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DUALITY THEORY FOR LINEAR n-TH ORDER INTEGRO-DIFFERENTIAL OPERATORS WITH DOMAIN IN L^p_m DETERMINED BY INTERFACE SIDE CONDITIONS

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0. INTRODUCTION

In this paper we develop a duality theory for linear integro-differential operators in the space L^p_m of m-vector valued functions L^p-integrable on $[0, 1]$ associated with the system

$$(0,1) \qquad (ly)(t) = \sum_{i=0}^{n} (A_i(t)\, y^{(i)}(t) + \int_0^1 K_i(t, s)\, y^{(i)}(s)\, \mathrm{d}s + \sum_{i=0}^{n-1} \sum_{j=1}^{k} C_{i,j}(t)\, y^{(i)}(t_{j-1}+) + D_{i,j}(t)\, y^{(i)}(t_j-) + f(t),$$

$$(0,2) \qquad Hy = \sum_{i=0}^{n-1} \sum_{j=1}^{k} (M_{i,j}\, y^{(i)}(t_{j-1}+) + N_{i,j}\, y^{(i)}(t_j-)) + \sum_{i=0}^{n} \int_0^1 Q_i\, y^{(i)}\, \mathrm{d}t = 0,$$

where $0 = t_0 < t_1 < \ldots t_k = 1$ is a fixed subdivision of $[0, 1]$ and y is an m-vector valued function which is together with its derivatives $y^{(i)}$ of the orders i, $i \leqq n - 1$ absolutely continuous on every subinterval (t_{j-1}, t_j), $j = 1, 2, \ldots, k$ and whose n-th order derivative $y^{(n)}$ is L^p-integrable on $[0, 1]$. Such systems are usually called *interface boundary value problems.* Parhimovič [13], [14] showed (for $p = 2$) that under certain natural assumptions on the coefficients such problems are normally solvable, and found their index. We shall give an explicit formula for the adjoint relation to the operator $L: D(L) \subset L^p_m \to L^p_m$ corresponding to (0,1), (0,2) which is in general unbounded and nondensely defined. Similarly as in Brown, Krall [1] the main tool is the Linear Dependence Principle. Boundary value problems for integro-differential operators have been recently treated e.g. by Maksimov [10], Maksimov and Rahmatullina [11], cf. also Schwabik, Tvrdý and Vejvoda [16] or [18], [19] and [20]. Interface problems for differential operators were considered e.g. by Bryan [3], Conti [5], Gonelli [6], Krall [9], Stallard [17] and Zettl [21]. Schwabik [15] disclosed the relationship between interface problems and linear generalized differential equations (in the sense of Kurzweil).

Throughout the paper the following notation and conventions are kept. For $-\infty < a < b < \infty$ the closed interval $a \leqq t \leqq b$ is denoted by $[a, b]$, its interior $a < t < b$ by (a, b) and the corresponding half-open intervals $a < t \leqq b$ and $a \leqq t < b$ by $(a, b]$ and $[a, b)$, respectively. Given an $m \times k$-matrix $A = (a_{i,j})_{i=1,\dots,m\ j=1,\dots,k}$, A^* denotes its transpose and $|A| = \max_{i=1,\dots,m} \sum_{j=1}^{k} |a_{i,j}|$. The symbol I stands everywhere for the unit matrix of the proper type and any zero matrix is denoted by 0. R_m is the space of real column m-vectors with the norm $|x| = \max_{j=1,\dots,m} |x_j|$ for $x = (x_1, x_2, \dots, x_m) \in R_m$ $(R_1 = R)$. $L^p_m(a, b)$ denotes the Banach space of functions $y\colon [a, b] \to R_m$ such that

$$\|y\|_{L^p} = \left(\int_0^1 |y(t)|^p \, dt\right)^{1/p} < \infty ,$$

$L^\infty_m(a, b)$ is the Banach space of functions $y : [a, b] \to R_m$ measurable and essentially bounded on $[a, b]$, i.e.

$$\|y\|_{L^\infty} = \sup_{t \in [a\ b]} \operatorname{ess} |y(t)| < \infty .$$

Instead of $L^p_m(0,1)$ we write only L^p_m.

Let $q = p/(p-1)$ if $p > 1$, $q = \infty$ if $p = 1$. Then $L^q_m(a, b)$ is isometrically isomorphic with the dual space of $L^p_m(a, b)$. Given $z \in L^q_m(a, b)$, the corresponding linear bounded functional $\langle \cdot, z \rangle_{L^p}$ on $L^p_m(a, b)$ is given by

$$\langle y, z \rangle_{L^p} = \int_a^b z^* y \, dt \quad \text{for} \quad y \in L^p_m(a, b) .$$

An $m \times k$-matrix valued function is said to be L^p-integrable on $[a, b]$ if every its column belongs to $L^p_m(a, b)$. (This concerns also the case $p = \infty$.)

Let X, Y be Banach spaces and let T be a linear operator acting from X into Y. Then $D(T)$ denotes the domain of definition of T in X, $R(T)$ is the range of T in Y and $N(T)$ is its null space. If the spaces X^* and Y^* are respectively dual spaces to X and Y and $\langle \cdot, u \rangle_X$, $\langle \cdot, v \rangle_Y$ denote the linear bounded functionals corresponding respectively to $u \in X^*$ and $v \in Y^*$, then $T^* \subset X^* \times Y^*$ stands for the adjoint of T defined by

$$(u, v) \in T^* \quad \text{iff} \quad \langle Tx, v \rangle_Y = \langle x, u \rangle_X \quad \text{for all} \quad x \in D(T) .$$

If $D(T)$ is dense in X and T is bounded, then T^* is a linear bounded operator $Y^* \to X^*$ defined on the whole $Y^*((u, 0) \in T^*$ iff $u = 0)$. In general, T^* is a linear relation with the domain of definition $D(T^*) = \{v \in Y^*\colon$ there exists $u \in X^*$ such that $(u, v) \in T^*\}$ and the range $R(T^*) = \{u \in X^*\colon$ there exists $v \in Y^*$ such that $(u, v) \in T^*\}$. Let us notice that if T has a closed range in Y, then the Fredholm alternatives

$$R(T) = N(T^*)^\perp = \{y \in Y : \langle y, v \rangle_Y = 0 \text{ for all } v \in N(T^*)\}$$

and

$$N(T^*) = {}^\perp R(T) = \{v \in Y^* : \langle y, v \rangle_Y = 0 \text{ for all } y \in R(T)\}$$

hold (where $N(T^*) = \{v \in Y^* : (0, v) \in T^*$ is the null space of $T^*\}$). For more details concerning linear relations see Coddington, Dijksma [4] Section 2.

1. THE SPACE $D_m^{n,p}$

Let $\{0 = t_0 < t_1 < \ldots < t_k = 1\}$ be an arbitrarily chosen fixed subdivision of the interval $[0,1]$ and let $1 \leqq p < \infty$.

Let us denote by $D_m^{n,p}$ the space of all functions $y : [0,1] \to R_m$ which together with their derivatives $y^{(i)}$ of the orders i, $i \leqq n - 1$, are absolutely continuous on every (t_{j-1}, t_j), $j = 1, 2, \ldots, k$, and whose n-th order derivative $y^{(n)}$ is L^p-integrable on $[0,1]$.

1.1. Lemma. *The mapping*

$$\begin{aligned}(1,1)\qquad \varkappa : y \in D_m^{n,p} \to (&y(t_0+), y(t_1+), \ldots, y(t_{k-1}+), y'(t_0+), \\ &y'(t_1+), \ldots, y'(t_{k-1}+), \ldots, y^{(n-1)}(t_0+), \\ &y^{(n-1)}(t_1+), \ldots, y^{(n-1)}(t_{k-1}+), y^{(n)}) \in R_{nmk} \times L_m^p\end{aligned}$$

is a one-to-one mapping of $D_m^{n,p}$ onto $R_{nmk} \times L_m^p$.

Proof. Given $\xi = (\alpha_{1,0}, \alpha_{1,1}, \ldots, \alpha_{1,k-1}, \ldots, \alpha_{n-1,0}, \alpha_{n-1,1}, \ldots, \alpha_{n-1,k-1}, z) \in$ $\in R_{nmk} \times L_m^p = Y$, let us put $\psi(\xi) = y$, where $y : [0,1] \to R_m$ is defined by

$$y^{(n)} = z \quad \text{a.e. on } [0,1] ,$$

$$y^{(n-1)}(t) = \alpha_{n-1,j} + \int_{t_{j-1}}^{t} z \, d\tau \quad \text{for} \quad t \in (t_{j-1}, t_j) , \quad j = 1, 2, \ldots, k ,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y'(t) = \alpha_{1,j} + \int_{t_{j-1}}^{t} y'' \, d\tau \quad \text{for} \quad t \in (t_{j-1}, t_j) , \quad j = 1, 2, \ldots, k ,$$

$$y(t) = \alpha_{0,j} + \int_{t_{j-1}}^{t} y' \, d\tau \quad \text{for} \quad t \in (t_{j-1}, t_j) , \quad j = 1, 2, \ldots, k$$

$(y(t_j)$, $j = 0, 1, \ldots, k$ may be arbitrary).

Then evidently $\psi(\xi) \in D_m^{n,p}$, $\varkappa(\psi(\xi)) = \xi$ for every $\xi \in Y$ and $\psi(\varkappa(y)) = y$ for every $y \in D_m^{n,p}$.

Let us put for $y \in D_m^{n,p}$

$$(1,2)\qquad \|y\|_D = \sum_{i=0}^{n-1} \sum_{j=1}^{k} |y^{(i)}(t_{j-1}+)| + \|y^{(n)}\|_{L^p} .$$

Then $\|\cdot\|_D$ is obviously a norm on $D_m^{n,p}$. Moreover, $\|y\|_D = \|\varkappa(y)\|_Y$ *) for every $y \in D_m^{n,p}$. Consequently, we have

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*) If $\|\cdot\|_X$ and $\|\cdot\|_Y$ are norms in X and Y, respectively, then the norm on the product space $X \times Y$ is defined by $(x, y) \in X \times Y$

$$\|(x, y)\|_{X \times Y} = \|x\|_X + \|y\|_Y .$$

1.2. Lemma. *$D_m^{n,p}$ equipped with the norm (1,2) becomes a Banach space isometrically isomorphic with the Banach space $Y = R_{nmk} \times L_m^p$.*

1.3. Remark. Let us notice that

$$\|y\|_D = \sum_{j=1}^{k} \|y_j\|_W \quad \text{for} \quad y \in D_m^{n,p},$$

where y_j ($j = 1, 2, \ldots, k$) denote respectively the restrictions of y on (t_{j-1}, t_j) ($j = 1, 2, \ldots, k$) and

$$\|y_j\|_W = \sum_{i=0}^{n-1} |y_j^{(i)}(t_{j-1}+)| + \left(\int_{t_{j-1}}^{t_j} |y_j^{(n)}|^p \, dt\right)^{1/p}$$

is the norm of y_j in the Sobolev space $W_m^{n,p}(t_{j-1}, t_j)$ (m-vector valued functions which together with their derivatives of the orders i, $i \leq n-1$, are absolutely continuous on (t_{j-1}, t_j) and their n-th order derivative is L^p-integrable on (t_{j-1}, t_j)).

The zero element in the space $D_m^{n,p}$ is the class of functions $z : [0,1] \to R_m$ which vanish on every subinterval (t_{j-1}, t_j), $j = 1, 2, \ldots, k$ (the values $z(t_j)$ may be arbitrary).

It follows from Lemma 1.2 that the dual space $(D_m^{n,p})^*$ of $D_m^{n,p}$ is isometrically isomorphic with the dual space $Y^* = R_{nmk} \times L_m^q$ ($q = p/(p-1)$ if $p > 1$, $q = \infty$ if $p = 1$) of $Y = R_{nmk} \times L_m^p$.

1.4. Lemma. *Given an arbitrary linear bounded operator $H : D_m^{n,p} \to R_h$, there exist $h \times m$-matrices $M_{i,j}$ ($i = 0, 1, \ldots, n-1; j = 0, 1, \ldots, k-1$) and an $h \times m$-matrix valued function Q, L^q-integrable on $[0,1]$ ($q = p/(p-1)$ if $p > 1$, $q = \infty$* if $p = 1$), *such that*

$$(1,3) \qquad Hy = \sum_{i=0}^{n-1} \sum_{j=1}^{k} M_{i,j} y^{(i)}(t_{j-1}+) + \int_0^1 Q y^{(n)} \, dt \quad \textit{for any} \quad y \in D_m^{n,p}.$$

Remark. In particular, any "side" operator H of the form (0,2) may be transformed to the form (1,3).

1.5. Linear differential operator in $D_m^{n,p}$. Let A_i ($i = 0, 1, \ldots, n$) be $m \times m$-matrix valued functions defined a.e. on $[0,1]$ and L^p – integrable on $[0,1]$, while A_n is essentially bounded on $[0,1]$ and possesses an essentially bounded on $[0,1]$ inverse A_n^{-1}. Let us consider the linear differential expression

$$\lambda y = \sum_{i=0}^{n} A_i y^{(i)}$$

on the space $D_m^{n,p}$.

Obviously $\lambda y \in L_m^p$ for any $y \in D_m^{n,p}$. Furthermore, it is well known that for any $j = 1, 2, \ldots, k$, $g_j \in L_m^p(t_{j-1}, t_j)$ and $d_j = (c_{0j}, c_{1j}, \ldots, c_{n-1,j}) \in R_{nm}$, there exists a unique function $y_j \in W_m^{n,p}(t_{j-1}, t_j)$ such that

$$\lambda y_j = g_j \quad \text{a.e. on} \quad (t_{j-1}, t_j), \quad y_j^{(i)}(t_{j-1}+) = c_{i,j} \quad (i = 0, 1, \ldots, n-1).$$

By the variation-of-constants formula these y_j may be expressed in the form

$$y_j = U_j d_j + V_j g_j ,$$

where $U_j : R_{nm} \to W_m^{n,p}(t_{j-1}, t_j)$ and $V_j : L_m^p(t_{j-1}, t_j) \to W_m^{n,p}(t_{j-1}, t_j)$ are linear bounded operators. Hence, for a given $g \in L_m^p$ and $d = (c_{i,j})_{i=0,1,\dots,n-1\ j=1,2,\dots,k} \in R_{nmk}$, there exists a unique function $y \in D_m^{n,p}$ left-continuous on every $(t_{j-1}, t_j]$, right-continuous at 0 and such that

$$\lambda y = g \quad \text{a.e. on } [0,1]$$

and

$$y^{(i)}(t_{j-1}+) = c_{i,j} \quad (i = 0, 1, \dots, n-1;\ j = 1, 2, \dots, k) .$$

The function y may be expressed in the form

$$(1,4) \qquad y = Ud + Vg ,$$

where $U : R_{nmk} \to D_m^{n,p}$ and $V : L_m^p \to D_m^{n,p}$ are linear bounded operators. In fact, we put $y(t) = y_j(t)$ on (t_{j-1}, t_j), $y(t_j) = y(t_j-)$, $j = 1, 2, \dots, k$, $y(0) = y(0+)$,

$$(Ud)(t) = (U_j d_j)(t) \quad \text{for} \quad t \in (t_{j-1}, t_j) ,$$
$$(Ud)(t_j) = (Ud)(t_j-),\ (Ud)(0) = (Ud)(0+) , \quad j = 1, 2, \dots, k$$

and

$$(Vg)(t) = (Vg_j)(t) \quad \text{for} \quad t \in (t_{j-1}, t_j) ,$$
$$(Vg)(t_j) = (Vg)(t_j-),\ (Vg)(0) = (Vg)(0+) , \quad j = 1, 2, \dots, k ,$$

where $d_j = (c_{0,j}, c_{1,j}, \dots, c_{n-1,j}) \in R_{nm}$, $d = (d_1, d_2, \dots, d_k) \in R_{nmk}$ and g_j is the restriction of g on (t_{j-1}, t_j) $(j = 1, 2, \dots, k)$. Thus

$$\|Ud\|_D = \sum_{j=1}^{k} \|U_j d_j\|_W \quad \text{and} \quad \|Vg\|_D = \sum_{j=1}^{k} \|V_j g_j\|_W .$$

2. LINEAR INTEGRO-DIFFERENTIAL OPERATORS ON $D_m^{n,p}$

Throughout the rest of the paper we assume

2.1. Assumptions. $0 = t_0 < t_1 < \dots < t_k = 1$ is a fixed subdivision of the interval $[0,1]$ and $D_m^{n,p}$ is the corresponding function space defined as in Section 1. $A_i(t)$, $C_{i,j}(t)$ $(i = 0, 1, \dots, n;\ j = 1, 2, \dots, k)$ are $m \times m$-matrix valued functions defined a.e. on $[0,1]$ and L^p-integrable on $[0,1]$, $1 \leqq p < \infty$, A_n is essentially bounded on $[0,1]$, $q = p/(p-1)$ if $p > 1$, $q = \infty$ if $p = 1$, $K(t, s)$ is an $m \times m$-matrix valued function measurable in (t, s) on $[0,1] \times [0,1]$ and such that $K(\cdot, s)$ is measurable on $[0,1]$ for a.e. $s \in [0,1]$, $K(t, \cdot)$ is L^q-integrable on $[0,1]$ for a.e. $t \in [0,1]$ and the function $t \in [0,1] \to \|K(t, \cdot)\|_{L^q}$ is L^p-integrable on $[0,1]$, i.e.

$$(2,1) \qquad \|K\|_{p,q} = \left(\int_0^1 \left(\int_0^1 |K(t, s)|^q \,\mathrm{d}s\right)^{p/q} \mathrm{d}t\right)^{1/p} < \infty .$$

Under the assumptions 2.1 the integro-differential expression

$$(2,2)\quad (\ell y)(t) = \sum_{i=0}^{n} A_i(t)\, y^{(i)}(t) + \sum_{j=1}^{k} \sum_{i=0}^{n-1} C_{i,j}(t)\, y^{(i)}(t_{j-1}+) + \int_0^1 K(t,s)\, y^{(n)}(s)\, \mathrm{d}s$$

is for every $y \in D_m^{n,p}$ defined a.e. on $[0,1]$. Moreover, as

$$(2,3)\qquad K : u \in L_m^p \to \int_0^1 K(t,s)\, u(s)\, \mathrm{d}s$$

defines a *Hille-Tamarkin operator* on L_m^p, K is linear and bounded (cf. [7], Theorems 11.5 and 11.1). Thus we have

2.2. Lemma. *$\ell y \in L_m^p$ for any $y \in D_m^{n,p}$ and the linear operator $\ell : y \in D_m^{n,p} \to \to \ell y \in L_m^p$ is bounded.*

Proof. It remains to show the boundedness of ℓ. In fact, using the Hölder inequality we have for any $y \in D_m^{n,p}$

$$\|\ell y\|_{L^p} = \left(\int_0^1 \left| \sum_{i=0}^{n} A_i y^{(i)} + \sum_{j=1}^{k} \sum_{i=0}^{n-1} C_{i,j} y^{(i)}(t_{j-1}+) + \int_0^1 K(t,s)\, y^{(n)}(s)\, \mathrm{d}s \right|^p \mathrm{d}t\right)^{1/p} \leqq$$

$$\leqq \|A_n\|_{L^\infty} \|y^{(n)}\|_{L^p} + \sum_{i=0}^{n-1} \|A_i\|_{L^p} \|y^{(i)}\|_{L^\infty} +$$

$$+ \sum_{j=1}^{k} \sum_{i=0}^{n-1} \|C_{i,j}\|_{L^p} |y^{(i)}(t_{j-1}+)| + \|K\|_{p,q} \|y^{(n)}\|_{L^p}.$$

Since for any $i = 0, 1, \ldots, n-1$ and $t \in (t_{j-1}, t_j)$, $j = 0, 1, \ldots, k$

$$|y^{(i)}(t)| = \left| \sum_{r=0}^{n-i-1} y^{(i+r)}(t_{j-1}+)\,(t - t_{j-1})^r \frac{1}{r!} + \right.$$

$$\left. + \int_{t_{j-1}}^{t} \left(\int_{t_{j-1}}^{\tau_1} \left(\ldots \left(\int_{t_{j-1}}^{\tau_{n-i-1}} y^{(n)}\, \mathrm{d}\tau_{n-i} \right) \mathrm{d}\tau_{n-i-1} \right) \ldots \right) \mathrm{d}\tau_1 \right| \leqq$$

$$\leqq \sum_{r=0}^{n-i-1} |y^{(i+r)}(t_{j-1}+)| + \|y^{(n)}\|_{L^1} \leqq$$

$$\leqq \sum_{r=0}^{n-i-1} |y^{(i+r)}(t_{j-1}+)| + \|y^{(n)}\|_{L^p} \leqq \|y\|_D,$$

it follows that

$$\|\ell y\|_{L^p} \leqq \{\|A_n\|_{L^\infty} + \sum_{i=0}^{n-1} (\|A_i\|_{L^p} + \sum_{j=1}^{k} \|C_{i,j}\|_{L^p}) + \|K\|_{p,q}\} \|y\|_D$$

for all $y \in D_m^{n,p}$.

Remark. Under reasonable assumptions the integro-differential expression on the left-hand side of (0,1) can be reduced by repeated integration by parts to the form (2,2).

3. LINEAR INTEGRO-DIFFERENTIAL OPERATORS UNDER LINEAR CONSTRAINTS ON $D_m^{n,p}$

Under the assumptions 2.1 the integro-differential expression (2,2) defines a function from L_m^p for every $y \in D_m^{n,p}$ (cf. 2.2).

Let $H : D_m^{n,p} \to R_h$ be an arbitrary linear bounded h-vector valued functional on $D_m^{n,p}$, i.e.

(3,1) $$Hy = \sum_{i=0}^{n-1} \sum_{j=1}^{k} M_{i,j} y^{(i)}(t_{j-1}+) + \int_0^1 Q y^{(n)} \, dt \quad \text{for} \quad y \in D_m^{n,p},$$

where

(3,2) $M_{i,j}$ $(i = 0, 1, \ldots, n-1;\ j = 0, 1, \ldots, k-1)$ are $h \times m$-matrices and Q is an $h \times m$-matrix valued function L^q-integrable on $[0,1]$

(cf. 1.3).

Endowed with the norm of L_m^p, $D_m^{n,p}$ becomes a dense subspace of L_m^p and ℓ may be considered a densely defined operator in L_m^p.

3.1. Definition. L is the linear operator with domain $D(L) = \{y \in D_m^{n,p} : Hy = 0\}$ in L_m^p and the range $R(L)$ in L_m^p defined by

$$L : y \in D(L) \subset L_m^p \to \ell y \in L_m^p .$$

(L is the restriction of ℓ to $D(L) = N(H)$.)

Since $D(L)$ need not be dense in L_m^p, the adjoint L^* to L is in general a linear relation in $L_m^q \times L_m^q$. To derive its explicit form we examine the expression

(3,3) $$\langle Ly, z \rangle_{L^p} = \int_0^1 z^*(\ell y) \, dt =$$

$$= \sum_{i=0}^{n} \int_0^1 z^*(A_i y^{(i)}) dt + \sum_{i=0}^{n-1} \sum_{j=1}^{k} \left(\int_0^1 z^* C_{i,j} \, dt \right) y^{(i)}(t_{j-1}+) +$$

$$+ \int_0^1 z^*(t) \left(\int_0^1 K(t,s) \, y^{(n)}(s) \, ds \right) dt$$

with $z \in L_m^q$ and $y \in D(L)$.

3.2. Lemma. *Given $z \in L_m^q$, $y \in D_m^{n,p}$ and $i = 0, 1, \ldots, n-1$, then*

(3,4) $$\sum_{i=0}^{n} \int_0^1 z^* A_i y^{(i)} \, dt = \sum_{j=1}^{k} \sum_{i=0}^{n-1} \left(\sum_{r=0}^{i} [J^{i-r+1}(z^* A_r)] (t_{j-1}) \, y^{(i)}(t_{j-1}+) + \right.$$

$$\left. + \sum_{i=0}^{n} \int_0^1 [J^{n-i}(z^* A_i)] \, y^{(n)} \, dt \right),$$

where

$$(3,5)\quad [J^r u](t) = \int_t^{t_j} \left(\int_{\tau_1}^{t_j} \left(\dots \left(\int_{\tau_{r-1}}^{t_j} u(\tau_r)\, d\tau_r \right) d\tau_{r-1} \right) \dots \right) d\tau_1 \quad \textit{for} \quad t \in (t_{j-1}, t_j)$$

$$\textit{and any} \quad u \in L_m^p, \quad r = 1, 2, \dots,$$

$$J^0 u = u\,.$$

Proof. By repeated integration by parts we obtain for any $z \in L_m^q$, $y \in D_m^{n,p}$ and $i = 0, 1, \dots, n-1$ successively

$$\sum_{i=0}^{n-1} \int_0^1 z^* A_i y^{(i)}\, dt = \sum_{i=0}^{n-1} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} z^* A_i y^{(i)}\, dt =$$

$$= \sum_{j=1}^{k} \sum_{i=0}^{n-1} \left(\int_{t_{j-1}}^{t_j} z^* A_i\, d\tau \right) y^{(i)}(t_{j-1}+) + \int_{t_{j-1}}^{t_j} \left(\int_t^{t_j} z^* A_i\, d\tau \right) y^{(i+1)}\, dt =$$

$$= \sum_{j=1}^{k} \sum_{i=0}^{n-1} \left(\int_{t_{j-1}}^{t_j} z^* A_i\, d\tau \right) y^{(i)}(t_{j-1}+) + \left(\int_{t_{j-1}}^{t_j} \left(\int_{\tau_1}^{t_j} z^* A_i\, d\tau_2 \right) d\tau_1 \right) y^{(i+1)}(t_{j-1}+) +$$

$$+ \int_{t_{j-1}}^{t_j} \left(\int_t^{t_j} \left(\int_{\tau_1}^{t_j} z^* A_i\, d\tau_2 \right) d\tau_1 \right) y^{(i+2)}\, dt = \dots =$$

$$= \sum_{j=1}^{k} \sum_{i=0}^{n-1} \left(\int_{t_{j-1}}^{t_j} z^* A_i\, d\tau \right) y^{(i)}(t_{j-1}+) + \left(\int_{t_{j-1}}^{t_j} \left(\int_{\tau_1}^{t_j} z^* A_i\, d\tau_2 \right) d\tau_1 \right) y^{(i+1)}(t_{j-1}+) +$$

$$+ \left(\int_{t_{j-1}}^{t_j} \left(\int_{\tau_1}^{t_j} \left(\dots \left(\int_{\tau_{n-i-1}}^{t_j} z^* A_i\, d\tau_{n-i} \right) d\tau_{n-i-1} \right) \dots \right) d\tau_1 \right) y^{(n-1)}(t_{j-1}+) +$$

$$+ \int_{t_{j-1}}^{t_j} \left(\int_t^{t_j} \left(\int_{\tau_1}^{t_j} \left(\dots \left(\int_{\tau_{n-i-1}}^{t_j} z^* A_i\, d\tau_{n-i} \right) d\tau_{n-i-1} \right) \dots \right) d\tau_1 \right) y^{(n)}\, dt =$$

$$= \sum_{j=1}^{k} \Big(\sum_{i=0}^{n-1} \sum_{r=i}^{n-1} [J^{r-i+1}(z^* A_i)]\, (t_{j-1})\, y^{(r)}(t_{j-1}+) \Big) + \sum_{i=0}^{n-1} \int_0^1 [J^{n-i}(z^* A_i)]\, y^{(n)}\, dt\,,$$

where the notation (3,5) was utilized. Changing the order of summation in the expression in the brackets we obtain the relation (3,4).

3.3. Lemma. *Given $z \in L_m^q$ and $u \in L_m^p$, then*

$$(3,6)\qquad \int_0^1 z^*(t) \left(\int_0^1 K(t,s)\, u(s)\, ds \right) dt = \int_0^1 \left(\int_0^1 |z^*(t)|\, K(t,s)\, dt \right) u(s)\, ds\,.$$

Proof. Since for any $z \in L_m^q$ and $u \in L_m^p$

$$\int_0^1 \left(\int_0^1 |z^*(t)\, K(t,s)\, u(s)|\, ds \right) dt \leqq \left(\int_0^1 |z^*(t)|\, \|K(t,\cdot)\|_{L^q}\, dt \right) \|u\|_{L^p} \leqq$$

$$\leqq \|z\|_{L^q}\, \|K\|_{p,q}\, \|u\|_{L^p} < \infty$$

and the function $z^*(t)K(t,s)u(s)$ is certainly measurable on $[0,1] \times [0,1]$, the relation (3,6) follows by the Tonelli-Hobson Theorem ([12], Corollary of Theorem XII.4.2).

By virtue of the formulas (3,4)–(3,6) the relation (3,3) will be reduced to

$$(3,7)\quad \langle Ly, z\rangle_{L^p} = \sum_{i=0}^{n-1}\sum_{j=1}^{k}\left(\int_0^1 z^* C_{i,j}\,dt + \sum_{r=0}^{i}[J^{i-r+1}(z^*A_r)](t_{j-1})\right) y^{(i)}(t_{j-1}+) + $$

$$+ \int_0^1\left(\int_0^1 z^*(s)\,K(s,t)\,ds + \sum_{i=0}^{n}[J^{n-i}(z^*A_i)]\,y^{(n)}\right)dt$$

for all $z \in L^q_m$ and $y \in D^{n,p}_m$.

The couple $(v, z) \in L^q_m \times L^q_m$ belongs to the graph of the adjoint relation L^* if and only if

$$\langle Ly, z\rangle_{L^p} = \langle y, v\rangle_{L^p} = \int_0^1 v^* y\,dt \quad \text{for all} \quad y \in D(L).$$

Similarly as the relation (3,4) was derived in Lemma 3.2 we may derive that

$$(3,8)\qquad \langle y, v\rangle_{L^p} = \sum_{i=0}^{n-1}\sum_{j=1}^{k}[J^{i+1}v^*](t_{j-1})\,y^{(i)}(t_{j-1}+) + \int_0^1 [J^n v^*]\,y^{(n)}\,dt$$

holds for all $y \in D^{n,p}_m$ and $v \in L^q_m$. This together with (3,7) yields that $(v, z) \in L^*$ if and only if

(3,9)

$$\sum_{i=0}^{n-1}\sum_{j=1}^{k}\left\{\int_0^1 z^*C_{i,j}\,dt + \sum_{r=0}^{i}[J^{i-r+1}(z^*A_r)](t_{j-1}) - [J^{i+1}v^*](t_{j-1})\right\} y^{(i)}(t_{j-1}+) +$$

$$+ \int_0^1\left\{\int_0^1 z^*(s)\,K(s,t)\,ds + \sum_{r=0}^{n}[J^{n-r}(z^*A_r)] - [J^n v^*]\right\} y^{(n)}\,dt = 0$$

holds for every $y \in D(L)$. Now, we can make use of the Linear Dependence Principle ([8], p. 7):

Suppose $\lambda, \psi_1, \psi_2, \ldots, \psi_N$ is a finite collection of linear functionals (possibly unbounded) defined on a linear space X and such that

$$\psi_j(x) = 0\,, \quad j = 1, 2, \ldots, N \quad \text{implies} \quad \lambda(x) = 0$$

($\bigcap_{j=1}^{N} N(\psi_j) \subset N(\lambda)$). Then on X λ is a linear combination of the functionals $\psi_1, \psi_2, \ldots$ $\ldots, \psi_N$ (i.e. there are $\varphi_1, \varphi_2, \ldots, \varphi_N \in R$ such that

$$\lambda(x) = \varphi_1\,\psi_1(x) + \varphi_2\,\psi_2(x) + \ldots + \varphi_N\,\psi_N(x) \text{ on } X).$$

From the definition of $D(L)$ and from the Linear Dependence Principle it is clear that (3,9) may occur if and only if there exists $\varphi \in R_h$ such that the relations

$$(3,10)\quad \int_0^1 z^*C_{i,j}\,dt + \sum_{r=0}^{i}[J^{i-r+1}(z^*A_r)](t_{j-1}) - [J^{i+1}v^*](t_{j-1}) = \varphi^* M_{i,j}\,,$$

$$i = 0, 1, \ldots, n-1\,; \quad j = 1, 2, \ldots, k$$

and

$$(3{,}11)\quad \int_0^1 z^*(s)K(s,t)\,\mathrm{d}s + \sum_{r=0}^{n}[J^{n-r}(z^*A_r)](t) - [J^n v^*](t) = \varphi^*\,Q(t) \text{ a.e. on } [0,1]$$

hold. In particular, if we denote

$$\ell_0^+(z,\varphi) = -\sum_{r=0}^{n-1}[J^{n-r}(A_r^* z)] + J^n v\,,$$

then $\ell_0^+(z,\varphi)$ is absolutely continuous on every (t_{j-1}, t_j), $j = 1, 2. \ldots, k$ and

$$(3{,}12)\qquad \ell_0^+(z,\varphi) = A_n^* z + \int_0^1 K^*(t,s)\,z(s)\,\mathrm{d}s - Q^*\varphi \quad \text{a.e. on} \quad [0,1]\,.$$

Let us notice that for a given $u \in L_m^q$, $[Ju]' = -u$ a.e. on $[0,1]$ and

$$[J^r u]' = -[J^{r-1}u]\,,\quad r = 2, 3, \ldots \quad \text{on each} \quad (t_{j-1}, t_j)\,,\quad j = 1, 2, \ldots, k\,.$$

Hence

$$[\ell_0^+(z,\varphi)]' = A_{n-1}^* z + \sum_{i=0}^{n-2}[J^{n-1-r}(A_r^* z)] - [J^{n-1}v] \quad \text{a.e. on} \quad [0,1]\,.$$

Denoting

$$\ell_1^+(z,\varphi) = -\sum_{i=0}^{n-2}[J^{n-1-r}(A_r^* z)] + [J^{n-1}v]\,,$$

we obtain

$$(3{,}13)\qquad \ell_1^+(z,\varphi) = -[\ell_0^+(z,\varphi)]' + A_{n-1}^* z \quad \text{a.e. on} \quad [0,1]\,,$$

with $\ell_1^+(z,\varphi)$ absolutely continuous on every (t_{j-1}, t_j). In general, we denote for $i = 0, 1, \ldots, n-1$

$$(3{,}14)\qquad \ell_i^+(z,\varphi) = -\sum_{r=0}^{n-1-i}[J^{n-i-r}(A_r^* z)] + [J^{n-i}v]\,.$$

Thus every $\ell_i^+(z,\varphi)$, $i = 0, 1, \ldots, n-1$ is absolutely continuous on each interval (t_{j-1}, t_j), $j = 1, 2, \ldots, k$. Moreover,

$$(3{,}15)\qquad \ell_i^+(z,\varphi) = -[\ell_{i-1}^+(z,\varphi)]' + A_{n-i}^* z \quad \text{a.e. on} \quad [0,1]\,,$$

$$i = 1, 2, \ldots, n-1$$

and

$$[\ell_{n-1}^+(z,\varphi)]' = A_0^* z + v \quad \text{a.e. on} \quad [0,1]\,.$$

It means that the relation (3,11) is equivalent to

$$v = -[\ell_{n-1}^+(z,\varphi)]' + A_0^* z \quad \text{a.e. on} \quad [0,1]\,.$$

In particular,

$$(3{,}16)\qquad \ell_n^+(z,\varphi) := -[\ell_{n-1}^+(z,\varphi)]' + A_0^* z \in L_m^q\,.$$

By (3,14) we have

(3,17) $\quad [\ell_i^+(z, \varphi)](t_j-) = 0$ for all $i = 0, 1, \ldots, n-1$; $j = 1, 2, \ldots, k$.

Furthermore,

$$\ell_i^+(z, \varphi)(t_{j-1}+) = -\sum_{r=0}^{n-1-i} [J^{n-i-r}(A_r^* z)](t_{j-1}) + [J^{n-i}v](t_{j-1}) .$$

By virtue of this identity the relation (3,10) becomes

(3,18) $$[\ell_i^+(z, \varphi)](t_{j-1}+) = \int_0^1 C_{n-1-i,j}^* z \, dt - M_{n-1-i,j}^* \varphi \quad \text{for all}$$

$$i = 0, 1, \ldots, n-1 \quad \text{and} \quad j = 1, 2, \ldots, k .$$

To summarize:

3.4. Theorem. *Let us assume 2.1 and (3,2) and let us denote by D^+ the set of all couples $(z, \varphi) \in L_m^q \times R_h$ such that there exist functions $\ell_i^+(z, \varphi)$, $i = 0, 1, \ldots, n-1$, absolutely continuous on every interval (t_{j-1}, t_j), $j = 1, 2, \ldots, k$, and fulfilling (3,12), (3,15), (3,17) and (3,18).*

Let $\ell_n^+(z, \varphi)$ be defined for $(z, \varphi) \in D^+$ by (3,16). Then the graph of the adjoint relation L^* to L consists of all couples $(\ell_n^+(z, \varphi), z)$ with $(z, \varphi) \in D^+$, i.e.

$$L^* = \{(\ell_n^+(z, \varphi), z) : (z, \varphi) \in D^+\} .$$

In particular, the domain $D(L^*)$ of L^* is the set of all $z \in L_m^q$ for which there exists $\varphi \in R_h$ such that $(z, \varphi) \in D^+$.

The "only if" part of Theorem 3.4 also follows from the following "Green's formula" which is easy to verify (cf. [1]).

3.5. Proposition. *Given $y \in D_m^{n,p}$ and $(z, \varphi) \in D^+$, then*

(3,19) $$\langle y, \ell_n^+(z, \varphi)\rangle_{L^p} = \langle \ell y, z\rangle_{L^p} - \varphi^*(Hy) .$$

Remark. If $1 < p < \infty$, then by [7], Theorem 11.6 the operator K given by (2,3) is compact. This enables us to show analogously as in [16] V.2 the closedness of the range $R(L)$ of the operator L. In fact, according to the variation-of-constants formula (1,3), for a couple $(f, r) \in L_m^p \times R_h$ there exists $y \in D_m^{n,p}$ such that $\ell y = f$ and $Hy = r$ if and only if for some $d = (c_{i,j})_{i=0,1,\ldots,n-1 \; j=1,2,\ldots,k} \in R_{nmk}$,

$$y = Ud + V(f - Cd - Ky) \quad \text{and} \quad H(U - VC)\, d - HVKy = r ,$$

where $C : R_{nmk} \to L_m^p$,

$$(Cd)(t) := \sum_{j=1}^{k} \sum_{i=0}^{n-1} C_{i,j}(t)\, c_{i,j} \quad \text{a.e. on} \quad [0,1]$$

for

$$d = (c_{i,j})_{i=0,1,\ldots,n-1 \; j=1,2,\ldots,k} \in R_{nmk} .$$

In other words, $(f, r) \in L^p_m \times R_h$ belongs to the range $R(\mathscr{L})$ of the operator

(3,20) $$\mathscr{L} : y \in D^{n,p}_m \to \begin{pmatrix} \ell y \\ Hy \end{pmatrix} \in L^p_m \times R_h$$

if and only if (Vf, r) belongs to the range $R(T)$ of the operator

$$T : (y, d) \in D^{n,p}_m \times R_{nmk} \to \begin{pmatrix} y - (U - VC)\, d + VKy \\ H(U - VC)\, d - HVKy \end{pmatrix} \in D^{n,p}_m \times R_h\,.$$

Since all the operators $U - VC$, VK, $H(U - VC)$ and HVK are compact and $\theta : (f, r) \in L^p_m \times R_h \to (Vf, r) \in D^{n,p}_m \times R_h$ is bounded, the closedness of the range $R(\mathscr{L}) = \theta_{-1}(R(T))$ of $\mathscr{L}$ follows from the following lemma.

3.7. Lemma. *Let X be a Banach space. Let the operators $Q : X \to X$, $P : X \to R_h$, $A : R_m \to X$ and $B : R_m \to R_h$ be linear and bounded. Then, provided that Q is compact, the operator*

$$W : (x, d) \in X \times R_m \to \begin{pmatrix} x - Ad - Qx \\ Bd + Px \end{pmatrix} \in X \times R_h$$

has closed range in $X \times R_h$.

Proof. a) If $m < h$, let us put for $d = \begin{pmatrix} c \\ d' \end{pmatrix} \in R_h$, $c \in R_m$

$$\tilde{A}d := Ac \in X\,, \quad \tilde{B}d := Bc \in R_h$$

and for $x \in X$

$$\tilde{W}(x, d) := \begin{pmatrix} \tilde{A}d + Qx \\ (-I + \tilde{B})\, d + Px \end{pmatrix} \in X \times R_h\,,$$

where I stands for the identity operator on R_h (the identity $h \times h$-matrix). Clearly, $\tilde{W}$ is linear, bounded and compact and consequently the range $R(W) = R(I - \tilde{W})$ (I the identity operator on $X \times R_h$) of both W and $I - \tilde{W}$ is closed in $X \times R_h$.

b) If $m > h$, we put

$$\tilde{B}d := \begin{pmatrix} Bd \\ 0 \end{pmatrix} \in R_m\,, \quad \tilde{P}x := \begin{pmatrix} Px \\ 0 \end{pmatrix} \in R_m$$

for $d \in R_m$ and $x \in X$. Then $(y, u) \in R(W)$ if and only if (y, v), where $v = \begin{pmatrix} u \\ 0 \end{pmatrix} \in R_m$, belongs to the range of the operator

$$I - \tilde{W} : (x, d) \in X \times R_m \to \begin{pmatrix} x \\ d \end{pmatrix} - \begin{pmatrix} Ad + Qx \\ (-I + \tilde{B})\, d + \tilde{P}x \end{pmatrix} \in X \times R_m\,.$$

Again $\tilde{W}$ is compact and consequently $R(I - \tilde{W})$ is closed in $X \times R_m$. Now it is easy to verify that also $R(W)$ is closed in $X \times R_h$.

c) The case $m = h$ is obvious.

3.8. Corollary. *The operator $\mathscr{L}$ given by* (3,20) *has closed range in* $L^p_m \times R_h$.

Since $f \in L^p_m$ belongs to the range of L if and only if $(f, 0) \in L^p_m \times R_h$ belongs to the range of $\mathscr{L}$, the closedness of the range of L in L^p_m follows immediately from 3.8.

3.9. Theorem. *Let us assume* 2.1 *and* (3,2) *and let* $1 < p < \infty$. *Then the operator L* (*defined in* 3.1) *has closed range in* L^p_m.

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