On the pumping effect in a pipe/tank flow configuration with friction

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Abstract

We provide sufficient conditions for the existence and the asymptotic stability of periodic positive solutions for a pipe/tank flow configuration. The model is a nonlinear ordinary second order differential equation with a singularity containing the second power of the derivative of the unknown function and with a friction. In this way we complement previous results obtained by G. Propst in 2006.

Keywords. Valveless pumping; singular differential equation; periodic boundary value problem.

1 Introduction

A periodically forced differential system is called a *pump* if it has an asymptotically stable periodic solution with a non-equilibrium mean. A precise general definition of a

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pump may be found in [10] (see Definition 1). In the special case, when the system is governed by a second order scalar differential equation and we require the existence of a T-periodic solution, this definition reads as follows.

1.1 Notation. For a given continuous function $h\colon [0,T] \to \mathbb{R}$, we denote

$$\bar{h} = \frac{1}{T}\int_0^T h(s)\,\mathrm{d}s\,,\; h_* = \min\{h(t) : t \in [0,T]\} \;\text{ and }\; h^* = \max\{h(t) : t \in [0,T]\}.$$

1.2 Definition. Let $T > 0$, $g : \mathbb{R}^3 \to \mathbb{R}$ and let $e : \mathbb{R} \to \mathbb{R}$ be a nonconstant T-periodic function, then we say that the equation

$$x'' = g(x, x', e(t)) \tag{1.1}$$

is a *periodically forced pump* if it has a T-periodic solution x such that

$$g(\bar{x}, 0, \bar{e}) \neq 0,$$

i.e. the mean value $\bar{x}$ of x is not an equilibrium of

$$x'' = g(x, x', \bar{e}).$$

Recently, G. Propst [10] presented an explanation of the pumping effect for flow configurations of 1–3 rigid tanks that are connected by rigid pipes. Moreover, he proved the existence of periodic solutions to the corresponding differential equations for systems of 2 or 3 tanks, while for the apparently simplest configuration consisting of 1 pipe and 1 tank, he guaranteed the existence of a positive periodic solution only in some particular cases.

1.3 Remark. The flow configurations considered by G. Propst in [10] are special cases of valveless systems of a moving fluid. The term valveless pumping refers to the conveyance of liquid fluids in mechanical systems that have no valves to ensure the preferential direction of flow. Such phenomena appear e.g. in the models of blood circulation in the cardiovascular system, in some other models from microfluidics, or, at large scales, in oceanic currents. Valveless pumping is also referred to as Liebau phenomenon, after the pioneering works by G. Liebau starting in 1954, see [1, 3, 7].

In the configuration of 1 pipe and 1 tank (see Figure 1), a horizontal pipe is connected to a vertical tank, both contain a fluid of density ρ, up to the level height h in the tank. In the pipe, at distance ℓ from the tank, the pressure outside the mass- and frictionless moveable piston is forced T-periodically in time t, while the environmental pressure above the fluid in the open tank is set to zero. The time derivative of the momentum of the mass of the fluid in the pipe between the piston and the tank equals the sum of the forces acting on it. These forces are due to the pressure p and the

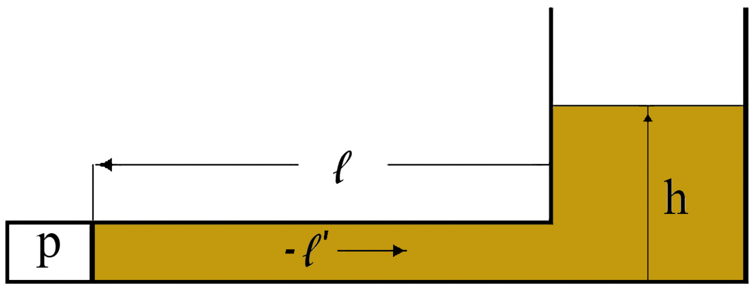

Figure 1: 1 pipe–1 tank configuration

hydrostatic pressure $\rho \cdot g \cdot h$ at the bottom of the tank, where g is the acceleration of gravity. The friction is modeled by Poiseuille's law with friction coefficient r_0. Furthermore, since the cross section A_P of the pipe is assumed to be small in comparison to the cross section A_T of the tank, the fluid in the tank is modeled to be at rest. Finally, by Bernoulli's equation, the pressure loss is due to the difference of the fluid velocity in the pipe and zero velocity in the tank, i.e., is given by $\zeta \frac{\rho}{2} w^2$, where $\zeta \geq 1$ is the junction coefficient depending on the particular geometry and smoothness of the junction of the tank and the pipe and $w = -\ell'$ is the fluid velocity in the pipe (oriented in the direction from the piston to the tank). Therefore, the fluid motion in the pipe is described by the relation

$$\rho\,(\ell(t)\,w(t))' = p(t) - r_0\,\ell(t)\,w(t) - \rho\,g\,h(t) + \zeta\frac{\rho}{2}\,(w(t))^2. \tag{1.2}$$

The level height h is coupled to ℓ by the momentum equation

$$A_P\,\ell(t) + A_T\,h(t) = V_0, \tag{1.3}$$

where the total volume V_0 of the fluid in the system is supposed to be constant. Inserting $w = -\ell'$ and (1.3) into (1.2), we get

$$\ell''(t) = -\frac{r_0}{\rho}\,\ell'(t) + \frac{1}{\ell(t)}\left(-\left(1+\frac{\zeta}{2}\right)(\ell'(t))^2 + \frac{g\,V_0}{A_T} - \frac{p(t)}{\rho}\right) - \frac{gA_P}{A_T}. \tag{1.4}$$

From the physical point of view we are interested in the search of positive solutions of the problem (1.4). The right-hand side of the differential equation in (1.4) is singular for $\ell = 0$ and, moreover, the singular term

$$\frac{1}{\ell(t)}\left(-\left(1+\frac{\zeta}{2}\right)(\ell'(t))^2\right)$$

involves also the second power of the derivative ℓ' of ℓ. As there is lack of general existence results for such problems, this makes the analysis of the given model rather difficult. To our knowledge, only Hakl, Torres and Zamora considered in [5] and [6] remotely resembling problems for equations of the form

$$u'' + f(u)\,u' + g(t,u) = h(t,u),$$

where both f and g can have a singularity at the origin. Our aim is to fill the gap providing existence and stability results for the quite general periodic forcing p.

For the sake of clarity, let us denote

$$a = \frac{r_0}{\rho}, \quad b = 1 + \frac{\zeta}{2}, \quad c = \frac{g A_P}{A_T}, \quad e(t) = \frac{g\,V_0}{A_T} - \frac{p(t)}{\rho}, \quad \text{and} \quad u = \ell. \tag{1.5}$$

Then the given problem can be reformulated as the periodic boundary value problem

$$u'' + a\,u' = \frac{1}{u}\Big(e(t) - b\,(u')^2\Big) - c, \tag{1.6}$$

$$u(0) = u(T), \quad u'(0) = u'(T). \tag{1.7}$$

According to the physical meaning of the involved parameters, we may assume

$$a \geq 0, \quad b > 1, \quad c > 0, \quad \text{and} \quad e \ \text{ is continuous and } T\text{-periodic on } \mathbb{R}. \tag{1.8}$$

1.4 Remark. The case $a = 0$ means that there is no Poiseuille friction, which might be considered an idealization but not meaningless from modeling point of view. However, for the application of the theory presented in Section 2 we need to have a positive coefficient at the linear first order term of the second order boundary value problem, that is $a > 0$.

Assuming that u is a positive solution to (1.6),(1.7), multiplying (1.6) by $u(t)$ and integrating over the interval $[0,T]$, we get that the relation

$$\bar{e} - c\,\bar{u} = (b-1)\overline{(u')^2} \tag{1.9}$$

must hold. This yields immediately the following necessary condition for the existence of a positive solution to (1.6),(1.7).

1.5 Theorem. *Let* (1.8) *hold and let problem* (1.6), (1.7) *have a positive solution. Then*

$$\bar{e} > 0. \tag{1.10}$$

1.6 Remark. By (1.5), condition (1.10) is satisfied if and only if

$$\bar{p} < \rho\, g\, \frac{V_0}{A_T}. \tag{1.11}$$

This means that, if A_T and V_0 are fixed and the periodic forcing p has a positive mean value $\bar{p}$, then problem (1.6),(1.7) has no solution whenever the fluid density ρ is small enough.

Furthermore, put $g(x,y,z) = -a\,y + \frac{1}{x}\,(z - b\,y^2) - c$ for $x \in (0,\infty)$, $y, z \in \mathbb{R}$. Then the equation (1.6) takes the form (1.1). Let u be an arbitrary positive nonconstant solution of (1.6), (1.7). By (1.8) and (1.9) we have

$$g(\bar{u}, 0, \bar{e}) = \frac{1}{\bar{u}}\,(\bar{e} - c\,\bar{u}) = \frac{1}{\bar{u}}\,(b-1)\overline{(u')^2} > 0,$$

i.e. $x = \bar{u}$ is not a zero of $g(x, 0, \bar{e})$. Therefore, the following simple observation is true, as well.

1.7 Theorem. *Let (1.8) hold and let problem (1.6), (1.7) have a nonconstant positive solution. Then equation (1.6) is a T-periodically forced pump.*

The main results of this paper concerning the existence and stability of positive periodic solutions for equation (1.6) are the following Theorems 1.8 and 1.10.

1.8 Theorem. *Assume (1.8) and $e_* > 0$. Then problem (1.6), (1.7) has a positive solution provided that the following inequality holds:*

$$\frac{(b+1)\,c^2}{4\,e_*} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}\,. \tag{1.12}$$

1.9 Remark. In particular, whenever $e_* > 0$, problem (1.6),(1.7) has a positive solution if a is large enough or T is small enough.

Notice that $e_* > 0$ if and only if

$$p^* < \rho\, g\, \frac{V_0}{A_T},$$

which means that the forcing pressure is not too large in comparison with the hydrostatic pressure at the bottom of the tank and with the mass of the fluid in the system. So it is a natural requirement preventing the emptying of the pipe.

1.10 Theorem. *Assume (1.8) and let $a > 0$ and $e_* > 0$. Then problem (1.6), (1.7) has at least one asymptotically stable positive solution provided that the following inequalities hold:*

$$\frac{c^2\,(b\,(e^*)^2 - (b-1)\,(e_*)^2)}{e_*\,(e^*)^2} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4} \tag{1.13}$$

and

$$(b-1)\,e^* < b\,e_*. \tag{1.14}$$

1.11 Remark. If $e(t)$ is a positive constant, that is $e^* = e_* > 0$, then condition (1.14) is always satisfied. Moreover, if in addition $1 < b \le 3$, then condition (1.12) is satisfied whenever (1.13) holds, while if $b \ge 3$, then condition (1.13) is satisfied whenever (1.12) holds. The size of $b = 1 + \frac{\zeta}{2}$ is related to the strength of the nonlinearity in the model (1.2), and in (1.15) below.

The rest of the paper is devoted to the proof of the previous two theorems: in Section 2 we present some lemmas needed in order to prove Theorem 1.8 in Section 3 and Theorem 1.10 in Section 4. Our main tool will be the use of lower and upper functions.

The change of variables $u=x^\mu$, where $\mu=\frac{1}{b+1}$, transforms the singular problem (1.6), (1.7) to the regular problem

$$x''(t)+a\,x'(t)+s(t)\,x^\beta-r(t)\,x^\alpha=0, \tag{1.15}$$

$$x(0)=x(T), \quad x'(0)=x'(T), \tag{1.16}$$

where

$$r(t)=\frac{e(t)}{\mu}, \quad s(t)=\frac{c}{\mu},\ \alpha=1-2\,\mu,\ \beta=1-\mu. \tag{1.17}$$

In particular, we have the following result.

1.12 Lemma. *The function $u:[0,T]\to\mathbb{R}$ is a positive solution to problem* (1.6), (1.7) *if and only if $x(t)=u(t)^{1/\mu}$ is a positive solution to problem* (1.15), (1.16).

1.13 Remark. Since $b>1$ by (1.8), we have $0<\mu<1/2$ and, hence, $0<\alpha<\beta<1$. In particular, this means that problem (1.15), (1.16) is regular.

2 Some auxiliary results

Assume

$$f:[0,T]\times\mathbb{R}\to\mathbb{R} \text{ is continuous} \tag{2.1}$$

and consider the auxiliary problem

$$x''(t)+a\,x'(t)+f(t,x)=0, \quad x(0)=x(T),\ x'(0)=x'(T). \tag{2.2}$$

The proofs of our main results rely on the method of lower and upper functions, see e.g. [4]. For our purposes the following definitions are satisfactory.

2.1 Definition. A T-periodic function σ_1 with a second order derivative continuous on $[0,T]$ is a *lower function* for (2.2) if

$$\sigma_1''(t)+a\,\sigma_1'(t)+f(t,\sigma_1(t))\ge 0 \quad \text{for} \ \ t\in[0,T],$$

while an *upper function* is defined analogously, but with reversed inequality.

If a lower function or an upper function is not a solution to (2.2), it is called a *strict* lower function or a strict upper function of (2.2), respectively.

The following two auxiliary results will be useful for the proof of Theorem 1.8. The former is a special case of [2, Theorem 2.4] and is due to D. Bonheure and C. de Coster.

2.2 Lemma. *In addition to* (2.1) *assume that there is a function p continuous on* $[0,T]$ *such that*

$$\limsup_{x\to-\infty} f(t,x) \le p(t) \quad \textit{and} \quad \limsup_{x\to\infty} \frac{f(t,x)}{x} \le \frac{\pi^2}{T^2} \tag{2.3}$$

uniformly in $t \in [0,T]$. *Moreoever, let* σ_1 *and* σ_2 *be respectively lower and upper functions of* (2.2) *such that* $\sigma_2 < \sigma_1$ *on* $[0,T]$.

Then problem (2.2) *possesses a solution x such that* $\sigma_2(t_1) \le x(t_1) \le \sigma_1(t_1)$ *for some* $t_1 \in [0,T]$.

2.3 Remark. Let us note that the proof of [2, Theorem 2.4] as given in [2] should be supplied by a proper a priori estimate of the derivatives of the possible solutions to (2.2). With respect to the first two parts of the proof in [2], this estimate can be, of course, obtained using some of the standard Nagumo type theorems (see e.g. [4, Proposition 4.3]).

The latter lemma relies on the antimaximum principle by P. Omari and M. Trombetta.

2.4 Lemma. [9, Proposition 2.2] *Let* $a, \lambda \in \mathbb{R}$ *be such that*

$$0 < \lambda \le \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4} \tag{2.4}$$

and let h be continuous on $[0,T]$. *Then any solution u of the linear periodic boundary value problem*

$$u'' + a\,u' + \lambda\,u = h(t), \quad u(0) = u(T),\ u'(T) = u'(T) \tag{2.5}$$

is nonnegative on $[0,T]$ *whenever h is nonnegative on* $[0,T]$.

In order to get stability of the periodic solution the following result will be useful.

2.5 Lemma. ([8, Theorem 1.2 and Remark 3.2]) *Assume* $a > 0$ *and let* σ_1 *be a strict lower and* σ_2 *a strict upper function of* (1.15), (1.7) *such that* $\sigma_2 < \sigma_1$ *on* $[0,T]$. *Furthermore, let f and* $\dfrac{\partial}{\partial x} f$ *be continuous and such that*

$$\frac{\partial}{\partial x} f(t,x) \le \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4} \quad \textit{for a.e. } t \in [0,T] \textit{ and every } x \in [\sigma_2(t), \sigma_1(t)]\,, \tag{2.6}$$

and

$$\left.\begin{array}{l} \textit{there is a continuous function } \ \gamma : [0,T] \to [0,\infty) \ \textit{ such that } \bar{\gamma} > 0 \ \textit{ and} \\ \dfrac{\partial}{\partial x} f(t,x) \ge \gamma(t) \textit{ for } t \in [0,T],\ x \in [\sigma_2(t), \sigma_1(t)]\,. \end{array}\right\} \tag{2.7}$$

Then problem (2.2) *has at least one asymptotically stable T-periodic solution x fulfilling*

$$\sigma_2 \le x \le \sigma_1 \quad \textit{on} \quad [0,T]\,. \tag{2.8}$$

3 Proof of Theorem 1.8

Assume that the assumptions of Theorem 1.8 are satisfied. We are going to prove that then problem (1.6), (1.7) has a positive solution. Put

$$\lambda^* = \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}\,. \tag{3.1}$$

With respect to Lemma 1.12 it suffices to show that problem (1.15), (1.16) has at least one positive solution provided the relations $a>0$, $r_*>0$, $s_*>0$, $\alpha=1-2\mu$, $\beta=1-\mu$ and

$$\frac{(s^*)^2}{4\,r_*} < \lambda^* \tag{3.2}$$

are true. (Notice that, by Notation 1.1 and definitions (1.17), conditions (3.2) and (1.12) are equivalent.)

Step 1. We have

$$0 < r_* \le r^* < \infty \ \text{ and } \ 0 < s_* \le s^* < \infty. \tag{3.3}$$

Put

$$f(t,x) = s(t)\,x^\beta - r(t)\,x^\alpha \quad \text{for} \ \ t\in[0,T] \ \text{ and } x\in\mathbb{R}\,. \tag{3.4}$$

Then

$$\left.\begin{aligned} & f(t,x) \ge x^\alpha\,(s_*\,x^{\beta-\alpha} - r^*) > 0 \ \text{ for } t\in[0,T] \ \text{ and } x > (r^*/s_*)^{1/(\beta-\alpha)}\,, \\ &\text{and} \\ & f(t,x) \le x^\alpha\,(s^*\,x^{\beta-\alpha} - r_*) < 0 \ \text{ for } t\in[0,T] \ \text{ and } x\in(0,(r_*/s^*)^{1/(\beta-\alpha)})\,. \end{aligned}\right\} \tag{3.5}$$

Therefore, for arbitrary $\rho_1\in((r^*/s_*)^{1/(\beta-\alpha)},\infty)$ and $\rho_2\in(0,(r_*/s^*)^{1/(\beta-\alpha)})$, the constant functions

$$\sigma_1(t)\equiv\rho_1 \ \text{ and } \ \sigma_2(t)\equiv\rho_2 \tag{3.6}$$

are, respectively, a strict lower and a strict upper function of problem (1.15), (1.7). (As $\rho_2<\rho_1$, we have a *reversely ordered* pair of strict lower and upper functions.)

Step 2. Let $\rho_1\in((r^*/s_*)^{1/(\beta-\alpha)},\infty)$ and $\rho_2\in(0,(r_*/s^*)^{1/(\beta-\alpha)})$ be fixed and σ_1 and σ_2 be given by (3.6). Let

$$\lambda_0 := \frac{(s^*)^2}{4r_*}. \tag{3.7}$$

We will prove that there is a $\delta_0\in(0,\rho_2)$ such that

$$\lambda\,(x-\delta) - f(t,x) \ge 0 \text{ for } t\in[0,T],\ \delta\in(0,\delta_0),\ \lambda\ge\lambda_0,\ t\in[0,T] \text{ and } x\in[\delta,\infty). \tag{3.8}$$

First, recall that by (3.5) and (1.17) we respectively have

$$-\,f(t,x) > 0 \ \text{ for } t\in[0,T] \ \text{ and } x\in(0,x_0), \quad \text{where} \ \ x_0 = \left(\frac{r_*}{s^*}\right)^{1/(\beta-\alpha)} \tag{3.9}$$

and

$$\beta-\alpha=1-\beta=\mu,\ 1-\alpha=2\,\mu \quad\text{and}\quad x_0=\left(\frac{r_*}{s^*}\right)^{1/\mu}, \tag{3.10}$$

Furthermore, for $x>0$ and $\lambda>0$ denote

$$\psi(x,\lambda):=\lambda\,x-s^*\,x^{1-\mu}+r_*\,x^{1-2\,\mu}$$

and

$$\varphi\,(x,\lambda):=\lambda\,x^{2\mu}-s^*\,x^{\mu}+r_*\,.$$

Then

$$\left.\begin{array}{c}\psi(x,\lambda)=x^{1-2\,\mu}\,\varphi\,(x,\lambda)\quad\text{and}\quad\lambda\,x-f(t,x)\geq\psi(x,\lambda)\\ \text{for } t\in[0,T],\ x>0 \text{ and } \lambda>0\end{array}\right\} \tag{3.11}$$

and (cf. (3.9))

$$\psi(\delta,\lambda)\geq\lambda\,\delta\quad\text{if and only if}\quad\delta\in[0,x_0]. \tag{3.12}$$

In particular, $\lambda\,x-f(t,x)>0$ whenever $\varphi\,(x,\lambda)>0$. Moreover, if

$$\lambda>\lambda_0\,, \tag{3.13}$$

then $\varphi\,(x,\lambda)>0$ and, hence, also $\lambda\,x-f(t,x)>0$ holds for all $t\in[0,T]$ and $x>0$. Furthermore,

$$\frac{\partial}{\partial\,x}\,\psi(x,\lambda)=x^{-2\mu}\left(\lambda\,x^{2\,\mu}-s^*\,(1-\mu)\,x^{\mu}+r_*\,(1-2\,\mu)\right) \tag{3.14}$$

and it is easy to check that

$$\frac{\partial}{\partial\,x}\,\psi(x,\lambda)\geq0\quad\text{for}\quad x>0\quad\text{and}\quad\lambda\geq\lambda_1:=\frac{(s^*)^2\,(1-\mu)^2}{4\,r_*\,(1-2\,\mu)}\,.$$

(Notice that $\lambda_1=\lambda_0\,\dfrac{(1-\mu)^2}{1-2\,\mu}>\lambda_0$.) Therefore, for any $\lambda\geq\lambda_1$, the function $\psi(\cdot,\lambda)$ is nondecreasing on $[0,\infty)$. So, with respect to (3.11) and (3.12), we can see that

$$\left.\begin{array}{l}\lambda\,(x-\delta)-f(t,x)\geq0\\ \quad\text{holds for}\quad\lambda\geq\lambda_1,\ \delta\in(0,x_0),\ x\in[\delta,\infty)\ \text{ and }\ t\in[0,T]\,.\end{array}\right\} \tag{3.15}$$

Next, assume that

$$\lambda\in(\lambda_0,\lambda_1)\,. \tag{3.16}$$

Since $0<\mu<\frac{1}{2}$ (cf. Remark 1.13), it follows that

$$(s^*)^2\,(1-2\,\mu)<4\,r_*\lambda\,(1-2\,\mu)<(s^*)^2\,(1-\mu^2)\,. \tag{3.17}$$

In particular,

$$\Delta := (s^*)^2\,(1-\mu)^2 - 4\,\lambda\,r_*\,(1-2\,\mu) > 0 \tag{3.18}$$

and

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\,\psi(x,\lambda) = 0 \quad \text{if and only if} \\ &x = \xi_1 := \left(\frac{s^*(1-\mu)-\sqrt{\Delta}}{2\,\lambda}\right)^{1/\mu} \quad \text{or} \quad x = \xi_2 := \left(\frac{s^*(1-\mu)+\sqrt{\Delta}}{2\,\lambda}\right)^{1/\mu}. \end{aligned}\right\} \tag{3.19}$$

It is easy to see that

$$\frac{\partial}{\partial x}\,\psi(x,\lambda) > 0 \;\; \text{for} \;\; x \in (0,\xi_1) \cup (\xi_2,\infty) \;\; \text{and} \;\; \frac{\partial}{\partial x}\,\psi(x,\lambda) < 0 \;\; \text{for} \;\; x \in (\xi_1,\xi_2)\,,$$

i.e. $x = \xi_1$ is the point of the local maximum of ψ on $(0,\infty)$ and $x = \xi_2$ is the point of the local minimum of ψ on $(0,\infty)$. In particular,

$$\left.\begin{aligned} &\psi(x,\lambda) \ge \psi(\delta,\lambda) \ge \lambda\,\delta \quad \text{for} \;\; \delta \in (0,x_0) \;\; \text{and} \quad x \in (\delta,\xi_1] \\ &\psi(x,\lambda) \ge \psi(\xi_2,\lambda) \quad \text{for} \;\; x \in (\xi_1,\infty)\,. \end{aligned}\right\} \tag{3.20}$$

By (3.14) and (3.19), we have

$$\lambda\,\xi_2^{2\mu} - s^*\,(1-\mu)\,\xi_2^{\mu} + r_*\,(1-2\,\mu) = 0\,. \tag{3.21}$$

Furthermore, the left-hand inequality in (3.17) can be rewritten as

$$(\mu-1)^2\,(s^*)^2 - 4\,\lambda\,r_*\,(1-2\,\mu) < \mu^2\,(s^*)^2\,,$$

wherefrom the inequality

$$\xi_2^{\mu} < 2\,\frac{r_*}{s^*} \tag{3.22}$$

follows (cf. (3.18)). Now, using (3.21) and (3.22), we get

$$\begin{aligned} \varphi\,(\xi_2,\lambda) &= \lambda\,\xi_2^{2\mu} - s^*\,\xi_2^{\mu} + r_* - \lambda\,\xi_2^{2\mu} + s^*\,(1-\mu)\,\xi_2^{\mu} - r_*\,(1-2\,\mu) \\ &= \mu\,(2\,r_* - s^*\,\xi_2^{\mu}) > 0\,, \end{aligned}$$

i.e. $\psi(\xi_2,\lambda) > 0$. In particular, the assertion

$$\psi(x,\lambda) \ge \lambda\,\delta \quad \text{holds for} \;\; \delta \in (0,\delta_0),\;\; \lambda \in (\lambda_0,\lambda_1) \;\; \text{and} \;\; x \in [\delta,\infty) \tag{3.23}$$

is true with $\delta_0 = \min\{x_0, \dfrac{\psi}{\lambda}\}$.

Now, (3.15) and (3.23) complete the proof of (3.8).

Step 3. Let λ_0 and δ_0 be given by the previous step and let an arbitrary $\delta \in (0, \delta_0)$ be given. Let λ^* be given by (3.1) and assume that $\lambda^* > \lambda_0$ (i.e. a is large enough or T is small enough). Define

$$\widetilde{f}(t,x) = \begin{cases} f(t,\delta) + \lambda^* (x-\delta) & \text{for } \ x < \delta, \\ f(t,x) & \text{for } \ x \geq \delta \end{cases}$$

and consider the auxiliary problem

$$x''(t) + a\,x'(t) + \widetilde{f}(t,x) = 0, \quad x(0) = x(T),\ x'(0) = x'(T). \tag{3.24}$$

Since $\delta \in (0, \delta_0)$ and δ_0 was chosen in Step 2 so that $0 < \delta_0 < \rho_2$, the functions σ_1 and σ_2 are respectively a lower and an upper function of problem (3.24) and the function $\widetilde{f}$ evidently satisfies the assumptions of Lemma 2.2 for f. Consequently, (3.24) has a solution x such that $\rho_2 \leq x(t_1) \leq \rho_1$ for some $t_1 \in [0,T]$.

Step 4. Finally, we will show that, under the assumptions of Step 3, the inequality $x \geq \delta$ on $[0,T]$ holds for any solution x of (3.24). This, of course, will imply that x is a positive solution of the given problem (1.15), (1.16).

Let x be an arbitrary solution to (3.24). Put $u(t) = x(t) - \delta$ for $t \in [0,T]$. Then u clearly satisfies the periodic conditions (1.7) and, in addition,

$$u''(t) + a\,u'(t) + \lambda^*\,u(t) = \lambda^*\,(x(t)-\delta) - \widetilde{f}(t,x(t)) \quad \text{for } \ t \in [0,T] \ \text{ and } x \in \mathbb{R}\,.$$

By Step 2, the right hand side $h(t) := \lambda^*\,(x(t)-\delta) - \widetilde{f}(t,x(t))$ is nonnegative for all $t \in [0,T]$. Consequently, by Lemma 2.4, u is nonnegative on $[0,T]$, i.e. $x \geq \delta$ on $[0,T]$. □

3.1 Remark. By an analogous reasoning we could show that if condition (1.17) is replaced by

$$r(t) = \frac{e(t)}{\mu}, \quad s(t) = \frac{c}{\mu},\ 0 < \alpha < \beta < 1\,, \tag{3.25}$$

then problem (2.2), with f given by (3.4), has at least one positive solution whenever a^2 is large enough or T is small enough. However, to establish an explicit bound for λ_0 like that provided by (3.7), we would need a (small as possible) upper estimate for the roots of the algebraic equation

$$\lambda\,x^{1-\alpha} - s^*\,\beta\,x^{\beta-\alpha} + r_*\,\alpha = 0\,.$$

4 Proof of Theorem 1.10

Assume that assumptions of Theorem 1.10 are satisfied. We are going to prove that then problem (1.6), (1.7) has at least one asymptotically stable positive solution.

With respect to Lemma 1.12 it suffices to show that problem (1.15), (1.16) has at least one asymptotically stable positive solution provided the relations $a > 0$, $r_* > 0$, $s_* > 0$, $0 < \alpha < \beta < 1$ and

$$\beta\, s^* \left(\frac{s^*}{r_*}\right)^{(1-\beta)/(\beta-\alpha)} - \alpha\, r_* \left(\frac{s_*}{r^*}\right)^{(1-\alpha)/(\beta-\alpha)} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}\,, \tag{4.1}$$

and

$$\frac{\alpha}{\beta}\,\frac{r^*}{s_*} < \frac{r_*}{s^*} \tag{4.2}$$

are satisfied. (Notice that, by Notation 1.1 and definitions (1.17), conditions (1.13), (1.14) and (4.1), (4.2) are equivalent.)

Step 1. Recall (see Step 1 in Section 3) that problem (1.15), (1.16) possesses a reversely ordered pair of lower and upper functions. Furthermore, put

$$\widetilde{\sigma}_1 = \left(\frac{r^*}{s_*}\right)^{1/(\beta-\alpha)} \quad \text{and} \quad \widetilde{\sigma}_2 = \left(\frac{r_*}{s^*}\right)^{1/(\beta-\alpha)}. \tag{4.3}$$

Then, for each $\eta > 0$, we can choose a constant strict lower function σ_1 of (1.15), (1.16) and a constant strict upper function σ_2 of (1.15), (1.16) in such a way that

$$0 < \widetilde{\sigma}_2 - \eta < \sigma_2 < \widetilde{\sigma}_2 \le \widetilde{\sigma}_1 < \sigma_1 < \widetilde{\sigma}_1 + \eta. \tag{4.4}$$

Step 2. Due to (4.2), we can choose $\eta > 0$ and σ_2 in such a way that the relations (4.4) and

$$\frac{\alpha}{\beta}\,\frac{r^*}{s_*} < \frac{r_*}{s^*} - \eta = \widetilde{\sigma}_2 - \eta < \sigma_2 \tag{4.5}$$

are satisfied. On the other hand, we have

$$\frac{\partial}{\partial x} f(t,x) \ge \beta\, s_*\, x^{\beta-1} - \alpha\, r^*\, x^{\alpha-1} \quad \text{for all} \;\; x > 0 \text{ and all } \;\; t \in [0,T]\,.$$

This means that $\dfrac{\partial}{\partial x} f(t,x) > 0$ holds for all $t \in [0,T]$ whenever $x > \left(\dfrac{\alpha\, r^*}{\beta\, s_*}\right)^{1/(\beta-\alpha)}$. In other words,

$$\frac{\partial}{\partial x} f(t,x) > 0 \quad \text{for all} \;\; x \ge \sigma_2\,.$$

In particular, with such a choice of σ_1, σ_2, condition (2.7) of Lemma 2.5 is satisfied.

Step 3. For each $\varepsilon > 0$ we can find $\eta > 0$ such that the relations (4.4), (4.5) and

$$\beta\, s^*\, (\widetilde{\sigma}_2 - \eta)^{\beta-1} - \alpha\, r_*\, (\widetilde{\sigma}_1 + \eta)^{\alpha-1} \le \beta\, s^*\, \widetilde{\sigma}_2^{\beta-1} - \alpha\, r_*\, \widetilde{\sigma}_1^{\alpha-1} + \varepsilon$$

are satisfied.

For each $t\in[0,T]$, $x\in[\sigma_2,\sigma_1]$ and $\eta>0$ we have

$$\begin{aligned}\frac{\partial}{\partial x}f(t,x) &= \beta\, s(t)\, x^{\beta-1}-\alpha\, r(t)\, x^{\alpha-1}\le \beta\, s^*\, \sigma_2^{\beta-1}-\alpha\, r_*\, \sigma_1^{\alpha-1}\\ &\le \beta\, s^*\,(\widetilde{\sigma}_2-\eta)^{\beta-1}-\alpha\, r_*\,(\widetilde{\sigma}_1+\eta)^{\alpha-1}\,.\end{aligned}$$

Thanks to the continuity of the functions $x^{\beta-1}$ and $x^{\alpha-1}$ on $(0,\infty)$, it follows easily by (4.1) and (4.3) that we can choose σ_1, σ_2 in such a way that also condition (2.6) of Lemma 2.5 is satisfied.

STEP 4. To summarize, problem (1.15), (1.16) has at least one asymptotically stable positive solution by Lemma 2.5. □

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