σ_α.

Lemma 3.3 *If σ is a unifier of $p \to \Box p$, and $n \in \omega$, the following are equivalent:*

(i) $\sigma \preceq \sigma_n$,

(ii) $\sigma \equiv \sigma \circ \sigma_n$,

(iii) $\vdash \sigma(p) \to \Box^n \bot$.

Proof: (ii) $\to$ (i) follows from the definition of $\preceq$.

(i) $\to$ (iii): If $\sigma \equiv \tau \circ \sigma_n$, then $\vdash \sigma_n(p) \to \Box^n \bot$ implies $\vdash \tau(\sigma_n(p)) \to \tau(\Box^n \bot)$, i.e., $\vdash \sigma(p) \to \Box^n \bot$.

(iii) $\to$ (ii): Put $\varphi = \sigma(p)$. Since σ is a unifier of $p \to \Box p$, we have $\vdash \varphi \to \Box \varphi$, hence $\vdash \varphi \to \Box^{<n} \varphi$ by induction on n. Since we also assume $\vdash \varphi \to \Box^n \bot$, we have $\vdash \sigma(p) \to \sigma(\sigma_n(p))$. The other implication is trivial as $\vdash \sigma_n(p) \to p$. $\Box$

Definition 3.4 For any substitution σ, let $\sigma \restriction p$ be the substitution τ such that $\tau(p) = \sigma(p)$, and $\tau(q) = q$ for every variable $q \neq p$.

Lemma 3.5 *If σ is a substitution, the following are equivalent:*

(i) $\sigma \preceq \sigma_\top$,

(ii) $\sigma \equiv \sigma \circ \sigma_\top$,

(iii) $\sigma \restriction p \equiv \sigma_\top$,

(iv) $\vdash \sigma(p)$.

Proof: (ii) $\leftrightarrow$ (iii) $\leftrightarrow$ (iv): If $q \neq p$ is a variable, we have $\sigma_\top(q) = (\sigma \restriction p)(q) = q$ and $(\sigma \circ \sigma_\top)(q) = \sigma(q)$, hence the corresponding equivalences in (ii) and (iii) are trivially valid. For p itself, we have $\sigma_\top(p) = (\sigma \circ \sigma_\top)(p) = \top$, hence (ii) and (iii) both amount to $\vdash \sigma(p) \leftrightarrow \top$, which is the same as $\vdash \sigma(p)$.

(ii) $\to$ (i) follows from the definition of $\preceq$. Conversely, if $\sigma \equiv \tau \circ \sigma_\top$, we have $\tau(\sigma_\top(p)) = \top$, thus $\vdash \sigma(p)$. $\Box$

The crucial element in the description of $U(p \to \Box p)$ is to show that one of the conditions in Lemma 3.3 or 3.5 applies to every unifier. This amounts to a variant of the rule of margins, as alluded to in the introduction. The basic idea is similar to Williamson's proof [18] of the rule of margins for **KD**: in order to invalidate $\varphi \to \Box \varphi$, we take two models satisfying φ and $\neg\varphi$, respectively, and join them by a path, while making sure this does not mess up the valuation of φ in the end-points. Then φ has to switch to $\neg\varphi$ somewhere along the path, at which point the formula $\varphi \to \Box \varphi$ will not hold.

Theorem 3.6 *If $\vdash \varphi \to \Box \varphi$, then $\vdash \varphi$ or $\vdash \varphi \to \Box^n \bot$, where $n = \mathrm{md}(\varphi)$.*

Proof: Assume $\nvdash \varphi$ and $\nvdash \varphi \to \Box^n \bot$. By Fact 2.2, the latter implies that there exists a finite irreflexive intransitive tree $\langle F, R, \vDash \rangle$ with root x_0 such that $F, x_0 \vDash \varphi \wedge \Diamond^n \top$. This means that there exists a sequence $x_0 \mathrel{R} x_1 \mathrel{R} \cdots \mathrel{R} x_n$ of elements of F, and as R is an intransitive tree, $x_n \notin R^{<n}(x_0)$. Since $\nvdash \varphi$, there exists a model $\langle G, S, \vDash \rangle$ and a point $x_{n+1} \in G$ such that $G, x_{n+1} \nvDash \varphi$. Let $\langle H, T, \vDash \rangle$ be the disjoint union of F and G, where we additionally put $x_n \mathrel{T} x_{n+1}$. Since $F \restriction R^{\leq n}(x_0) = H \restriction T^{\leq n}(x_0)$, we have $H, x_0 \vDash \varphi$ by Fact 2.3. On the other hand, $H, x_{n+1} \nvDash \varphi$, hence there exists $i \leq n$ such that $H, x_i \vDash \varphi$ and $H, x_{i+1} \nvDash \varphi$. Then $H, x_i \nvDash \varphi \to \Box \varphi$. $\Box$

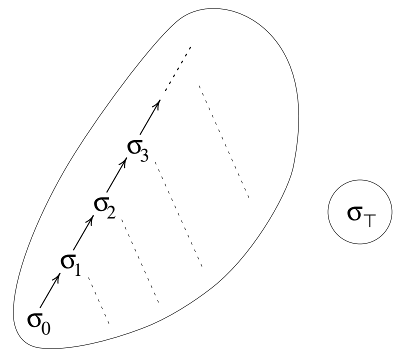

Figure 1: Unifiers of $p \to \Box p$.

Ignoring the explicit dependence of n on φ, we can rephrase Theorem 3.6 by saying that the infinitary multiple-conclusion rule

$$p \to \Box p \;/\; \{p \to \Box^n \bot : n \in \omega\} \cup \{p\} \tag{2}$$

is admissible in **K**. Let us mention that a similar proof also shows that **K** satisfies the following variant of Williamson's alternative rule of disjunction: if $n_0 \geq \mathrm{md}(\varphi_0)$, $n_1, \ldots, n_k > \mathrm{md}(\varphi_0)$, and $\vdash \varphi_0 \vee \Box^{n_1}\varphi_1 \vee \cdots \vee \Box^{n_k}\varphi_k$, then $\vdash \varphi_0 \vee \Box^{n_0}\bot$ or $\vdash \varphi_i$ for some $i = 1, \ldots, k$. We leave the details to the interested reader as we have no further use for this property.

Corollary 3.7 *The substitutions $\{\sigma_\alpha : \alpha \in \omega_+\}$ form a complete set of unifiers of the formula $p \to \Box p$.*

Proof: By Lemmas 3.2, 3.3, and 3.5, and Theorem 3.6. □

Theorem 3.8 *Unification in* **K** *is nullary.*

Proof: Since $\vdash \sigma_n(p) \to \Box^{n+1}\bot$ and $\nvdash \sigma_{n+1}(p) \to \Box^n\bot$, Lemma 3.3 shows that $\sigma_n \prec \sigma_{n+1}$. Similarly, $\nvdash \sigma_n(p)$ and $\nvdash \top \to \Box^n\bot$, hence σ_n and $\sigma_\top$ are incomparable by Lemmas 3.3 and 3.5. By Corollary 3.7, every maximal element of $U(p \to \Box p)$ is equivalent to some σ_α, and in view of $\sigma_n \prec \sigma_{n+1}$, we must have $\alpha = \top$. Thus, none of the unifiers σ_n is majorized by a maximal element in $U(p \to \Box p)$. □

The preorder of unifiers of $p \to \Box p$ is depicted in Figure 1. (We consider substitutions defined only for the p variable in the diagram, which is why there are no unifiers strictly below $\sigma_\top$ or σ_0.)

The basic description logic $\mathcal{ALC}$ [2, 1] is a notational variant of multimodal **K**, with concept names corresponding to propositional variables, and universal and existential restrictions corresponding to boxes and diamonds, one pair for each role name. We obtain immediately the following.

Corollary 3.9 *Unification in $\mathcal{ALC}$ is nullary, even for formulas with only one role name and one concept name*[1]. $\Box$

Now we turn to the (non)equivalence of exact and admissibly saturated formulas. That $p \to \Box p$ is inexact follows easily from Theorem 3.6:

Proposition 3.10 *The formula $p \to \Box p$ is not exact, and a fortiori not projective.*

Proof: Assume for contradiction that σ is a substitution such that

$$p \to \Box p \vdash \psi \quad \text{iff} \quad \vdash \sigma(\psi)$$

for every ψ. In particular, σ is a unifier of $p \to \Box p$, hence $\vdash \sigma(p)$ or $\vdash \sigma(p) \to \Box^n \bot$ for some n by Theorem 3.6. However, $p \to \Box p \nvdash p$ and $p \to \Box p \nvdash p \to \Box^n \bot$, a contradiction. $\Box$

We remark that σ_n and $\sigma_\top$ are projective unifiers of the formulas $p \to \Box p \wedge \Box^n \bot$ and p, respectively.

We complement Proposition 3.10 by showing that $p \to \Box p$ is admissibly saturated. We mention another pathological property of $p \to \Box p$ which will arise from the proof. Intuitively, it is not so surprising that a formula φ with an infinite cofinal chain of unifiers like σ_n (or more generally, a formula whose preorder of unifiers is directed, even if it has no maximal element) can be admissibly saturated, as the unifiers high enough in the chain eventually become "indistinguishable" when applied to any particular formula ψ. However, if a formula has two incomparable maximal unifiers, say σ, σ', we would expect it *not* to be admissibly saturated: presumably, we can find formulas ψ, ψ' unified by σ and σ', respectively, but not vice versa. Then $\varphi \mathrel{|\!\sim} \psi, \psi'$, but not $\varphi \mathrel{|\!\sim} \psi$ or $\varphi \mathrel{|\!\sim} \psi'$. By the same intuition, we would expect that a formula like $p \to \Box p$, whose set of unifiers consists of two incomparable parts (a chain and a maximal unifier, in our case), is not admissibly saturated either.

What happens here is that when we apply the unifiers σ_n to a particular formula, they not only become "indistinguishable" from each other for n large enough, but they also "cover" the unifier $\sigma_\top$, despite that it is not comparable to any element of the chain. Returning to our weak rule of margins, one can imagine that the margins of error about the approximate falsities $\Box^n \bot$ gradually blend into the margin about the truth $\top$ as n goes to infinity.

Proposition 3.11 *The formula $p \to \Box p$ is admissibly saturated.*

Proof: Assume $p \to \Box p \mathrel{|\!\sim} \Delta$, and pick $n > \max\{\mathrm{md}(\psi) : \psi \in \Delta\}$. Since σ_n unifies $p \to \Box p$, there exists $\psi \in \Delta$ such that $\vdash \sigma_n(\psi)$. We claim

$$p \to \Box p \vdash \psi.$$

If not, there exists a Kripke model $\langle F, R, \vDash \rangle$ such that $F \vDash p \to \Box p$ and $F, x_0 \nvDash \psi$ for some $x_0 \in F$. First, we unravel F to a tree (cf. [6, Prop. 2.15], [8, Thm. 3.18]): let $\langle G, S, \vDash \rangle$ be the model where G consists of sequences $\langle x_0, \ldots, x_m \rangle$ such that $m \in \omega$, $x_i \in F$, $x_i \mathrel{R} x_{i+1}$;

[1] That is, one concept variable and no concept constants. We employ no unification problems with constants in this paper.

we put $\langle x_0, \dots, x_m \rangle \; S \; \langle x_0, \dots, x_m, x_{m+1} \rangle$; and $G, \langle x_0, \dots, x_m \rangle \vDash p_j$ iff $F, x_m \vDash p_j$ for each variable p_j. The mapping $f \colon G \to F$ given by $f(\langle x_0, \dots, x_m \rangle) = x_m$ is a p-morphism, hence it preserves the valuation of formulas by Fact 2.4. In particular, $G \vDash p \to \Box p$ and $G, \langle x_0 \rangle \nvDash \psi$.

Let H be the submodel of G consisting of sequences $\langle x_0, \dots, x_m \rangle$ where $m < n$. We still have $H \vDash p \to \Box p$: if $\vec{x} = \langle x_0, \dots, x_m \rangle$ with $m < n - 1$, then $G \restriction S^{\le 1}(\vec{x}) = H \restriction S^{\le 1}(\vec{x})$, hence $H, \vec{x} \vDash p \to \Box p$ by Fact 2.3; on the other hand, if $m = n - 1$, then $H, \vec{x} \vDash \Box\bot$, and a fortiori $H, \vec{x} \vDash p \to \Box p$. It follows that $H \vDash p \to \Box^{<n} p$, and moreover $H \vDash \Box^n \bot$, hence $H \vDash p \leftrightarrow \sigma_n(p)$. However, $G \supseteq H \supseteq G \restriction S^{\le \mathrm{md}(\psi)}(\langle x_0 \rangle)$, hence $H, \langle x_0 \rangle \nvDash \psi$ by Fact 2.3. These properties together imply $H, \langle x_0 \rangle \nvDash \sigma_n(\psi)$, contradicting $\vdash \sigma_n(\psi)$. $\Box$

We remark that unlike Theorem 3.6, we could not directly take a finite irreflexive intransitive tree for F in the proof above, because **K** is not finitely strongly complete with respect to such frames. (Every finite irreflexive tree is converse well-founded, and therefore validates Löb's rule $\Box p \to p \;/\; p$, which is admissible but not derivable in **K**.)

Corollary 3.12 *The formula $p \to \Box p$ has no projective approximation.*

Proof: In view of Propositions 3.10 and 3.11, this follows from the discussion leading to Fact 2.6. $\Box$

4 Conclusion

We have provided examples confirming that unification and admissibility in the basic modal logic **K** involves peculiar phenomena not encountered in the familiar case of transitive modal logics with frame extension properties: the fact that **K** has the worst possible unification type, even for very simple formulas in one variable like $p \to \Box p$, is a problem by itself; as we have seen, this formula is also a counterexample to other structural properties vital for the kind of analysis of admissibility and unification that has been applied in the transitive case, namely it is neither projective nor exact despite being admissibly saturated, it has no projective approximation, and it is admissibly saturated even though its preorder of unifiers is not directed (it consists of two disjoint connected components).

The major remaining problem in this area is whether admissibility or unifiability in **K** is decidable. Our results might be seen as hinting towards the possibility that these tasks are undecidable. (The results of Wolter and Zakharyaschev [21] also point in this direction.) For example, (2) means that a rule of the form $\Gamma, p \to \Box p \;/\; \psi$ is admissible iff $\Gamma, p \;/\; \psi$ and $\Gamma, p \to \Box^n \bot \;/\; \psi$ are admissible for every $n \in \omega$. Note that $p \to \Box^n \bot$ holds in a model iff the submodel generated by points satisfying p is well-founded of finite depth at most n; one can imagine that the discrete nature of such models could be used to encode finite computation or some kind of finite combinatorial structures. Since n can be arbitrarily large irrespective of the size of Γ or ψ, this might lead to an undecidable problem.

On the other hand, should admissibility in **K** be decidable after all, our results show that proving this will require methods more powerful and more delicate than what we are used to from the transitive case, as current techniques are not ready to cope with obstacles exhibited by the behaviour of $p \to \Box p$.

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References

[1] Franz Baader and Silvio Ghilardi, *Unification in modal and description logics*, Logic Journal of the IGPL 19 (2011), no. 6, pp. 705–730.

[2] Franz Baader, Ian Horrocks, and Ulrike Sattler, *Description logics*, in: Handbook of Knowledge Representation (F. van Harmelen, V. Lifschitz, and B. Porter, eds.), Foundations of Artificial Intelligence vol. 3, Elsevier, 2008, pp. 135–179.

[3] Franz Baader and Barbara Morawska, *Unification in the Description Logic $\mathcal{EL}$*, Logical Methods in Computer Science 6 (2010), no. 3, paper no. 17.

[4] Franz Baader and Paliath Narendran, *Unification of concept terms in description logics*, Journal of Symbolic Computation 31 (2001), pp. 277–305.

[5] Franz Baader and Wayne Snyder, *Unification theory*, in: Handbook of Automated Reasoning (A. Robinson and A. Voronkov, eds.), vol. I, Elsevier, 2001, ch. 8, pp. 445–533.

[6] Patrick Blackburn, Maarten de Rijke, and Yde Venema, *Modal logic*, Cambridge Tracts in Theoretical Computer Science vol. 53, Cambridge University Press, 2001.

[7] Leonardo M. Cabrer, *Simplicial geometry of unital lattice-ordered abelian groups*, Forum Mathematicum (2013), to appear.

[8] Alexander V. Chagrov and Michael Zakharyaschev, *Modal logic*, Oxford Logic Guides vol. 35, Oxford University Press, 1997.

[9] Wojciech Dzik, *Transparent unifiers in modal logics with self-conjugate operators*, Bulletin of the Section of Logic 35 (2006), no. 2/3, pp. 73–83.

[10] Silvio Ghilardi, *Unification in intuitionistic logic*, Journal of Symbolic Logic 64 (1999), no. 2, pp. 859–880.

[11] ———, *Best solving modal equations*, Annals of Pure and Applied Logic 102 (2000), no. 3, pp. 183–198.

[12] Rosalie Iemhoff, *On the admissible rules of intuitionistic propositional logic*, Journal of Symbolic Logic 66 (2001), no. 1, pp. 281–294.

[13] Emil Jeřábek, *Admissible rules of modal logics*, Journal of Logic and Computation 15 (2005), no. 4, pp. 411–431.

[14] ———, *Bases of admissible rules of Łukasiewicz logic*, Journal of Logic and Computation 20 (2010), no. 6, pp. 1149–1163.

[15] Dick H. J. de Jongh, *Formulas of one propositional variable in intuitionistic arithmetic*, in: The L. E. J. Brouwer Centenary Symposium (A. S. Troelstra and D. van Dalen, eds.), Studies in Logic and the Foundations of Mathematics vol. 110, North-Holland, 1982, pp. 51–64.

[16] Vincenzo Marra and Luca Spada, *Duality, projectivity, and unification in Łukasiewicz logic and MV-algebras*, Annals of Pure and Applied Logic 164 (2013), no. 3, pp. 192–210.

[17] Vladimir V. Rybakov, *Admissibility of logical inference rules*, Studies in Logic and the Foundations of Mathematics vol. 136, Elsevier, 1997.

[18] Timothy Williamson, *An alternative rule of disjunction in modal logic*, Notre Dame Journal of Formal Logic 33 (1992), no. 1, pp. 89–100.

[19] ———, *Some admissible rules in nonnormal modal systems*, Notre Dame Journal of Formal Logic 34 (1993), no. 3, pp. 378–400.

[20] ———, *Non-genuine MacIntosh logics*, Journal of Philosophical Logic 23 (1994), no. 1, pp. 87–101.

[21] Frank Wolter and Michael Zakharyaschev, *Undecidability of the unification and admissibility problems for modal and description logics*, ACM Transactions on Computational Logic 9 (2008), no. 4, article no. 25.