FORCING

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1. The axioms of ZFC

We recall the list of axioms of ZF and the Axiom of Choice (AC) which we will deal with.

A1: Extensionality.

$$\forall\, x_1, x_2 (\forall\, y\ (y \in x_1 \iff y \in x_2)) \implies x_1 = x_2).$$

A2: Empty Set.

$$\exists\, x\, \forall\, y\, \neg(y \in x).$$

The set x satisfying this axiom is unique by A1 and will be denoted by $\emptyset$.

A3: Pairing.

$$\forall\, x, y\, \exists\, z\, \forall\, t\ (t \in z \iff (t = x \vee t = y)).$$

A4: Union.

$$\forall\, a\, \exists\, b\, \forall\, t\ (t \in b \iff \exists\, x\ (t \in x \wedge x \in b)).$$

A5: Power Set.

$$\forall\, a\, \exists\, b\, \forall\, t\ (t \in b \iff t \subset a),$$

where $t \subset a$ means $\forall\, s \in t\ (s \in a)$.

A6: Infinity.

$$\exists\, x\ (\emptyset \in x \wedge \forall\, y \in x\ (y \cup \{y\} \in x)).$$

A7: Regularity.

$$\forall\, x\ (x \neq \emptyset \implies \exists\, y \in x\, \neg(\exists\, t\ (t \in y \wedge t \in x))).$$

A8: Comprehension Axiom Scheme. If $\varphi(x, y_1, \ldots, y_n)$ is a formula with all free variables shown, then the following is an axiom.

$$\forall\, a\, \forall\, s_1, \ldots, s_n\, \exists\, b\, \forall\, x\ (x \in b \iff x \in a \wedge \varphi(x, s_1, \ldots, s_n)).$$

A9: Replacement Axiom Scheme. If $\varphi(x, y, t_1, \ldots, t_n)$ is a formula with all free variables shown, then the following is an axiom.

$$\forall\, a\, \forall\, s_1, \ldots, s_n\, \Big((\forall\, x \in a\, \exists!\, y\ \varphi(x, y, s_1, \ldots, s_n)) \implies \\ \implies \exists\, b\ (\forall\, x \in a\, \exists\, y \in b\ \varphi(x, y, s_1, \ldots, s_n))\Big).$$

AC: The Axiom of Choice.

$$\forall\, x\, \Big((\forall\, y \in x\ (y \neq \emptyset) \wedge \forall\, y_1, y_2 \in x\ (y_1 \neq y_2 \implies y_1 \cap y_2 = \emptyset)) \implies \\ \implies \exists\, z\, \forall\, y \in x\, \exists!\, t\ (t \in z \wedge t \in y)\Big).$$

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2. Some cardinal arithmetic

Proposition 2.1. *If κ, λ are infinite cardinals, $\kappa > 1$ and $\lambda \geqslant \kappa + \omega$ then $\kappa^\lambda = 2^\lambda$.*

Proof. We have $2^\lambda \leqslant \kappa^\lambda \leqslant \lambda^\lambda \leqslant (2^\lambda)^\lambda = 2^\lambda$. □

If $\{\kappa_i\}_{i\in I}$ is a collection of cardinals then we define $\sum_{i\in I} \kappa_i$ as the cardinality of $\bigcup_{i\in I} \kappa_i \times \{i\}$ and $\prod_{i\in I} \kappa_i$ as the cardinality of the product of κ_i's.

Proposition 2.2. *If $\beta > 0$ is a limit ordinal and $\{\lambda_\alpha\}_{\alpha<\beta}$ is a strictly increasing sequence of cardinals then $\sum_{\alpha<\beta} \lambda_\alpha = \sup_{\alpha<\beta} \lambda_\alpha$.*

Proof. Let $\kappa = \sup_{\alpha<\beta} \lambda_\alpha$. We have $\kappa \leqslant \sum_{\alpha<\beta} \lambda_\alpha \leqslant \sum_{\alpha<\beta} \kappa = \kappa|\beta|$. By induction we can show that $\lambda_\alpha \geqslant \alpha$ for every $\alpha < \beta$, since our sequence is strictly increasing. Hence $\beta \leqslant \kappa$ and $\kappa|\beta| = \kappa$. □

Theorem 2.3 (König)**.** *Let $\{\lambda_i\}_{i\in I}$ and $\{\kappa_i\}_{i\in I}$ be two collections of cardinals such that $\lambda_i < \kappa_i$ for every $i \in I$. Then*

$$\sum_{i\in I} \lambda_i < \prod_{i\in I} \kappa_i.$$

Proof. Let $\varphi\colon \bigcup_{i\in I} \lambda_i \times \{i\} \to \prod_{i\in I} \kappa_i$ be a map. We show that φ is not onto. Set $A_i = \varphi[\lambda_i \times \{i\}]$, $B_i = \{f(i)\colon f \in A_i\}$. Then $B_i \in [\kappa_i]^{\leqslant\lambda_i}$ so there is $x_i \in \kappa_i \setminus B_i$, since $\lambda_i < \kappa_i$. Let $h \in \prod_{i\in I} \kappa_i$ be defined as $h(i) = x_i$, $i \in I$. Observe that $h \notin \bigcup_{i\in I} A_i$. Thus $\operatorname{rng}(\varphi) \neq \prod_{i\in I} \kappa_i$. □

Corollary 2.4. *For every infinite cardinal κ, $\operatorname{cf}(2^\kappa) > \kappa$.*

Proof. Applying König's theorem for $I = \kappa$, $\lambda_i = 1, \kappa_i = 2$ we get $\kappa < 2^\kappa$. Suppose $\operatorname{cf}(2^\kappa) \leqslant \kappa$ and let $\{\lambda_\alpha\}_{\alpha<\kappa}$ be a sequence of cardinals such that $\lambda_\alpha < 2^\kappa$ and $\sup_{\alpha<\kappa} \lambda_\alpha = 2^\kappa$. Applying König's theorem once more for $I = \kappa$ and $\kappa_\alpha = 2^\kappa$, we obtain $2^\kappa = \sum_{\alpha<\kappa} \lambda_\alpha < (2^\kappa)^\kappa = 2^\kappa$, a contradiction. □

Corollary 2.5 (König)**.** *For every infinite cardinal κ we have $\kappa^{\operatorname{cf}(\kappa)} > \kappa$.*

Proof. If $\kappa = \operatorname{cf}(\kappa)$ then $\kappa^{\operatorname{cf}(\kappa)} = \kappa^\kappa = 2^\kappa > \kappa$. Suppose that $\operatorname{cf}(\kappa) < \kappa$ and fix a strictly increasing sequence of cardinals $\{\lambda_\alpha\}_{\alpha<\operatorname{cf}(\kappa)}$ with $\sup_{\alpha<\operatorname{cf}(\kappa)} = \kappa$. Applying König's theorem we get $\kappa = \sum_{\alpha<\operatorname{cf}(\kappa)} \lambda_\alpha < \kappa^{\operatorname{cf}(\kappa)}$. □

Theorem 2.6 (Hausdorff)**.** *If κ, λ are such cardinals that $\kappa > 1, \lambda > 0$ and $\kappa + \lambda$ is infinite then $(\kappa^+)^\lambda = \kappa^+\kappa^\lambda$.*

Proof. Suppose first that $\lambda \geqslant \kappa^+$. Then λ is infinite and, by Proposition 2.1, we get $(\kappa^+)^\lambda = 2^\lambda = \kappa^\lambda = \kappa^+\kappa^\lambda$. Suppose now that $\lambda < \kappa^+$. Then κ is infinite. Observe that if $f \in (\kappa^+)^\lambda$ then $\operatorname{rng}(f)$ is bounded in κ^+ so there is $\alpha < \kappa^+$ such that $f \in \alpha^\lambda$. Hence

$$(\kappa^+)^\lambda \leqslant \Big|\bigcup_{\alpha<\kappa^+} \alpha^\lambda\Big| \leqslant \kappa^+\kappa^\lambda.$$

The reverse inequality also holds, since $\lambda > 0$. □

Theorem 2.7. *Assume GCH. If λ, κ are infinite cardinals then*

$$\kappa^\lambda = \begin{cases} \kappa & \text{if } \lambda < \mathrm{cf}(\kappa), \\ \kappa^+ & \text{if } \mathrm{cf}(\kappa) \leqslant \lambda \leqslant \kappa, \\ \lambda^+ & \text{if } \kappa < \lambda. \end{cases}$$

Proof. If $\kappa < \lambda$ then $\lambda^+ = 2^\lambda \leqslant \kappa^\lambda \leqslant (2^\kappa)^\lambda = \lambda^+$. If $\mathrm{cf}(\kappa) \leqslant \lambda \leqslant \kappa$ then $\kappa < \kappa^{\mathrm{cf}(\kappa)} \leqslant \kappa^\lambda \leqslant (2^\kappa)^\lambda = \kappa^+$ so $\kappa^\lambda = \kappa^+$. Suppose that $\lambda < \mathrm{cf}(\kappa)$. If $\kappa = \delta^+$ then $\kappa^\lambda = (2^\delta)^\lambda = 2^\delta = \kappa$. Suppose now that κ is a limit cardinal. There exists an increasing sequence of cardinals $\{\kappa_\alpha\}_{\alpha<\mathrm{cf}(\kappa)}$ with $\lambda < \kappa_0$ and $\sup_{\alpha<\mathrm{cf}(\kappa)} \kappa_\alpha = \kappa$. We have

$$\kappa^\lambda = \Big| \bigcup_{\alpha<\mathrm{cf}(\kappa)} (\kappa_\alpha^+)^\lambda \Big| = \sum_{\alpha<\mathrm{cf}(\kappa)} (\kappa_\alpha^+)^\lambda = \sum_{\alpha<\mathrm{cf}(\kappa)} \kappa_\alpha^+ = \kappa.$$

This completes the proof. □

3. Partial orders

By a *partially ordered set* (or a *poset*) we mean a triple $\mathbb{P} = (P, \leqslant, 1_\mathbb{P})$ where $\leqslant$ is a partial order on a set P and $p \leqslant 1_\mathbb{P}$ holds for every $p \in P$. We consider partial orders with greatest elements for the sake of convenience only. We write $p \perp q$ whenever $p, q \in P$ are *incompatible*, i.e. there is no $r \in P$ with $r \leqslant p$ and $r \leqslant q$. We write $p \parallel q$ whenever p, q are *compatible*, i.e. $\neg(p \perp q)$. A subset $D \subset P$ is *dense* in $\mathbb{P}$ provided for every $p \in P$ there is $d \in D$ with $d \leqslant p$. A subset $F \subset P$ is a *filter* if

(1) $p, q \in F \implies (\exists\, r \in F)\, r \leqslant p \;\&\; r \leqslant q$,
(2) $p \in F \;\&\; q \geqslant p \implies q \in F$.

When we apply these notions for Boolean algebras, we consider the set of all positive elements; for instance elements a, b in a Boolean algebra $\mathbb{B}$ are compatible iff $a \cdot b > 0_\mathbb{B}$.

Fix a partially ordered set $\mathbb{P} = (P, \leqslant, 1_\mathbb{P})$. Define *the left topology* on P as the topology generated by all sets of the form $(p] = \{x \in P \colon x \leqslant p\}$, where $p \in P$. Observe that $(p]$ is the smallest neighborhood of p with respect to this topology. Let $\mathrm{RO}(\mathbb{P})$ denote the Boolean algebra of regular open subsets of P with respect to the left topology. Define $i_\mathbb{P} \colon P \to \mathrm{RO}(\mathbb{P})$ by setting $i_\mathbb{P}(p) = \mathrm{int}\,\mathrm{cl}(p]$. We have the following easy fact.

Proposition 3.1. *Let $\mathbb{P}$ be a partial order and let $\mathbb{B} = \mathrm{RO}(\mathbb{P})$. The map $i = i_\mathbb{P} \colon \mathbb{P} \to \mathbb{B}$ has the following properties:*

(1) *i is order preserving.*
(2) *$i[P]$ is dense in $\mathbb{B}^+$ and $i(1_\mathbb{P}) = 1_\mathbb{B}$.*
(3) *If $p, q \in P$ and $p \perp q$ then $i(p) \perp i(q)$.*

The map $i_\mathbb{P}$ will be referred to as the *canonical order preserving map*. The algebra $\mathrm{RO}(\mathbb{P})$ is called the *completion of* $\mathbb{P}$. The next theorem says that Proposition 3.1 characterizes the completion of a poset.

Theorem 3.2. *For any complete Boolean algebra $\mathbb{B}$ and an order preserving and $\perp$-preserving map $f \colon \mathbb{P} \to \mathbb{B}^+$ such that $f[\mathbb{P}]$ is dense in $\mathbb{B}$ and $f(1_\mathbb{P}) = 1_\mathbb{B}$, there exists a unique complete Boolean isomorphism $h \colon \mathrm{RO}(\mathbb{P}) \to \mathbb{B}$ such that $h \circ i_\mathbb{P} = f$.*

Proof. Set $i = i_{\mathbb{P}}$. Define $h\colon \operatorname{RO}(\mathbb{P}) \to \mathbb{B}$ and $h^*\colon \mathbb{B} \to \operatorname{RO}(\mathbb{P})$ by setting

$$h(a) = \sum\nolimits^{\mathbb{B}} \{f(p)\colon p \in \mathbb{P} \ \&\ i(p) \leqslant a\}$$
$$h^*(b) = \sum\nolimits^{\operatorname{RO}(\mathbb{P})} \{i(p)\colon p \in \mathbb{P} \ \&\ f(p) \leqslant b\}.$$

Clearly, h and h^* are order preserving and $h \circ i = f$. We show that $h \circ h^* = \mathrm{id}_{\mathbb{B}}$ and $h^* \circ h = \mathrm{id}_{\operatorname{RO}(\mathbb{P})}$ which implies that h is an isomorphism of partial orders and therefore it is a complete Boolean isomorphism.

Fix $a \in \operatorname{RO}(\mathbb{P})$ and consider $a' = h^*(h(a))$. If $i(p) \leqslant a$ then $f(p) \leqslant h(a)$ and hence $i(p) \leqslant a'$. This shows that $a \leqslant a'$. Suppose $a' \cdot \neg a > 0$ and let $p \in \mathbb{P}$ be such that $i(p) \leqslant a'$ and $i(p) \cdot a = 0$. By the definition of h^*, there is $q \in \mathbb{P}$ with $f(q) \leqslant h(a)$ and $i(p) \cdot i(q) > 0$. Let $r \in \mathbb{P}$ be below p and q. Now $f(r) \leqslant h(a)$ and, by the definition of h, there is $q' \in \mathbb{P}$ with $i(q') \leqslant a$ and $f(r) \cdot f(q') > 0$. Let $r' \in \mathbb{P}$ be below r and q'. Then $i(r') \leqslant i(q') \leqslant a$ and $i(r') \cdot a \leqslant i(p) \cdot a = 0$, a contradiction. This shows that $a = a'$.

Thus we have proved that $h^* \circ h = \mathrm{id}_{\operatorname{RO}(\mathbb{P})}$. By the same arguments, $h \circ h^* = \mathrm{id}_{\mathbb{B}}$. □

A partially ordered set $\mathbb{P}$ is *separative* if

$$\forall\, x, y \in \mathbb{P}(\neg(x \leqslant y) \implies \exists z \leqslant x\ (z \perp y)).$$

Theorem 3.3. *For a partially ordered set $\mathbb{P}$ the following are equivalent:*

(a) *$\mathbb{P}$ is separative.*
(b) *The map $i_{\mathbb{P}}\colon \mathbb{P} \to i_{\mathbb{P}}[\mathbb{P}]$ is an order isomorphism and $(p] \in \operatorname{RO}(\mathbb{P})$ for every $p \in \mathbb{P}$.*
(c) *There exists a complete Boolean algebra $\mathbb{B}$ and an order preserving embedding $i\colon \mathbb{P} \to \mathbb{B}$ such that $i[\mathbb{P}]$ is dense in $\mathbb{B}$.*

Proof. Implication (c) $\implies$ (a) is trivial and (b) $\implies$ (c) follows from Proposition 3.1. It remains to show (a) $\implies$ (b).

We first check that $(p] \in \operatorname{RO}(\mathbb{P})$. Clearly $(p] \subset \operatorname{int}\operatorname{cl}(p]$. Let $q \in \operatorname{int}\operatorname{cl}(p]$. Then $(q] \subset \operatorname{cl}(p]$ which implies that $(r] \cap (p] \neq \emptyset$ whenever $r \in (q]$. In other words, $r \parallel p$ whenever $r \leqslant q$. By the fact that $\mathbb{P}$ is separative, we deduce that $q \leqslant p$ which means $q \in (p]$.

Now, if $p, q \in \mathbb{P}$ and $\neg(p \leqslant q)$ then $i_{\mathbb{P}}(p) = (p] \not\subset (q] = i_{\mathbb{P}}(q)$. It follows that $i_{\mathbb{P}}$ is an order isomorphism. □

4. Generic filters

A filter G on a partially ordered set $\mathbb{P}$ is $\mathbb{P}$-*generic over M* if for any set $D \in M$ which is dense in $\mathbb{P}$ we have $G \cap D \neq \emptyset$. Usually, M will be a fixed countable transitive model of ZFC (called the *ground model*). The next lemma says that in this case a generic filter over M exists.

Lemma 4.1 (Rasiowa-Sikorski)**.** *Let M be a countable set and let $\mathbb{P}$ be a poset. Then for every $p \in \mathbb{P}$ there exists a $\mathbb{P}$-generic filter G over M with $p \in G$.*

Proof. Enumerate as $\{D_n\}_{n\in\omega}$ the collection of all dense sets from M. Define inductively $p_n \in \mathbb{P}$ such that $p_0 = p$, $p_{n+1} \leqslant p_n$ and $p_{n+1} \in D_n$. Now let $G = \{p \in \mathbb{P}\colon (\exists\, n \in \omega)\ p_n \leqslant p\}$. Clearly, G is a filter and $G \cap D_n \neq \emptyset$ for every $n \in \omega$. □

Here we give some basic facts about generic filters. We always assume that M denotes a fixed countable transitive model (briefly *ctm*) of ZFC.

Proposition 4.2. *Let G be a $\mathbb{P}$-generic filter over a ZFC model M and assume that $H \subset M$ is a subset of $\mathbb{P}$ containing G and consisting of pairwise compatible elements. Then $G = H$.*

Proof. Fix a $q \in H$. Define $D = \{p \in \mathbb{P} \colon p \leqslant q \vee p \perp q\}$. Observe that D is dense and $D \in M$. Thus there exists $p \in G \cap D$ which means $p \leqslant q$ and consequently $q \in G$. □

Proposition 4.3. *Let $\mathbb{P}$ be a poset in the ground model M and assume that $G \subset \mathbb{P}$ consists of pairwise compatible elements and meets every set from M which is dense in $\mathbb{P}$. Then G is a generic filter.*

Proof. We have to show that G is a filter. Fix $p, q \in G$ and consider

$$D = \{r \in \mathbb{P} \colon (r \leqslant p \,\&\, r \leqslant q) \vee (r \perp p) \vee (r \perp q)\}.$$

If $x \in \mathbb{P}$ and $x \notin D$ then $x \parallel p$ so there is $r_1 \leqslant x$ with $r_1 \leqslant p$. If $r_1 \notin D$ then $r_1 \parallel q$ so there is $r_2 \leqslant r_1$ with $r_2 \leqslant q$. Thus $r_2 \in D$. It follows that D is dense in $\mathbb{P}$. Clearly, $D \in M$. If $r \in D \cap G$ then r is below p and q, since any two elements of G are compatible. Hence G is a filter. □

Proposition 4.4. *Let $\mathbb{B}$ be a complete Boolean algebra in the ground model M and let G be a $\mathbb{B}$-generic filter over M. Then G is an ultrafilter and for each $S \in \mathcal{P}^M(\mathbb{B})$ the following holds:*

(i) *If $\sum S \in G$ then there is $p \in S$ with $p \in G$.*
(ii) *If $S \subset G$ then $\prod S \in G$.*

Proof. Clearly, G is a filter. Fix $p \in \mathbb{B}$ and consider

$$D_p = \{x \in \mathbb{B} \colon \text{either } x \leqslant p \text{ or } x \leqslant \neg p\}.$$

Then D_p is dense and in M, so $G \cap D \neq \emptyset$. Hence either $p \in G$ or $\neg p \in G$. Thus G is an ultrafilter.

For the proof of (i), consider the set $D = \{x \in \mathbb{B} \colon (\exists\, q \in S)\; x \leqslant q\}$. Clearly, $D \in M$ and D is dense below $\sum S$. Thus $D \cap G \neq \emptyset$, that is $p \in G$ for some $p \in S$. Statement (ii) follows from (i) since G is an ultrafilter. □

Theorem 4.5. *Let $\mathbb{P}$ be a poset in the ground model M and let G be a $\mathbb{P}$-generic filter over M. Consider the canonical order preserving map $i = i_{\mathbb{P}} \colon \mathbb{P} \to \mathrm{RO}(\mathbb{P})$. Then*

$$\overline{G} = \{b \in \mathrm{RO}(\mathbb{P}) \colon (\exists\, p \in G)\; i(p) \leqslant b\}$$

is an $\mathrm{RO}(\mathbb{P})$-generic filter over M. Conversely, if G is $\mathrm{RO}(\mathbb{P})$-generic then $i^{-1}[G]$ is $\mathbb{P}$-generic over M.

Proof. It is easy to see that $\overline{G}$ is a filter. Let $D \in M$ be dense in $\mathrm{RO}(\mathbb{P})$. Define

$$E = \{p \in \mathbb{P} \colon (\exists\, d \in D)\; i(p) \leqslant d\}.$$

Clearly $E \in M$. We check that E is dense in $\mathbb{P}$. Fix $q \in \mathbb{P}$. As D is dense in $\mathrm{RO}(\mathbb{P})$, there is $d \in D$ with $d \leqslant i(q)$. Furthermore, there is $q' \in \mathbb{P}$ with $i(q') \leqslant d$, since $i[\mathbb{P}]$ is dense in $\mathrm{RO}(\mathbb{P})$. Now observe that $q \parallel q'$ by Proposition 3.1(2). Let $p \in \mathbb{P}$ be below q, q'. Then $i(p) \leqslant i(q') \leqslant d$ so $p \in E$ and $p \leqslant q$. Thus E is dense in $\mathbb{P}$. Let $p \in E \cap G$. Then $i(p) \leqslant d$ for some $d \in D$ which implies that $d \in D \cap \overline{G}$. Thus we have shown that $\overline{G}$ intersects all sets from M which are dense in $\mathrm{RO}(\mathbb{P})$.

For the reverse statement, consider an $\mathrm{RO}(\mathbb{P})$-generic filter G. Let $D \in M$ be dense in $\mathbb{P}$. Then $i[D]$ is dense in $\mathrm{RO}(\mathbb{P})$ so $G \cap i[D] \neq \emptyset$ which means that $i^{-1}[G] \cap D \neq \emptyset$. Now observe that $i^{-1}[G]$ consists of pairwise compatible elements. By Proposition 4.3, $i^{-1}[G]$ is a generic filter. □

An *antichain* in $\mathbb{P}$ is a subset of $\mathbb{P}$ consisting of pairwise compatible elements. By the Kuratowski-Zorn Lemma, every antichain is contained in a maximal antichain. Note that a maximal antichain in a dense subset of $\mathbb{P}$ is also a maximal antichain in $\mathbb{P}$. Generic filters can be defined as those filters which intersect all maximal antichains from the ground model.

Proposition 4.6. *Let $\mathbb{P}$ be a poset in a ctm M. A set $G \subset \mathbb{P}$ consisting of pairwise compatible elements is a $\mathbb{P}$-generic filter over M iff G intersects all maximal antichains in $\mathbb{P}$ which are in M.*

Proof. Let G be $\mathbb{P}$-generic over M and fix an antichain $A \subset \mathbb{P}$ with $A \in M$. Define $D = \{x \in \mathbb{P} \colon (\exists\, a \in A)\ x \leqslant a\}$. By the maximality of A, $D \in M$ and D is dense, so $D \cap G \neq \emptyset$. Thus also $A \cap G \neq \emptyset$.

Now assume that G consists of pairwise compatible elements and G intersects all maximal antichains in $\mathbb{P}$ which are in M. Fix a dense set $D \subset \mathbb{P}$ with $D \in M$. Applying the Kuratowski-Zorn Lemma in M, we can find a maximal antichain $A \subset D$ which is also a maximal antichain in $\mathbb{P}$. Thus $A \cap G \neq \emptyset$. By Proposition 4.3, G is a $\mathbb{P}$-generic filter. □

5. Complete embeddings

In this section we discuss the relationship between embeddings of posets and their completions, using the results on generic filters.

Let $\mathbb{P}$, $\mathbb{Q}$ be two posets. For convenience, we assume that they are separative. A map $f \colon \mathbb{P} \to \mathbb{Q}$ will be called a *complete embedding* if f is order preserving and for every maximal antichain A in $\mathbb{P}$ the image $f[A]$ is a maximal antichain in $\mathbb{Q}$. These properties imply that $f(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$ and $f(p_1) \perp f(p_2)$ whenever $p_1 \perp p_2$. Indeed, $\{1_{\mathbb{P}}\}$ is the unique one-element maximal antichain and every antichain can be extended to a maximal antichain. Let us note that a complete embedding is not nessecarily an embedding, but it is an embedding whenever the domain is a separative poset.

A natural example of a complete embedding is the canonical map $i_{\mathbb{P}} \colon \mathbb{P} \to \operatorname{RO}(\mathbb{P})$. In case where $\mathbb{P}$ is separative, we will identify $p \in \mathbb{P}$ with its image $i_{\mathbb{P}}(p) \in \operatorname{RO}(\mathbb{P})$.

For Boolean algebras, one considers the notion of a *complete homomorphism.* A homomorphism of Boolean algebras $h \colon \mathbb{A} \to \mathbb{B}$ is *complete* if for every set $S \subset \mathbb{A}$ such that $\sum^{\mathbb{A}} S = 1_{\mathbb{A}}$ we have $\sum^{\mathbb{B}} f[S] = 1_{\mathbb{B}}$. Below we show that a complete embedding of posets extends to a complete homomorphism of their completions.

Proposition 5.1. *Let $\mathbb{P}$, $\mathbb{Q}$ be two posets and let $f \colon \mathbb{P} \to \mathbb{Q}$ be an order preserving and $\perp$-preserving map. The following properties are equivalent:*

(a) *For every maximal antichain $A \subset \mathbb{P}$, $f[A]$ is a maximal antichain in $\mathbb{Q}$.*
(b) *For every $\mathbb{Q}$-generic filter G, $f^{-1}[G]$ is a $\mathbb{P}$-generic filter.*
(c) *For every $S \subset \operatorname{RO}(\mathbb{P})$ with $\sum^{\operatorname{RO}(\mathbb{P})} S = 1_{\operatorname{RO}(\mathbb{P})}$ we have $\sum^{\operatorname{RO}(\mathbb{Q})} f[S] = 1_{\operatorname{RO}(\mathbb{Q})}$.*

Proof. (a) $\implies$ (b) By Proposition 4.6.

(b) $\implies$ (c) Fix $S \subset \operatorname{RO}(\mathbb{P})$ with $\sum S = 1_{\operatorname{RO}(\mathbb{P})}$ and suppose that $\sum f[S] < 1_{\operatorname{RO}(\mathbb{Q})}$. Fix $q \in \mathbb{Q}$ such that $q \cdot \sum f[S] = 0_{\operatorname{RO}(\mathbb{Q})}$. Let G be $\mathbb{Q}$ generic with $q \in G$. By (b), Theorem 4.5 and Proposition 4.4, there exists $p \in \mathbb{P}$ such that $p \in f^{-1}[G]$ and $p \leqslant s$ for some $s \in S$.

Thus $f(p) \in G$ but also $f(p) \perp q$; a contradiction. That f extends to a complete Boolean homomorphism is a standard fact and can be proved using similar arguments like in the proof of Theorem 3.2.

(c) $\implies$ (a) If $A \subset \mathbb{P}$ is a maximal antichain then $\sum^{\mathrm{RO}(\mathbb{P})} A = 1$ so $\sum^{\mathrm{RO}(\mathbb{Q})} f[A] = 1$ and therefore $f[A]$ is a maximal antichain in $\mathbb{Q}$. $\square$

Corollary 5.2. *Let $f\colon \mathbb{P} \to \mathbb{Q}$ be a complete embedding of separative posets. Then the formula*

$$\overline{f}(a) = \sum\nolimits^{\mathrm{RO}(\mathbb{Q})} \{f(p)\colon p \in \mathbb{P} \ \&\ p \leqslant a\}$$

defines a complete Boolean homomorphism $\overline{f}\colon \mathrm{RO}(\mathbb{P}) \to \mathrm{RO}(\mathbb{Q})$ which extends f.

Proof. Clearly, $\overline{f}$ is an order preserving map which extends f. We first show that $\overline{f}(a)\cdot\overline{f}(\neg a) = 0$. Suppose not, take a $\mathbb{Q}$-generic filter G such that some element below $b = \overline{f}(a)\cdot\overline{f}(\neg a)$ is in G. By Proposition 5.1, $f^{-1}[G]$ is $\mathbb{P}$-generic, let H be the filter in $\mathrm{RO}(\mathbb{P})$ generated by $f^{-1}[G]$. Then H is $\mathrm{RO}(\mathbb{P})$-generic by Theorem 4.5. However, we have $a \in H$ and $\neg a \in H$, which is a contradiction.

Now it suffices to show that $\overline{f}(\sum^{\mathrm{RO}(\mathbb{P})} S) = \sum^{\mathrm{RO}(\mathbb{Q})} \overline{f}[S]$ for every $S \subset \mathrm{RO}(\mathbb{P})$. Suppose it is not true and let S be such that $b = f(\sum^{\mathrm{RO}(\mathbb{P})} S) \cdot \neg \sum^{\mathrm{RO}(\mathbb{Q})} \overline{f}[S] > 0$. Let G and H be as before. Then $\sum S \in H$ and therefore some $s \in S$ is in H, so $f(p) \in G$ for some $p \leqslant s$, $p \in \mathbb{P}$. This is a contradiction to the fact that $b \cdot f(p) = 0$. $\square$

6. Generic extensions

Let $\mathbb{P}$ be a poset in the ground model M. Let G be a $\mathbb{P}$-generic filter over M. We define, using $\in$-recursion,

$$\mathrm{val}_G(x) = \{\mathrm{val}_G(t)\colon (\exists\, p \in G)\ (t,p) \in x\}.$$

Observe that $\mathrm{rank}(t) < \mathrm{rank}(x)$ whenever $(t,p) \in x$ for some p. Thus, val_G is well-defined. We will also write x_G instead of val_G (in the literature, there are also used symbols I_G, K_G or Int_G). $\mathrm{val}_G(x)$ is called the *G-interpretation* of x. The set

$$M[G] = \{\mathrm{val}_G(x)\colon x \in M\}.$$

is called the *G-extension* (or a *generic extension*) of M. Observe that $M[G]$ is transitive, by the definition of val_G. Now define in M, using $\in$-recursion,

$$\widehat{x} = \{(\widehat{t}, 1_{\mathbb{P}})\colon t \in x\}.$$

Define also $\Gamma = \{(\widehat{p}, p)\colon p \in \mathbb{P}\}$. Clearly, $\Gamma \in M$. Any $a \in M$ such that $b = \mathrm{val}_G(a)$, is called a *name* for b; $\widehat{x}$ is called the *standard name* for x.

In what follows, we always assume that M is a transitive model of ZFC, G is a $\mathbb{P}$-generic filter over M, where $\mathbb{P}$ is a poset in M.

Proposition 6.1. *For every $x \in M$ we have $\mathrm{val}_G(\widehat{x}) = x$ and $\mathrm{val}_G(\Gamma) = G$. Consequently, $M \subset M[G]$ and $G \in M[G]$.*

Proof. We use $\in$-induction. Clearly $\mathrm{val}_G(\widehat{\emptyset}) = \emptyset$. If $\mathrm{val}_G(\widehat{t}) = t$ whenever $t \in x$ then $\mathrm{val}_G(\widehat{x}) = \{\mathrm{val}_G(\widehat{t})\colon t \in x\} = x$. Now we have $\mathrm{val}_G(\Gamma) = \{\mathrm{val}_G(\widehat{p})\colon p \in G\} = G$. $\square$

Theorem 6.2. *Let N be a transitive model of ZF such that $M \subset N$ and $G \in N$. Then $M[G] \subset N$.*

Proof. Applying $\in$-recursion in N, we can define $\mathrm{val}_G^N(x)$ in the same way as val_G. Now, by $\in$-induction, we show that $\mathrm{val}_G^N(x) = \mathrm{val}_G(x)$ for every $x \in M$, since the formula defining val_G is absolute for transitive sets (it is even a Δ_0 formula). $\square$

Proposition 6.3. *For every $x \in M$, $\mathrm{rank}(\mathrm{val}_G(x)) \leqslant \mathrm{rank}(x)$. In particular, $ON^{M[G]} = ON^M$.*

Proof. The first statement can be proved by easy induction (recall that rank is absolute for transitive models). Clearly $ON^M \subset ON^{M[G]}$ since ordinals are absolute. Suppose $\lambda \in ON^{M[G]}$ is not in M. Then $\mathrm{rank}(\lambda) \leqslant \mathrm{rank}(\overline{\lambda})$ where $\overline{\lambda}$ is a name for λ. This is a contradiction, since $\mathrm{rank}(\lambda) = \lambda$. $\square$

7. Boolean value of a formula

Fix a partially ordered set $\mathbb{P}$ and let $\mathbb{B}$ denote the complete Boolean algebra $\mathrm{RO}(\mathbb{P})$. For each formula of set theory $\varphi(v_1, \dots, v_n)$ with parameters $x_1, \dots, x_n$ we will define its Boolean value $\|\varphi(x_1, \dots, x_n)\|_{\mathbb{P}} \in \mathbb{B}$ which "measures the probability of truth" of the interpretation of this formula in $\mathbb{P}$-generic extensions. Let $i \colon \mathbb{P} \to \mathrm{RO}(\mathbb{P})$ be the canonical order preserving map.

The definition of $\|\varphi\|_{\mathbb{P}}$ proceeds by recursion on the length of the formula and, for atomic formulas, by induction on rank. For atomic formulas $x \in y$ and $x = y$ define

$$\|x \in y\|_{\mathbb{P}} = \sum_{(t,p) \in y, p \in \mathbb{P}} i(p) \cdot \|t = x\|_{\mathbb{P}},$$

and

$$\|x = y\|_{\mathbb{P}} = \prod_{(t,p) \in x, p \in \mathbb{P}} (\neg i(p) + \|t \in y\|_{\mathbb{P}}) \cdot \prod_{(t,p) \in y, p \in \mathbb{P}} (\neg i(p) + \|t \in x\|_{\mathbb{P}})$$

Next, we define

$$\begin{aligned} \|\neg\varphi\|_{\mathbb{P}} &= \neg\|\varphi\|_{\mathbb{P}}, \\ \|\varphi \vee \psi\|_{\mathbb{P}} &= \|\varphi\|_{\mathbb{P}} + \|\psi\|_{\mathbb{P}}, \\ \|\exists\, x\ \varphi\|_{\mathbb{P}} &= \sum \{b \in \mathbb{B} \colon (\exists\, x)\ b = \|\varphi(x)\|_{\mathbb{P}}\}. \end{aligned}$$

The definition of the Boolean value for atomic formulas is in fact recursive with respect to a well-founded set-like relation E on all unordered pairs, namely $a\ E\ b$ iff there are x, y, y' such that $a = \{x, y\}$, $b = \{x, y'\}$ and $\mathrm{rank}(y) < \mathrm{rank}(y')$.

Observe that $\|\emptyset = \emptyset\|_{\mathbb{P}} = 1_{\mathbb{B}}$ and $\|\emptyset \in \emptyset\|_{\mathbb{P}} = 0_{\mathbb{B}}$ and, inductively, $\|x = x\|_{\mathbb{P}} = 1_{\mathbb{B}}$ and $\|x \in x\|_{\mathbb{P}} = 0_{\mathbb{B}}$. We will write $\|\varphi\|$ instead of $\|\varphi\|_{\mathbb{P}}$ whenever it will be clear what poset is under consideration.

8. The Truth Lemma

Let $\mathbb{P}$ be a partially ordered set and let $\varphi(x_1, \dots, x_n)$ be a formula with all free variables shown. Let $t_1, \dots, t_n$ and $p \in \mathbb{P}$ be fixed. We say that p *forces* $\varphi(t_1, \dots, t_n)$ and we write $p \Vdash \varphi(t_1, \dots, t_n)$, provided $i_{\mathbb{P}}(p) \leqslant \|\varphi(t_1, \dots, t_n)\|$, where $i_{\mathbb{P}}$ is the canonical order preserving map. The definition of $\Vdash$ does not mention models. The aim of this section is to show that the relation $\Vdash$ tells us about interpretations of φ in generic extensions. Some authors define the relation of forcing by condition (b) in Corollary 8.2 below; in this case it is very important to show that there is a formula in the ground model which defines the same relation (this is the Definability Lemma).

Proposition 8.1. *Let $\mathbb{P}$ be a poset in a ctm M, $p \in \mathbb{P}$. Then for any formula with parameters φ, $p \Vdash \varphi$ iff for every $\mathbb{P}$-generic filter G with $p \in G$ we have $\|\varphi\| \in \overline{G}$, where $\overline{G}$ is the filter generated by $i_{\mathbb{P}}[G]$ in* $\mathrm{RO}(\mathbb{P})$.

Proof. The "only if" part is trivial. For the "if" part, suppose $p \nVdash \varphi$, i.e. $i_{\mathbb{P}}(p) \cdot \neg\|\varphi\| > 0_{\mathrm{RO}(\mathbb{P})}$. There is $q \leqslant p$ such that $i_{\mathbb{P}}(q) \leqslant \neg\|\varphi\|$. Now, by the theorem of Rasiowa-Sikorski, there is a $\mathbb{P}$-generic filter G with $q \in G$. Finally, $p \in G$ and $\|\varphi\| \notin G$. $\square$

The Truth Lemma. *Let $\mathbb{P}$ be a poset in a transitive ZF model M and let G be a $\mathbb{P}$-generic filter over M. For any formula $\varphi(x_1, \dots, x_n)$ with all free variables shown, for any $v_1, \dots, v_n \in M$ the following are equivalent:*

(a) $M[G] \models \varphi(\mathrm{val}_G(v_1), \dots, \mathrm{val}_G(v_n))$.
(b) *There exists $p \in G$ with $p \Vdash \varphi(v_1, \dots, v_n)$.*

Proof. Denote by i the canonical order preserving map $i_{\mathbb{P}} \colon \mathbb{P} \to \mathrm{RO}(\mathbb{P})$ and denote by $\overline{G}$ the filter generated by $i_{\mathbb{P}}[G]$ in $\mathrm{RO}(\mathbb{P})$. Observe that $p \Vdash \varphi$ iff $\|\varphi\| \in \overline{G}$ (see Theorem 4.5).

We first prove the lemma for atomic formulas, i.e. formulas of the form $x = y$ and $x \in y$. We use induction on the well-founded relation E defined in Section 7. The equivalence (a)$\Longleftrightarrow$(b) is obvious for the formulas $\emptyset = \emptyset$ and $\emptyset \in \emptyset$. Fix $x, y \in M$, assume $\{x, y\} \neq \{\emptyset\}$ and assume that we have proved the equivalence (a)$\Longleftrightarrow$(b) for atomic formulas with pairs of parameters of E-rank less the E-rank of $\{x, y\}$. Consider first the formula $x = y$.

Assume $\mathrm{val}_G(x) = \mathrm{val}_G(y)$. Fix $(t, p) \in x$. If $p \in G$ then, by the assumption, $\mathrm{val}_G(t) \in \mathrm{val}_G(y)$. By induction hypothesis, $\|t \in y\| \in \overline{G}$. Thus $\neg i(p) + \|t \in y\| \in \overline{G}$ for each $(t, p) \in x$, $p \in \mathbb{P}$. Similarly, $\neg i(p) + \|t \in x\| \in \overline{G}$ for $(t, p) \in y$, $p \in \mathbb{P}$. By Proposition 4.4, $\|x = y\| \in \overline{G}$. Conversely, assume that $\|x = y\| \in \overline{G}$. Then for $(t, p) \in x$, $p \in \mathbb{P}$ we have $\neg i(p) + \|t \in y\| \in \overline{G}$. Thus, if $p \in G$ and $(t, p) \in x$ then $\|t \in y\| \in \overline{G}$ and, by induction hypothesis, $\mathrm{val}_G(t) \in \mathrm{val}_G(y)$. Hence $\mathrm{val}_G(x) \subset \mathrm{val}_G(y)$. Similarly $\mathrm{val}_G(y) \subset \mathrm{val}_G(x)$.

Now consider the formula $x \in y$. Assume $\mathrm{val}_G(x) \in \mathrm{val}_G(y)$. Then there is $p \in G$ and $(t, p) \in y$ such that $\mathrm{val}_G(x) = \mathrm{val}_G(t)$. By induction hypothesis, $\|t = x\| \in \overline{G}$. It follows that $\|x \in y\| \in \overline{G}$. Conversely, assume that $\|x \in y\| \in \overline{G}$. By Proposition 4.4, there is $(t, p) \in y$, $p \in \mathbb{P}$ with $i(p) \cdot \|t = x\| \in \overline{G}$. Hence $p \in G$ and $\mathrm{val}_G(t) \in \mathrm{val}_G(y)$. By induction hypothesis, $\mathrm{val}_G(t) = \mathrm{val}_G(x)$ so $\mathrm{val}_G(x) \in \mathrm{val}_G(y)$.

Suppose now that $\varphi(x_1, \dots, x_n)$ is a non-atomic formula and assume that we have already proved the equivalence (a)$\Longleftrightarrow$(b) for all formulas of length less than the length of φ. Fix parameters $v_1, \dots, v_n$ for φ. We will write φ instead of $\varphi(v_1, \dots, v_n)$ and φ_G instead of $\varphi(\mathrm{val}_G(v_1), \dots, \mathrm{val}_G(v_n))$. We have three cases.

Case 1. $\varphi = \neg\psi$. Then $M \models \varphi_G$ iff it is not true that $M \models \psi_G$ which, by induction hypothesis, is equivalent to $\|\psi\| \notin \overline{G}$; this means $\|\varphi\| = \neg\|\psi\| \in \overline{G}$, since $\overline{G}$ is an ultrafilter.

Case 2. $\varphi = \psi \wedge \chi$. Using induction hypothesis, we have $M \models \psi_G \wedge \chi_G$ iff ($M \models \psi_G$ and $M \models \chi_G$) iff ($\|\psi\| \in \overline{G}$ and $\|\chi\| \in \overline{G}$) iff $\|\psi \wedge \chi\| = \|\psi\| \cdot \|\chi\| \in \overline{G}$.

Case 3. $\varphi = \exists\ x\ \psi(x)$. In M define $A = \{a \in \mathrm{RO}(\mathbb{P}) \colon (\exists\ x)\ a = \|\psi(x)\|\}$. Now, using induction hypothesis and Proposition 4.4, we have $M \models \varphi_G$ iff $(\exists\ x \in M)\ M \models \psi_G(x)$ iff $(\exists\ x \in M)\ \|\psi(x)\| \in \overline{G}$ iff $(\exists\ a \in A)\ a \in \overline{G}$ iff $\|\varphi\| = \sum A \in \overline{G}$.

This completes the proof. $\square$

Corollary 8.2 (The Definability Lemma)**.** *Let $\mathbb{P}$ be a poset in a ctm M with $M \models ZF$. For any formula $\varphi(x_1, \dots, x_n)$ with all free variables shown, for any $v_1, \dots, v_n \in M$ and for every $p \in \mathbb{P}$ and $v_1, \dots, v_n \in M$ the following are equivalent:*

(a) $p \Vdash \varphi(v_1, \dots, v_n)$.
(b) *For each $\mathbb{P}$-generic filter G over M with $p \in G$ we have*

$$M[G] \models \varphi(\mathrm{val}_G(v_1), \dots, \mathrm{val}_G(v_n)).$$

Proof. We will write φ instead of $\varphi(v_1, \dots, v_n)$ or $\varphi(\mathrm{val}_G(v_1), \dots, \mathrm{val}_G(v_n))$.
(a) $\Longrightarrow$ (b) Suppose $p \Vdash \varphi$ and $M[G] \models \neg\varphi$ for some generic G with $p \in G$. By the Truth Lemma, there is $q \in G$ with $q \Vdash \neg\varphi$. Thus also $q \Vdash \neg\varphi$, whence $i_{\mathbb{P}}(q) \leqslant \|\varphi\| \cdot \neg\|\varphi\| = 0_{\mathrm{RO}(\mathbb{P})}$, a contradiction.
(b) $\Longrightarrow$ (a) Suppose $p \not\Vdash \varphi$. There is $q_1 \in \mathbb{P}$ with $i_{\mathbb{P}}(q_1) \leqslant i_{\mathbb{P}}(p) \cdot \neg\|\varphi\|$. Now $q_1 \parallel p$ so there is $q \leqslant q_1, p$. Let G be $\mathbb{P}$-generic over M with $q \in G$ (here we use the theorem of Rasiowa-Sikorski). Now $q \Vdash \neg\varphi$ so $M[G] \models \neg\varphi$, since we have proved that (a) $\Longrightarrow$ (b). It follows that $p \in G$ and $M[G] \not\models \varphi$. $\square$

Corollary 8.3. *If φ is a tautology of logic then $\|\varphi\| = 1_{\mathrm{RO}(\mathbb{P})}$ for every poset $\mathbb{P}$.*

The Definability Lemma shows that the relation $\Vdash$, defined by us in the ground model, defines indeed the forcing relation in the sense that it “forces” truth in generic extensions. Later on, we will use the Truth and Definability Lemmas also for proving some results in ZF or ZFC. Specifically, if we want to prove that $ZFC \vdash \varphi$, where φ is a sentence, then we can argue as follows. Suppose $ZFC \not\vdash \varphi$. Then, by Gödel's theorem, there is a ZFC model N with $N \models \neg\varphi$. Next, applying the Löwenheim-Skolem theorem and Mostowski's theorem on collapsing, we can find a countable transitive ZFC model M with $M \models \neg\varphi$. Now we can use generic filters to obtain a contradiction, by using some information about generic extensions. This is useful for instance when φ is “$\|\psi\| = 1_{\mathrm{RO}(\mathbb{P})}$” for some poset $\mathbb{P}$ (see e.g. the proof of Theorem 9.2 below).

9. The maximal principle

We give an important application of the Truth Lemma for computing Boolean values of formulas, called *the maximal principle.*

Lemma 9.1. *Let $\mathbb{P}$ be a partially ordered set and let $\{u_i\colon i \in J\}$ be an antichain in the Boolean algebra $\mathrm{RO}(\mathbb{P})$. Then for each collection $\{t_i\colon i \in J\}$ there exists t such that $(\forall\, i \in J)\ u_i \leqslant \|t = t_i\|$.*

Proof. Let $i = i_{\mathbb{P}}\colon \mathbb{P} \to \mathrm{RO}(\mathbb{P})$ be the canonical order preserving map (see Section 3). Let

$$t = \{(s,p) \in \big(\bigcup_{i \in J} \mathrm{dom}(t_i)\big) \times \mathbb{P}\colon (\exists\, i \in J)\ (p \Vdash s \in t_i)\ \&\ i(p) \leqslant u_i\}.$$

We check that t is as desired. Fix $i \in J$ and $p \in \mathbb{P}$ such that $i(p) \leqslant u_i$. Let G be a $\mathbb{P}$-generic filter with $p \in G$. If $\mathrm{val}_G(s) \in \mathrm{val}_G(t)$ and $(s,q) \in t$, $q \in G$ then $\mathrm{val}_G(s) \in \mathrm{val}_G(t_j)$ and $i(q) \leqslant u_j$ for some $j \in J$. As u_i, u_j are incompatible whenever $i \neq j$, we deduce that $i = j$ and $\mathrm{val}_G(s) \in \mathrm{val}_G(t_i)$. Conversely, if $x \in \mathrm{val}_G(t_i)$ then there is $q \in G$ and $(s,q) \in t_i$ with $x = \mathrm{val}_G(s)$. By the Truth Lemma, there is $r \in G$ with $r \leqslant p$ and $r \Vdash s \in t_i$. Hence $(s,r) \in t$ and $x \in \mathrm{val}_G(t)$. It follows that $\mathrm{val}_G(t) = \mathrm{val}_G(t_i)$. By the Definability Lemma, $p \Vdash t = t_i$. As p was chosen arbitrarily (with respect to the condition $i(p) \leqslant u_i$), $u_i \leqslant \|t = t_i\|$. $\square$

Theorem 9.2. *Let $\mathbb{P}$ be a poset. For each formula $\varphi(x)$ there exists t such that*

$$\|\exists\, x\ \varphi(x)\| = \|\varphi(t)\|.$$

Proof. Applying AC we can find $\{x_\alpha\colon \alpha<\kappa\}$, where κ is a cardinal and

$$b := \|\exists\, x\ \varphi(x)\| = \sum_{\alpha<\kappa} \|\varphi(x_\alpha)\|.$$

Define

$$a_\alpha = \sum_{\xi<\alpha} \|\varphi(x_\xi)\|, \qquad b_\alpha = a_\alpha \cdot \neg \sum_{\xi<\alpha} a_\xi.$$

Then $\{b_\alpha\}_{\alpha<\kappa}$ is an anitchain in $\mathrm{RO}(\mathbb{P})$ and $\sum_{\alpha<\kappa} a_\alpha = \sum_{\alpha<\kappa} b_\alpha = b$. Applying Lemma 9.1 we find t such that $a_\alpha \leqslant \|t = x_\alpha\|$ for every $\alpha<\kappa$. Observe that by Corollary 8.3, $\|\varphi(t)\| \leqslant \|\exists\, x\ \varphi(x)\|$. It remains to check the reverse inequality. Fix a $\mathbb{P}$-generic filter G with $b \in \overline{G}$, where $\overline{G}$ denotes the ultrafilter in $\mathrm{RO}(\mathbb{P})$ generated by $i_{\mathbb{P}}[G]$. By Proposition 4.4 there exists $\xi<\kappa$ with $b_\xi \in \overline{G}$. Let $\alpha = \min\{\xi<\kappa\colon b_\xi \in \overline{G}\}$. Then $a_\alpha \in \overline{G}$ and $a_\xi \notin \overline{G}$ whenever $\xi<\alpha$. It follows that $\|\varphi(x_\alpha)\| \in \overline{G}$ and $\|\varphi(x_\xi)\| \notin \overline{G}$ for $\xi<\alpha$. Hence $\mathrm{val}_G(t) = \mathrm{val}_G(x_\alpha)$ and $M[G] \models \varphi(\mathrm{val}_G(x_\alpha))$. Thus $M[G] \models \varphi(\mathrm{val}_G(t))$. It follows that $\|\varphi(t)\| \in \overline{G}$. This completes the proof. $\square$

For some purposes, we shall need a weaker version of Theorem 9.2.

Lemma 9.3. *Let $\mathbb{P}$ be a poset and let $\varphi(x)$ be formula with x the only free variable. For each a we have*

$$\|(\exists\, x \in a)\ \varphi(x)\| = \sum_{(s,p)\in a,\, p\in\mathbb{P}} i_{\mathbb{P}}(p)\cdot\|\varphi(s)\|.$$

Proof. The inequality "$\geqslant$" is trivial, since if $(s,p)\in a$ then $p \Vdash s\in a$ and hence $i_{\mathbb{P}}(p)\cdot\|\varphi(s)\| \leqslant \|s\in a\ \&\ \varphi(s)\| \leqslant \|(\exists\, x\in a)\ \varphi(x)\|$.
We show the reverse inequality. Let G be an $\mathrm{RO}(\mathbb{P})$-generic filter which contains $\|(\exists\, x \in a)\ \varphi(x)\|$. By the definition of the Boolean value and by Proposition 4.4, there is x such that $\|x\in a\ \&\ \varphi(x)\| \in G$. Using the definition of $\|x\in a\|$ and Proposition 4.4 again, we find $(s,p)\in a$ with $p\in\mathbb{P}$ and $i(p)\cdot\|s = x\|\cdot\|\varphi(x)\| \in G$. Hence also $i(p)\cdot\|\varphi(s)\|\in G$ since it is a tautology of logic that $s = x\ \&\ \varphi(x) \implies \varphi(s)$ (see Corollary 8.3). This completes the proof. $\square$

Theorem 9.4. *Let $\mathbb{P}$ be a poset and let $\varphi(x)$ be a formula of set theory. Then for each a and $p\in\mathbb{P}$ such that $p \Vdash (\ \exists\, x\in a)\ \varphi(x)$ there exists $q\leqslant p$ and there exists $(s,r)\in a$ such that $q\leqslant r$ and $q\Vdash\varphi(s)$.*

Proof. Let $i\colon \mathbb{P}\to\mathrm{RO}(\mathbb{P})$ be the canonical map. By Lemma 9.3, there exists $(s,r)\in a$ such that $i(p)\cdot i(r)\cdot\|\varphi(s)\| > 0_{\mathrm{RO}(\mathbb{P})}$. Let $q\in\mathbb{P}$ be below p, r with $i(q)\leqslant\|\varphi(s)\|$. Then $q\Vdash\varphi(s)$. $\square$

10. More about names

Let M be a ctm of ZFC and let $\mathbb{P}\in M$ be a poset. Observe that if $a\in M$ and $b\subset\widehat{a}\times\mathbb{P}$ in M then for each $\mathbb{P}$-generic filter G we have $\mathrm{val}_G(b)\subset\mathrm{val}_G(a)$. We show that $\mathcal{P}^M(\widehat{a}\times\mathbb{P})$ contains all names for possible subsets of a.

Lemma 10.1. *Let $\mathbb{P}$ be a poset and let $a\in M$ be fixed. Then for each x there exists $y\subset\widehat{a}\times\mathbb{P}$ such that $\|x\subset\widehat{a} \implies x = y\|_{\mathbb{P}} = 1_{\mathbb{P}}$.*

Proof. Set
$$y=\{(\widehat{s},p)\in\widehat{a}\times\mathbb{P}\colon (\exists\,(t,q)\in x)\; p\leqslant q\;\&\; p\Vdash t=\widehat{s}\}.$$
Let G be a $\mathbb{P}$-generic filter with $\mathrm{val}_G(x)\subset a$. We show that $\mathrm{val}_G(y)=\mathrm{val}_G(x)$.
Let $z\in\mathrm{val}_G(y)$. Then $z\in a$ and there is $p\in G$ and there is $(t,q)\in x$ with $p\leqslant q$ and $p\Vdash t=\widehat{z}$. Thus $q\in G$ and $z=\mathrm{val}_G(t)\in\mathrm{val}_G(x)$. This shows that $\mathrm{val}_G(y)\subset\mathrm{val}_G(x)$. Now fix $z=\mathrm{val}_G(t)\in\mathrm{val}_G(x)$, where $(t,q)\in x$ and $q\in G$. Then $z\in a$ so, by the Truth Lemma, there exists $p_0\in G$ with $p_0\Vdash\widehat{z}=t$. Let $p\in G$ be below p_0 and q. Then $(\widehat{z},p)\in y$ and $z\in\mathrm{val}_G(y)$. Hence $\mathrm{val}_G(x)\subset\mathrm{val}_G(y)$. □

11. ZFC in generic extensions

In this section we assume, as usual, that M is a fixed transitive model of ZFC, $\mathbb{P}$ is a partially ordered set in M and G is a fixed $\mathbb{P}$-generic filter over M. The aim of this section is to show that $M[G]\models ZFC$. This will done by several lemmas.

Lemma 11.1. $M[G]\models A1+A2+A3+A6+A7$ *(Extensionality, Empty Set, Pairing, Infinity, Regularity).*

Proof. Every nonempty transitive set satisfies $A1+A2+A7$. That $M[G]\models A6$ follows from the fact that $\omega\in M[G]$. It remains to check that $M[G]$ satisfies the Pairing Axiom. Fix $a,b\in M[G]$, $a=\mathrm{val}_G(a'),b=\mathrm{val}_G(b')$. Set $c'=\{(a',1_{\mathbb{P}}),(b',1_{\mathbb{P}})\}$. Then $\mathrm{val}_G(c')=\{a,b\}$. □

Lemma 11.2. $M[G]\models A4$ *(Union).*

Proof. Fix $a=\mathrm{val}_G(a')\in M[G]$. In M define
$$b'=\Big\{(t,p)\in\big(\bigcup_{s\in\mathrm{dom}(a')}\mathrm{dom}(s)\big)\times\mathbb{P}\colon p\Vdash(\exists\,x\in a')\;t\in x\Big\}.$$
We check that $\mathrm{val}_G(b')=\bigcup a$. If $\mathrm{val}_G(t)\in a$, where $(t,p)\in b'$ and $p\in G$ then $M[G]\models(\exists\,x\in a)\;\mathrm{val}_G(t)\in x$, i.e. $\mathrm{val}_G(t)\in\bigcup a$. Conversely, if $z\in\bigcup a$ then there is $(s,q)\in a'$, $q\in G$ with $z\in\mathrm{val}_G(s)$ and there is $(t,r)\in s$, $r\in G$ with $z=\mathrm{val}_G(t)$. Thus $t\in\bigcup_{s\in\mathrm{dom}(a')}\mathrm{dom}(s)$ and, by the Truth Lemma, there is $p\in G$ with $p\Vdash(\exists\,x\in a')\;t\in x$. Hence $(t,p)\in b'$ and $z=\mathrm{val}_G(t)\in\mathrm{val}_G(b')$. □

Lemma 11.3. $M[G]\models A5$ *(Power Set).*

Proof. Fix $a=\mathrm{val}_G(a')\in M[G]$. Using the Axiom of Replacement in M we can define
$$u=\{(t,1_{\mathbb{P}})\colon t\subset\mathrm{dom}(a')\;\&\;(\forall\,(s,p)\in t)\;p\Vdash s\in a'\}.$$
We show that $M[G]\models\mathrm{val}_G(u)=\mathcal{P}(a)$. If $y\in\mathrm{val}_G(u)$ and $y=\mathrm{val}_G(t)$, where $(t,1_{\mathbb{P}})\in u$ then for all $z\in y$ we have $z\in a$ since there is $p\in G$ with $(s,p)\in t$, $z=\mathrm{val}_G(s)$ and $p\Vdash s\in a'$. Thus $y\subset a$.
Now fix $y\subset a$, $y=\mathrm{val}_G(y')\in M[G]$. In M, define
$$y''=\{(s,p)\in\mathrm{dom}(a')\times\mathbb{P}\colon p\Vdash(s\in y'\;\&\;s\in a')\}.$$
Then $(y'',1_{\mathbb{P}})\in u$. It remains to check that $\mathrm{val}_G(y'')=y$. If $x\in\mathrm{val}_G(y'')$ then $x=\mathrm{val}_G(s)$, where $(s,p)\in y''$ and $p\in G$. Thus $p\Vdash s\in y'$ so $x\in y$. Conversely, if $x\in y$ then $x\in a$ so there is $(s,q)\in a'$, $q\in G$ such that $x=\mathrm{val}_G(s)$. By the Truth Lemma, there is $p\in G$ with $p\Vdash(s\in y'\;\&\;s\in a')$. Hence $(s,p)\in y''$ and $x\in\mathrm{val}_G(y'')$. □

Lemma 11.4. $M[G]\models A8$ *(Comprehension Axiom Scheme).*

Proof. Let $\varphi(v_0, v_1, \ldots, v_n)$ be a formula with all free variables shown, fix $a \in M[G]$ and $x_1, \ldots, x_n \in M[G]$. Let $x_i = \mathrm{val}_G(x_i')$, $a = \mathrm{val}_G(a')$. Using Comprehension in M, define

$$b' = \{(t, p) \in \mathrm{dom}(a') \times \mathbb{P} \colon p \Vdash \varphi(t, x_1', \ldots, x_n')\}.$$

Let $b = \mathrm{val}_G(b')$. We check that $b = \{z \in a \colon \varphi^{M[G]}(z, x_1, \ldots, x_n)\}$.
If $z \in b$ and $z = \mathrm{val}_G(t)$, where $(t, p) \in b'$, $p \in G$ then $M[G] \models \varphi(z, x_1, \ldots, x_n)$. Suppose that $M[G] \models z \in a \ \& \ \varphi(z, x_1, \ldots, x_n)$. Then $z = \mathrm{val}_G(t)$, where $(t, q) \in a'$ and $q \in G$. By the Truth Lemma, there is $p \in G$ such that $p \Vdash \varphi(t, x_1', \ldots, x_n')$. Hence $(t, p) \in b'$ and $z \in b$. $\square$

Lemma 11.5. *$M[G] \models A9$ (Replacement Axiom Scheme).*

Proof. Let $\varphi(x, y, v_1, \ldots, v_n)$ be a formula with all free variables shown and fix $a, x_1, \ldots, x_n \in M[G]$ such that $M[G] \models \forall\, x \in a\ \exists!\ y\ \varphi(x, y, x_1, \ldots, x_n)$. Let $a = \mathrm{val}_G(a')$, $x_i = \mathrm{val}_G(x_i')$. Applying AC and Theorem 9.2 in M, we can choose for each $s \in \mathrm{dom}(a')$ an element $t_s \in M$ such that $\|\exists\, y\ \varphi(s, y, x_1', \ldots, x_n')\| = \|\varphi(s, t_s, x_1', \ldots, x_n')\|$. Let

$$b' = \{(t_s, p) \colon (s, p) \in a'\}$$

and let $b = \mathrm{val}_G(b')$. Fix $x \in a$. Then $x = \mathrm{val}_G(s)$ where $(s, p) \in a'$ and $p \in G$. Now $M[G] \models \exists\ y\ \varphi(x, y, x_1, \ldots, x_n)$. Hence also $M[G] \models \varphi(x, \mathrm{val}_G(t_s), x_1, \ldots, x_n)$. This shows that $M[G] \models \forall\, x \in a\ \exists\, y \in b\ \varphi(x, y, x_1, \ldots, x_n)$. $\square$

Lemma 11.6. *$M[G] \models AC$ (the Axiom of Choice).*

Proof. Fix $a = \mathrm{val}_G(a') \in M[G]$. Applying AC in M, we can find a bijection $f \colon \alpha \to \mathrm{dom}(a')$, where α is an ordinal in M. Define $F \colon \alpha \to a$ by setting $F(\xi) = \mathrm{val}_G(f(\xi))$. Then $F \in M[G]$ and $a \subset \mathrm{rng}(F)$. It follows that a can be well-ordered in $M[G]$, since a map $g \colon a \to \alpha$ defined by $g(t) = \min F^{-1}(t)$ is an injection. $\square$

The above lemmas together give the following.

Theorem 11.7. *Let $\mathbb{P}$ be a partially ordered set in a transitive model M and let G be a $\mathbb{P}$-generic filter over M. If $M \models ZFC$ then also $M[G] \models ZFC$.*

12. Chain conditions

Lemma 12.1. *Let $\mathbb{P}$ be a θ-cc poset in a tcm M and let G be a $\mathbb{P}$-generic filter over M. Then for each $f \in \kappa^\lambda$ in $M[G]$ there exists a map $F \colon \lambda \to [\kappa]^{<\theta}$ in M with*

$$M[G] \models (\forall\, \alpha \in \lambda) f(\alpha) \in F(\alpha).$$

Proof. Fix $\alpha \in \lambda$ and define

$$F(\alpha) = \{\beta \in \kappa \colon (\exists\, p \in \mathbb{P})\ p \Vdash \overline{f}(\widehat{\alpha}) = \widehat{\beta}\},$$

where $\overline{f}$ denotes a name for f. For each $\beta \in F(\alpha)$ choose $p_\beta \in \mathbb{P}$ with $p_\beta \Vdash \overline{f}(\widehat{\alpha}) = \widehat{\beta}$. Then $\{p_\beta\}_{\beta \in F(\alpha)}$ is in M and consists of pairwise incompatible elements of $\mathbb{P}$. Thus

$$M \models |F(\alpha)| < \theta$$

and, by the Truth Lemma, $f(\alpha) \in F(\alpha)$ for every $\alpha \in \lambda$. $\square$

Theorem 12.2. *Under the above assumptions, for any $\kappa \in \mathrm{Card}^M$, $\kappa > \theta$ implies $\kappa \in \mathrm{Card}^{M[G]}$. Moreover, if $\delta = \mathrm{cf}^M(\kappa) \geqslant \theta$ then $\delta = \mathrm{cf}^{M[G]}(\kappa)$.*

Proof. Let $\delta = \mathrm{cf}^M(\kappa) \geqslant \theta$ and suppose that

$$M[G] \models f\colon \lambda \to \kappa \text{ is cofinal,}$$

where $\lambda < \delta$. By Lemma 12.1 there is $F \in M$ such that $F\colon \lambda \to [\kappa]^{<\theta}$ and $f(\alpha) \in F(\alpha)$ for $\alpha \in \lambda$. As $M \models \mathrm{cf}(\kappa) \geqslant \theta$, we can define in M a map $g\colon \lambda \to \kappa$ by setting $g(\alpha) = \sup F(\alpha)$. Clearly g is cofinal, so $M \models \mathrm{cf}(\kappa) \leqslant \lambda$, a contradiction.

Now, if $\kappa \in \mathrm{Card}^M$ is regular and $\kappa \geqslant \theta$ then $\kappa = \mathrm{cf}^{M[G]}(\kappa) \in \mathrm{Card}^{M[G]}$. If $\kappa > \theta$ is a singular cardinal in M then $\kappa = \sup_{\alpha<\lambda} \kappa_\alpha$ where $\{\kappa_\alpha\}_{\alpha<\lambda}$ is an increasing sequence of regular cardinals in M and $\theta \leqslant \kappa_0$. Thus $\kappa \in \mathrm{Card}^{M[G]}$ as the supremum of cardinals. □

Corollary 12.3. *If $\mathbb{P}$ is a ccc partial order then for every $\mathbb{P}$-generic filter G over a ctm M we have* $\mathrm{Card}^M = \mathrm{Card}^{M[G]}$.

13. Distributivity laws

Recall that a complete Boolean algebra $\mathbb{B}$ is (κ,λ)-distributive if for each indexed collection $\{a_{\alpha,\beta}\colon \alpha<\kappa, \beta<\lambda\}$ the following equality holds:

$$\prod_{\alpha<\kappa}\sum_{\beta<\lambda} a_{\alpha,\beta} = \sum_{\varphi\in\lambda^\kappa}\prod_{\alpha<\kappa} a_{\alpha,\varphi(\alpha)}.$$

Let us note that the inequality $\geqslant$ always holds.

Theorem 13.1. *Let M be a ctm and let $\mathbb{B}$ be a complete Boolean algebra in M. If $\mathbb{B}$ is (κ,λ)-distributive in M then $(\lambda^\kappa)^M = (\lambda^\kappa)^{M[G]}$ for each $\mathbb{B}$-generic filter G over M. Conversely, if for every $\mathbb{B}$-generic filter G we have $(\lambda^\kappa)^{M[G]} = (\lambda^\kappa)^M$ then $M \models$ "$\mathbb{B}$ is (κ,λ)-distributive".*

Proof. Suppose that "$\mathbb{B}$ is (κ,λ)-distributive" holds in M. Let $f = \mathrm{val}_G(\overline{f}) \in \lambda^\kappa$ in $M[G]$. Fix $\alpha\in\kappa$. For each $\beta\in\lambda$ define

$$a_{\alpha,\beta} = \sum\{p \in \mathbb{B}^+\colon p \Vdash \overline{f}(\widehat{\alpha}) = \widehat{\beta}\}.$$

Observe that $a_{\alpha,\beta} \in G$ for $\beta = f(\alpha)$. It follows that $\sum_{\beta\in\lambda} a_{\alpha,\beta} \in G$. By Proposition 4.4 also $b = \prod_{\alpha\in\kappa}\sum_{\beta\in\lambda} a_{\alpha,\beta} \in G$. By the (κ,λ)-distributivity law we get

$$M \models b = \sum_{\varphi\in\lambda^\kappa}\prod_{\alpha\in\kappa} a_{\alpha,\varphi(\alpha)}.$$

Applying Proposition 4.4 again we obtain $\prod_{\alpha\in\kappa} a_{\alpha,\varphi(\alpha)} \in G$ for some $\varphi\in\lambda^\kappa$ in M. Finally, by the definition of $a_{\alpha,\beta}$ we get $M \models f(\alpha) = \varphi(\alpha)$ for every $\alpha\in\kappa$. Hence $f \in M$.

Suppose now that $M \models$ "$\mathbb{B}$ is not (κ,λ)-distributive" and let $\{a_{\alpha,\beta}\colon \alpha<\kappa,\beta<\lambda\} \in \mathcal{P}^M(\mathbb{B})$ be such that

$$M \models l := \prod_{\alpha<\kappa}\sum_{\beta<\lambda} a_{\alpha,\beta} > \sum_{\varphi\in\lambda^\kappa}\prod_{\alpha<\kappa} a_{\alpha,\varphi(\alpha)} =: r.$$

Let G be a $\mathbb{B}$-generic filter containing $l\cdot\neg r$. In $M[G]$ we can define a function $f\colon \kappa\to\lambda$ by letting $f(\alpha) = \min\{\beta<\lambda\colon a_{\alpha,\beta}\in G\}$. This is well-defined since $\sum_{\beta<\lambda} a_{\alpha,\beta} \in G$ and $\{a_{\alpha,\beta}\}_{\beta<\lambda} \in M$. Suppose that $f \in M$. Then $\{a_{\alpha,f(\alpha)}\colon \alpha<\kappa\} \in M$ and consequently $\prod_{\alpha<\kappa} a_{\alpha,f(\alpha)} \in G$. On the other hand, this element is below r and $r \notin G$, a contradiction. This completes the proof. □

Now we show that some distributivity implies that some cardinals are preserved in generic extensions. First observe that (κ, λ)-distributivity implies (κ', λ')-distributivity for $\kappa' \leqslant \kappa$ and $\lambda' \leqslant \lambda$. This fact can be easily seen using Theorem 13.1. Indeed, if $f\colon \kappa' \to \lambda'$ is a "new" function then any function $F\colon \kappa \to \lambda$, which extends f, is also "new". Thus, in particular, (κ, λ)-distributivity implies $(\kappa, 2)$-distributivity (provided $\lambda \geqslant 2$).

Theorem 13.2. *Let $\mathbb{P}$ be a poset and let κ be an infinite cardinal. If* $\mathrm{RO}(\mathbb{P})$ *is $(\kappa, 2)$-distributive then for each cardinal $\lambda \leqslant \kappa^+$, $1_{\mathbb{P}} \Vdash$ "$\widehat{\lambda}$ is a cardinal".*

Proof. Let G be a $\mathbb{P}$-generic filter over a ctm model M of ZFC. Assume first that $\lambda \leqslant \kappa$. Suppose that $f \in M[G]$ is a bijection from δ onto λ, where $\delta < \lambda$. Then $f \subset \delta \times \lambda$ so, using Theorem 13.1, we have $f \in \mathcal{P}^{M[G]}(\delta \times \lambda) = \mathcal{P}^M(\delta \times \lambda)$ since $\lambda \times \delta$ has cardinality at most κ in M. Thus $f \in M$ and f is a bijection from δ onto λ in M, because this property is absolute; a contradiction. Thus λ is a cardinal in $M[G]$.

Assume now that $\lambda = \kappa^+$ and suppose that λ is not a cardinal in $M[G]$. Then $|\lambda|^{M[G]} = \kappa$ and hence in $M[G]$ there is a bijection from κ onto λ. This bijection induces a well-order $\prec$ on κ. Then $\prec \in M$ because $\prec \in \mathcal{P}^{M[G]}(\kappa) = \mathcal{P}^M(\kappa)$. Hence $(\kappa, \prec)$ is isomorphic in M to $(\delta, \in)$ for some ordinal δ, which is the order type of $\prec$. But the fact that two well-ordered sets are isomorphic is absolute, so $\delta = \kappa^+$ in M. This is a contradiction since $\kappa < \kappa^+$. □

There is an important property of partial orders which implies some distributive laws for their completions. A partially ordered set $\mathbb{P}$ is *κ-closed* (κ is an infinite cardinal), if for any $\lambda < \kappa$, for any decreasing sequence $\{p_\alpha\}_{\alpha<\lambda} \subset \mathbb{P}$ there exists $p \in \mathbb{P}$ such that $p \leqslant p_\alpha$ holds for every $\alpha < \lambda$. A complete Boolean algebra is (κ, ∞)-distributive if it is (κ, λ)-distributive for every cardinal λ.

Theorem 13.3. *Let κ be an infinite cardinal and let $\mathbb{P}$ be a κ-closed poset. Then for each $\lambda < \kappa$,* $\mathrm{RO}(\mathbb{P})$ *is (λ, ∞)-distributive.*

Proof. Suppose $\mathrm{RO}(\mathbb{P})$ is not (λ, μ)-distributive for some μ. Consider the canonical map $i = i_{\mathbb{P}}\colon \mathbb{P} \to \mathrm{RO}(\mathbb{P})$. There is $p \in \mathbb{P}$ with $i(p) \leqslant \prod_{\alpha<\lambda} \sum_{\beta<\mu} a_{\alpha,\beta}$ and $i(p) \cdot \sum_{f \in \mu^\lambda} \prod_{\alpha<\lambda} a_{\alpha,f(\alpha)} = 0_{\mathrm{RO}(\mathbb{P})}$. Construct inductively a decreasing sequence $\{p_\alpha\}_{\alpha<\lambda} \subset \mathbb{P}$ and a sequence $\{g(\alpha)\}_{\alpha<\lambda} \subset \mu$ such that $i(p_\alpha) \leqslant a_{\alpha,g(\alpha)}$. On limit ordinals we use the fact that $\mathbb{P}$ is λ-closed. Now, as $\mathbb{P}$ is κ-closed, there is $q \in \mathbb{P}$ with $q \leqslant p_\alpha$ for each $\alpha < \lambda$. We have $i(q) \leqslant \prod_{\alpha<\lambda} a_{\alpha,g(\alpha)}$ and $q \leqslant p$, a contradiction. □

Combining the two last theorems we see that a κ-closed poset does not collapse cardinals up to κ. Cardinals greater than κ can be collapsed, see Section 15.

14. Continuum Hypothesis

Lemma 14.1. *Assume that $\mathbb{P}$ is a partial order in a ctm M, $\kappa \in M$ and in M define $S_{\mathbb{P}}(\kappa) = {}^{\kappa}\,\mathrm{RO}(\mathbb{P})$. Then*

$$M[G] \models |\mathcal{P}(\kappa)| \leqslant |S_{\mathbb{P}}(\kappa)|$$

for every $\mathbb{P}$-generic filter G over M.

Proof. (a) Let $A = \mathcal{P}^M(\widehat{\kappa} \times \mathbb{P})$ and let $i\colon \mathbb{P} \to \mathrm{RO}(\mathbb{P})$ be the canonical map. By Lemma 10.1, $\mathcal{P}^{M[G]}(\kappa) = \{\mathrm{val}_G(x)\colon x \in A\}$. Fix $x \in A$. In M define

$$f_x(\alpha) = \sum\nolimits^{\mathrm{RO}(\mathbb{P})} \{i(p)\colon p \in \mathbb{P} \;\&\; p \Vdash \widehat{\alpha} \in x\},$$

for $\alpha \in \kappa$. Then $f_x \in M$ and $f_x \colon \kappa \to \mathrm{RO}(\mathbb{P})$.

(b) Suppose $x, y \in A$ and $\mathrm{val}_G(x) \neq \mathrm{val}_G(y)$. If e.g. $\alpha \in \mathrm{val}_G(x) \setminus \mathrm{val}_G(y)$ then there are $p, q \in G$ such that $p \Vdash \widehat{\alpha} \in x$ and $q \Vdash \widehat{\alpha} \notin y$. Consequently $0_{\mathrm{RO}(\mathbb{P})} < i(p) \cdot i(q) \leqslant f_x(\alpha) \cdot \neg f_y(\alpha)$. It follows that $f_x \neq f_y$.

(c) Now for $a \in \mathcal{P}^{M[G]}(\kappa)$ in $M[G]$ define

$$\theta(a) = \{f_x \colon \mathrm{val}_G(x) = a \;\&\; x \in A\}.$$

Observe that each $\theta(a)$ is a nonempty subset of $S_{\mathbb{P}}(\kappa)$. Applying AC in $M[G]$ we can find a function $\varphi \in M[G]$ such that $\varphi(a) \in \theta(a)$ for $a \in \mathcal{P}^{M[G]}(\kappa)$. By (b) φ is 1-1, which completes the proof. □

For an infinite cardinal cardinal κ define $C_\kappa = \{p \subset \kappa \times 2 \colon \mathrm{Func}(p) \;\&\; |p| < \omega\}$, i.e. C_κ is the set of all functions defined on finite subsets of κ with values in $2 = \{0, 1\}$. This is called the *Cohen forcing* of size κ. The partial order of C_κ is just the reverse inclusion. Observe that C_κ is isomorphic to a dense subset of the free Boolean algebra of size κ, hence it is ccc. Moreover $|\mathrm{RO}(C_\kappa)| = \kappa^\omega$. Note that $1_{C_\kappa} = \emptyset$.

Lemma 14.2. *Let κ be such a cardinal that $\kappa^\omega = \kappa$. Then $1_{C_\kappa} \Vdash 2^{\widehat{\omega}} = \widehat{\kappa}$.*

Proof. Let M be a fixed ctm. Fix a bijection $\varrho \colon \kappa \times \omega \to \kappa$. Let G be a C_κ-generic filter. Set $g = \bigcup G$. Then g is a function in $M[G]$. For any $\alpha < \kappa$ the set

$$D_\alpha = \{p \in C_\kappa \colon \alpha \in \mathrm{dom}(p)\}$$

is dense and in M, so $D_\alpha \cap G \neq \emptyset$. It follows that $\mathrm{dom}(g) = \kappa$. Now set

$$E_\alpha = \{p \in C_\kappa \colon (\exists\, n \in \omega)\; p(\varrho(\alpha, n)) \neq p(\varrho(\alpha, m))\}.$$

Observe that $E_\alpha \in M$ is dense. Hence $G \cap E_\alpha \neq \emptyset$ so for each $\alpha < \kappa$ there is $n \in \omega$ with $g(\varrho(\alpha, n)) \neq g(\varrho(\alpha, n))$. This means that a map $\varphi \colon \kappa \to \mathcal{P}(\omega)$ defined by

$$\varphi(\alpha) = \{n \in \omega \colon g(\varrho(\alpha, n)) = 1\}$$

is 1-1. As C_κ is ccc, κ is a cardinal in $M[G]$ (see Corollary 12.3). It follows that $M[G] \models \kappa \leqslant 2^\omega$. On the other hand

$$M \models |^\omega \mathrm{RO}(C_\kappa)| = \kappa^\omega = \kappa,$$

so by Lemma 14.1 we obtain $M[G] \models 2^\omega = \kappa$. □

Corollary 14.3. $\mathrm{Con}(ZFC) \implies \mathrm{Con}(ZFC + \neg CH)$.

15. Collapsing cardinals

Lemma 15.1. *Let $\mathbb{P}$ be a κ^+-cc poset and assume that $|\mathbb{P}| \leqslant 2^\kappa$. If $p \in \mathbb{P}$ and $p \Vdash |\widehat{\kappa}| = \widehat{\omega}$ then $p \Vdash 2^{\widehat{\omega}} = \widehat{2^\kappa}$.*

Proof. First observe that every element of $\mathrm{RO}(\mathbb{P})$ can be represented as the supremum of an antichain, hence $|\mathrm{RO}(\mathbb{P})| \leqslant (2^\kappa)^\kappa = 2^\kappa$. By Lemma 14.1, $1_{\mathbb{P}} \Vdash 2^{\widehat{\omega}} \leqslant \widehat{2^\kappa}$.

Fix a $\mathbb{P}$-generic filter G with $p \in G$, set $\delta = (2^\kappa)^M$. In the ground model M, there is a 1-1 map $f \colon \delta \to (2^\kappa)^M \subset (2^\kappa)^{M[G]}$. In $M[G]$, there is a bijection $g \colon (2^\kappa)^{M[G]} \to (2^\omega)^{M[G]}$. Setting $h = gf$, we obtain a 1-1 map from δ into $(2^\omega)^{M[G]}$. This shows that $M[G] \models 2^\omega \leqslant \delta$. □

As an application, consider $\mathbb{P} = (\kappa^{<\omega}, \supset, \emptyset)$. Let G be a $\mathbb{P}$-generic filter over M and set $g = \bigcup G$. Observe that g is a function with $\mathrm{dom}(g) = \omega$. For $\alpha < \kappa$ define

$$D_\alpha = \{\sigma \in \kappa^{<\omega} \colon \alpha \in \mathrm{rng}(\sigma)\}.$$

It is obvious that D_α is dense, hence $D_\alpha \cap G \neq \emptyset$ which means $\alpha \in \mathrm{rng}(g)$. Thus $\mathrm{rng}(g) = \kappa$. It follows that $1_{\mathbb{P}} \Vdash |\widehat{\kappa}| = \widehat{\omega}$. By Lemma 15.1, $M[G] \models 2^\omega = |(2^\kappa)^M|$. Note that if $M \models GCH$ then $M[G] \models GCH + V \neq L$.

Now consider the poset $\mathbb{P} = (\kappa^{<\omega_1}, \supset)$. Observe that $\mathbb{P}$ is ω_1-closed. As above, we can easily show that if G is a $\mathbb{P}$-generic filter over M then $\bigcup G$ is a function from ω_1 onto κ. On the other hand, by Theorems 13.1 and 13.3, $\mathcal{P}^{M[G]}(\omega) = \mathcal{P}^M(\omega)$. If for instance $\kappa = (2^\omega)^M$ then $M[G] \models 2^\omega = \omega_1$. It follows that $\mathrm{Con}(ZFC) \implies \mathrm{Con}(ZFC + CH)$.

16. Weak distributivity

A complete Boolean algebra $\mathbb{B}$ is *weakly* (κ, λ)*-distributive* provided

$$\prod_{\alpha<\kappa} \sum_{\beta<\lambda} a_{\alpha,\beta} = \sum_{f\in\lambda^\kappa} \prod_{\alpha<\kappa} \sum_{\beta<f(\alpha)} a_{\alpha,\beta}$$

holds for each indexed collection $\{a_{\alpha,\beta} \colon \alpha < \kappa, \beta < \lambda\} \subset \mathbb{B}$. Observe that the inequality $\geqslant$ always holds.

Theorem 16.1. *A complete Boolean algebra $\mathbb{B}$ is weakly (κ, λ)-distributive iff*

$$\| \forall\, f \in \widehat{\lambda^\kappa}\ \exists\, g \in \widehat{\lambda^\kappa}\ \forall\, \alpha \in \widehat{\kappa}\ (f(\alpha) < g(\alpha))\|_{\mathbb{B}} = 1_{\mathbb{B}}.$$

Proof. Let M be a ctm of ZFC and let $M \models$ "$\mathbb{B}$ is weakly (κ, λ)-distributive". Fix a $\mathbb{B}$-generic filter G over M and $f \in (\lambda^\kappa)^{M[G]}$. Set $a_{\alpha,\beta} = \|\overline{f}(\widehat{\alpha}) = \widehat{\beta}\|_{\mathbb{B}}$ for $\alpha < \kappa$, $\beta < \lambda$, where $\overline{f}$ is a name for f. Observe that $\sum_{\beta<\lambda} a_{\alpha,\beta} \in G$ since $a_{\alpha,\beta} \in G$ for $\beta = f(\alpha)$. By Proposition 4.4, $\prod_{\alpha<\kappa} \sum_{\beta<\lambda} a_{\alpha,\beta} \in G$. Using weak (κ, λ)-distributivity of $\mathbb{B}$ in M we get

$$\sum_{g\in(\lambda^\kappa)^M} \prod_{\alpha<\kappa} \sum_{\beta<g(\alpha)} a_{\alpha,\beta} \in G.$$

Thus there exists $g \in (\lambda^\kappa)^M$ such that $a_{\alpha,\beta} \in G$ for $\alpha < \kappa$ and $\beta < g(\alpha)$. It follows that $M[G] \models\ (\forall\, \alpha < \kappa)\ f(\alpha) < g(\alpha)$.

Conversely, suppose that $M \models$ "$\mathbb{B}$ is not weakly (κ, λ)-distributive" and let $\{a_{\alpha,\beta} \colon\ \alpha < \kappa, \beta < \lambda\} \in \mathcal{P}^M(\mathbb{B})$ be such that

$$l := \prod_{\alpha<\kappa} \sum_{\beta<\lambda} a_{\alpha,\beta} > \sum_{g\in(\lambda^\kappa)^M} \prod_{\alpha<\kappa} \sum_{\beta<g(\alpha)} a_{\alpha,\beta} =: r.$$

Let G be a $\mathbb{B}$-generic filter over M with $l \cdot \neg r \in G$. Define

$$f(\alpha) = \min\{\beta < \lambda \colon a_{\alpha,\beta} \in G\}.$$

Then $f \in (\lambda^\kappa)^{M[G]}$. Suppose $g \in (\lambda^\kappa)^M$ is such that $f(\alpha) < g(\alpha)$ for every $\alpha < \kappa$. Then $r \geqslant \prod_{\alpha<\kappa} \sum_{\beta<g(\alpha)} a_{\alpha,\beta} \in G$, a contradiction. This completes the proof. $\square$

17. Maximal almost disjoint families

A collection $\mathcal{A} \subset \mathcal{P}(\kappa)$ (κ an infinite cardinal) is called *almost disjoint* (briefly: *a.d.*) on κ, if $\mathcal{A}$ consists of sets of cardinality κ and $|a \cap b| < \kappa$ whenever $a, b \in \mathcal{A}$ are distinct. A family $\mathcal{A} \subset \mathcal{P}(\kappa)$ is *maximal almost disjoint* (briefly: a *m.a.d. family*) if it is a maximal with respect to inclusion a.d. family on κ. It is easy and well-known that there is an a.d. family of size 2^ω on ω. More generally, if $2^{<\kappa} = \kappa$ then there is an a.d. family on κ of size 2^κ. Indeed, if we identify κ with $2^{<\kappa}$ then setting $a_f = \{\sigma \in 2^{<\kappa} \colon \sigma \subset f\}$, where $f \colon \kappa \to 2$, we get an a.d. family $\{a_f\}_{f \in \{0,1\}^\kappa}$ on κ of size 2^κ. Thus, if CH is true then there is an a.d. family on ω_1 of size 2^{ω_1}. On the other hand, it is well-known that every m.a.d. family on κ has size $> \kappa$ provided κ is regular. Indeed, if $\{a_\alpha\}_{\alpha<\kappa}$ is an a.d. family on a regular cardinal κ then picking $x_\alpha \in a_\alpha \setminus \bigcup_{\xi<\alpha} a_\xi$ we obtain a set $b = \{x_\alpha \colon \alpha < \kappa\}$ which has cardinality κ and which is almost disjoint from each a_α.

We show that the sentence "there exists an a.d. family on ω_1 of size 2^{ω_1}" is independent of ZFC+$2^\omega = 2^{\omega_1} = \omega_3$.

Theorem 17.1 (Baumgartner). *If* Con(ZFC) *then* Con(ZFC+ *"every a.d. family on* ω_1 *has size* $< 2^{\omega_1}$*"*).

Proof. Let M be a ctm of ZFC+GCH and let $\mathbb{P}$ be the Cohen forcing of size ω_3. Let G be a $\mathbb{P}$-generic filter over M. Then $\mathbb{P}$ preserves cardinals since it is ccc and $M[G] \models 2^\omega_1 \geqslant \omega_3$. Suppose that $\mathcal{A} \subset \mathcal{P}^{M[G]}(\omega_1)$ is an a.d. family in $M[G]$ of size ω_3. Let $\mathcal{A}'$ be a name for $\mathcal{A}$. Choose $\tau \in M$ and $q \in G$ such that $q \Vdash$"$\tau \colon \widehat{\omega_3} \to \mathcal{A}'$ is a bijection and $\mathcal{A}'$ is an a.d. family on $\widehat{\omega_3}$". Fix $\{\alpha, \beta\} \in [\omega_3]^2$. Set $T = \{\gamma < \omega_1 \colon i_{\mathbb{P}}(q) \cdot \| \sup(\tau(\widehat{\alpha}) \cap \tau(\widehat{\beta})) = \widehat{\gamma}\| > 0\}$. For each $\gamma \in T$ choose $p_\gamma \leqslant q$ with $p_\gamma \Vdash \sup(\tau(\widehat{\alpha}) \cap \tau(\widehat{\beta})) = \widehat{\gamma}$. Then $\{p_\gamma\}_{\gamma \in T}$ forms an antichain in $\mathbb{P}$. Thus $|T| \leqslant \omega$. Let $\varphi(\{\alpha, \beta\}) = \sup T$. Thus we have defined in M a map $\varphi \colon [\omega_3]^2 \to \omega_1$. Let $f = \mathrm{val}_G(\tau)$. Observe that

$$M[G] \models f(\alpha) \cap f(\beta) \subset \varphi(\{\alpha, \beta\})$$

for $\{\alpha, \beta\} \in [\omega_3]^2$. As $M \models \omega_3 = (2^{\omega_1})^+$, we can apply the theorem of Erdös-Rado in M to obtain a set $K \in [\omega_3]^{\omega_2}$ and $\xi < \omega_1$ such that $\varphi(\{\alpha, \beta\}) = \xi$ for all $\{\alpha, \beta\} \in [K]^2$. Then $\mathcal{B} = \{f(\alpha)\}_{\alpha \in K}$ is an a.d. family on ω_1 in $M[G]$ of size ω_2. Furthermore the intersection of each two distinct elements of $\mathcal{B}$ is contained in ξ. Define

$$g(\alpha) = f(\alpha) \setminus \bigcup_{\eta \in \alpha \cap K} (f(\alpha) \cap f(\eta)),$$

for $\alpha \in K$. We get a disjoint collection of nonempty subsets of ω_1 of size ω_2, a contradiction. $\square$

18. Kurepa trees

Recall that a *Kurepa tree* is a tree T with height ω_1, such that each level of T is countable and there are at least ω_2 paths through T. Here a *path* through T is a linearly ordered subset of T which intersects each nonempty level of T, a maximal linearly ordered subset of T is called a *branch*. We denote by $\mathrm{Lev}_\alpha(T)$ the α-th level of T, i.e. the set of all elements $x \in T$ such that the order type of $\{y \in T \colon y < x\}$ is α. The height of T is denoted by $\mathrm{ht}(T)$, this is the minimum of ordinals $\alpha \geqslant 0$ such that $\mathrm{Lev}_\alpha(T) = \emptyset$. For $x \in T$ we denote by $\mathrm{ht}_T(x)$ the unique ordinal α such that $x \in \mathrm{Lev}_\alpha(T)$. A *Kurepa tree* is a tree T of height ω_1, with countable levels and with at least ω_2 paths.

We define two posets, the second one "adds a Kurepa tree" that is, assuming CH, in a generic extension there exists a Kurepa tree. However, there is no proof of $\mathrm{Con}(ZF) \implies \mathrm{Con}(ZFC+$"there are no Kurepa trees"$)$ since the last sentence implies that ω_2 is inaccessible in L (see Kunen [1, Exercise (B9) on page 240]).

A natural example of a tree is $T = \lambda^{<\kappa}$ with inclusion of maps. If $\lambda = 2$ then it is called the *complete binary tree* of height κ. By a *subtree* of a tree T we mean a subset $P \subset T$ with the property $x \in P \,\&\, y < x \implies y \in P$.

Fix an uncountable cardinal κ. The *Jech κ-poset* $\mathbb{J}_\kappa$ is the set of all subtrees p of $2^{<\kappa}$ such that there exists $\alpha < \kappa$ with the following properties

$$\mathrm{ht}(p) = \alpha + 1 \,\&\, \forall\, \xi < \alpha\ (\forall\, s \in \mathrm{Lev}_\xi(p)\ (s^\frown(0), s^\frown(1) \in p) \,\&\, |\mathrm{Lev}_\xi(p)| < \kappa)$$

and

$$\forall\, s \in p\ \exists\, t \in \mathrm{Lev}_\alpha(p)(s \subset t).$$

For $p, q \in \mathbb{J}_\kappa$ define $p \leqslant q$ iff $q = \{s \in p\colon\ \mathrm{ht}_p(s) < \mathrm{ht}(q)\}$. Observe that if κ is regular then $|p| < \kappa$ for each $p \in \mathbb{J}_\kappa$.

Next we define the *Jensen $\diamondsuit^+$ poset* as

$$\mathbb{J}^+ = \{(p, \mathcal{S})\colon p \in \mathbb{J}_{\omega_1} \,\&\, \mathcal{S} \in [2^{\omega_1}]^{\leqslant\omega} \,\&\, (\forall\, f \in \mathcal{S})\ f \mid (\mathrm{ht}(p) - 1) \in p\}.$$

For $(p, \mathcal{S}), (p'\, \mathcal{S}') \in \mathbb{J}^+$ define $(p, \mathcal{S}) \leqslant (p', \mathcal{S}')$ iff $p \leqslant p'$ and $\mathcal{S}' \subset \mathcal{S}$.

We show that in a $\mathbb{J}^+$-generic extension there exists a Kurepa tree, provided that in the ground model CH holds.

Proposition 18.1. *For each cardinal κ of uncountable cofinality, $\mathbb{J}_\kappa$ is ω_1-closed.*

Proof. Let $\{p_n\}_{n\in\omega}$ be a strictly decreasing chain in $\mathbb{J}_\kappa$, let $\alpha_n + 1 = \mathrm{ht}(p_n)$, $\beta = \sup_{n\in\omega} \alpha_n$ and $q = \bigcup_{n\in\omega} p_n$. Then q is a subtree of $2^{<\kappa}$ of height β with all levels of size $< \kappa$. Observe that for every $s \in q$ there is $f_s\colon \beta \to 2$ such that $\{f \mid \xi\colon \xi < \beta\}$ is a path in q which contains s. Such a function f_s can be constructed by simple induction, using the fact that all p_n's have successor height. Set $d = \bigcup_{n\in\omega} \mathrm{Lev}_{\alpha_n}(q)$. Observe that $|d| < \kappa$ since $\mathrm{cf}(\kappa) > \omega$. Define $q' = q \cup \{f_s\colon s \in d\}$. Then $q' \in \mathbb{J}_\kappa$ and $p_n \geqslant q'$ for every $n \in \omega$. □

Proposition 18.2. *$\mathbb{J}^+$ is ω_1-closed.*

Proof. Fix a decreasing chain $\{(p_n, \mathcal{S}_n)\}_{n\in\omega}$ in $\mathbb{J}^+$. Set $\mathcal{S} = \bigcup_{n\in\omega} \mathcal{S}_n$ and let $\alpha_n + 1 = \mathrm{ht}(p_n)$, $\beta = \sup_{n\in\omega} \alpha_n$. By Proposition 18.1 there is $q \in \mathbb{J}_{\omega_1}$ such that $p_n \geqslant q_0$ for $n \in \omega$. We may assume that $\mathrm{ht}(q) = \beta + 1$. Observe that for $f \in \mathcal{S}$ and $n \in \omega$ we have $f \mid \alpha_n \in p_n \subset q$. Thus if we define

$$q' = q \cup \{f \mid \beta\colon f \in \mathcal{S}\},$$

then $(q', \mathcal{S}) \in \mathbb{J}^+$ and $(p_n, \mathcal{S}_n) \geqslant (q', \mathcal{S})$ for every $n \in \omega$. □

Proposition 18.3. *For each $\alpha < \omega_1$ the set $D_\alpha = \{(p, \mathcal{S}) \in \mathbb{J}^+\colon\ \mathrm{ht}(p) > \alpha\}$ is dense in $\mathbb{J}^+$.*

Proof. Fix $(p, \mathcal{S}) \in \mathbb{J}^+$ and let $\mathrm{ht}(p) = \beta + 1$. Using Proposition 18.2, we can define inductively a sequence $\{p_\xi\}_{\xi\leqslant\alpha}$ such that $(p_\xi, \mathcal{S}) \in \mathbb{J}^+$, $\mathrm{ht}(p_\xi) \geqslant \xi + 1$ and $p_\xi \leqslant p$ for $\xi \leqslant \alpha$. Then $(p_\alpha, \mathcal{S}) \leqslant (p, \mathcal{S})$ and $(p_\alpha, \mathcal{S}) \in D_\alpha$. □

Proposition 18.4. *If CH holds, then $\mathbb{J}^+$ is ω_2-cc.*

Proof. First note that if $(p,\mathcal{S}),(p',\mathcal{S}')\in\mathbb{J}^+$ are incompatible then $p\neq p'$. Now observe that $\mathbb{J}_{\omega_1}\subset[2^{<\omega_1}]^{\leqslant\omega}$ and, under CH, the last set has cardinality ω_1. Thus $|\mathbb{J}_{\omega_1}|=\omega_1$ and hence there are no antichains in $\mathbb{J}^+$ of size $>\omega_1$. □

Theorem 18.5. *Let $M\models ZFC+CH$ and let G be a $\mathbb{J}^+$-generic filter over M, set $T=\bigcup\{p\colon(\exists\,\mathcal{S})\ (p,\mathcal{S})\in G\}$. Then $M[G]\models$ "T is a Kurepa tree".*

Proof. By Proposition 18.2 and 18.4, $\mathbb{J}^+$ adds no new subsets of ω and $\mathbb{J}^+$ preserves cardinals, so $\omega_1^{M[G]}=\omega_1^M$ $(2^\omega)^{M[G]}=(2^\omega)^M=\omega_1$ and T is a subtree of $2^{<\omega_1}$ with countable levels. By Proposition 18.3, $\operatorname{ht}(T)=\omega_1$. It remains to check that there are at least ω_2 paths through T. Define

$$\mathcal{B}=\bigcup\{\mathcal{S}\colon\ (\exists\,p)\ (p,\mathcal{S})\in G\}.$$

Fix $f\in\mathcal{B}$. If $\alpha<\omega_1$ then there is p of height $\geqslant\alpha+1$ such that $(p,\mathcal{S})\in G$ for some $\mathcal{S}$. Thus $f\mid\alpha\in p\subset T$. It follows that each $f\in\mathcal{B}$ determines a path through T. Now it suffices to show that $|\mathcal{B}|\geqslant\omega_2$.

In $M[G]$ define an equivalence relation $\sim$ on 2^{ω_1} as $f\sim g$ iff there is $\alpha<\omega_1$ with $f\mid(\omega_1\setminus\alpha)=g\mid(\omega_1\setminus\alpha)$. For $f\in(2^{\omega_1})^M$ define in M,

$$E_f=\{(p,\mathcal{S})\in\mathbb{J}^+\colon\ (\exists\,\alpha<\omega_1)\ (\exists\,g\in\mathcal{S})\ f\sim g\}.$$

Note that $\sim$ is absolute for M and $M[G]$. Observe that E_f is dense in $\mathbb{J}^+$. Indeed, if $(p,\mathcal{S})\in\mathbb{J}^+$, $h\in\mathcal{S}$ and $\alpha+1=\operatorname{ht}(p)$ then $(p,\mathcal{S})\geqslant(p,\mathcal{S}\cup\{f'\})\in E_f$ where $f'(\xi)=f(\xi)$ for $\xi>\alpha$ and $f'(\xi)=h(\xi)$ for $\xi\leqslant\alpha$. Thus $G\cap E_f\neq\emptyset$ which means that

$$M[G]\models(\ \forall\,f\in(2^{\omega_1})^M)\ (\exists\,g\in\mathcal{B})\ f\sim g. \tag{*}$$

Now observe that each equivalence class under $\sim$ has cardinality $|2^{<\omega_1}|=2^\omega=\omega_1$ in $M[G]$. Let $\pi\colon 2_1^\omega\to 2_1^\omega/_\sim$ be tha canonical surjection. By (*), $\pi[(2^{\omega_1})^M]\subset\pi[\mathcal{B}]$. Furthermore, in $M[G]$ we have $|\pi[(2^{\omega_1})^M]|=|(2^{\omega_1})^M|\geqslant\omega_2$. Thus $|\pi[\mathcal{B}]|\geqslant\omega_2$ and also $\mathcal{B}$ has size at least ω_2. This completes the proof. □

19. More about Cohen forcing

Recall that by C_κ we denote the Cohen forcing of size κ, in particular C_ω can be regarded as any non-atomic countable partial order.

Theorem 19.1. $1_{C_\omega}\Vdash(\forall\,f\in\widehat{\omega}^{\widehat{\omega}})\ (\exists\,g\in\widehat{\omega^\omega})\ (\forall\,n\in\widehat{\omega})\ (\exists\,k>n)\ f(k)=g(k).$

Proof. Fix a C_ω-generic filter G over M and fix $f\in(\omega^\omega)^{M[G]}$. Suppose that $p\in C_\omega$ is such that $p\Vdash\overline{f}\in\widehat{\omega}^{\widehat{\omega}}$ and $p\Vdash(\forall\,g\in\widehat{\omega^\omega})\ (\exists\,n\in\widehat{\omega})\ (\forall\,k>n)\ \overline{f}(k)\neq g(k)$, where $\overline{f}$ is a name for f. Let $\{q\in C_\omega\colon q\leqslant p\}=\{p_n\}_{n\in\omega}$, where $p_0=p$.

Now, in M define inductively $q_n\leqslant p_n$ and $m_n\in\omega$ such that $q_n\Vdash\overline{f}(\widehat{n})=\widehat{m_n}$. Let $g=\{(n,m_n)\colon n\in\omega\}$. Let $N\in\omega$ be such that $p_N\Vdash(\forall\,k\geqslant\widehat{N})\ \overline{f}(k)\neq\widehat{g}(k)$. Then, also q_N forces the same formula, but on the other hand, $q_N\Vdash\overline{f}(\widehat{N})=\widehat{g}(\widehat{N})$, a contradiction. □

Now, let us assume that C_ω consists of all finite functions s with $\operatorname{dom}(s)\subset\omega$ and $\operatorname{rng}(s)\subset 2$. Let $M\subset N$ be two transitive models of ZFC. We say that $x\in(2^\omega)^N$ is a *Cohen real over* M if for every dense set $D\subset C_\omega$ such that $D\in M$, there exists $d\in D$ with $d\subset x$. In other words, x is a Cohen real over M iff the set $G=\{p\in C_\omega\colon p\subset x\}$ is a C_ω-generic flter over M.

Observe that the sentence "p forces that r is a Cohen real" is a sentence of the language of ZF, because it can be written formally as:

$$p \Vdash (\forall\, D \in \widehat{P})\ D \text{ is dense in } \widehat{C} \implies (\exists\, d \in D)\ d \subset r,$$

where $C = C_\omega$ and $P = \mathcal{P}(C)$.

Theorem 19.2. *Let $\mathbb{P}$ be a poset such that $1_{\mathbb{P}} \Vdash$ "there exists a Cohen real". Then there exists a complete embedding of C_ω in $\mathrm{RO}(\mathbb{P})$ and consequently $\mathrm{RO}(C_\omega)$ is a complete subalgebra of $\mathrm{RO}(\mathbb{P})$.*

Proof. By the Maximal Principle, there exists r such that $1_{\mathbb{P}} \Vdash$ "r is a Cohen real". Let

$$F = \{n \in \omega \colon (\exists\, k < 2)\ 1_{\mathbb{P}} \Vdash r(\widehat{n}) = \widehat{k}\}.$$

We claim that F is finite. Indeed, otherwise defining $f\colon F \to 2$ so that $1_{\mathbb{P}} \Vdash r(\widehat{n}) = \widehat{f(n)}$ and setting

$$D = \{s \in C_\omega \colon (\exists\, n \in F)\ s(n) = 1 - f(n)\}$$

we define a dense subset of C_ω and $1_{\mathbb{P}} \nVdash \widehat{s} \subset r$ for any $s \in D$, which is a contradiction. Thus F is finite, so without loss of generality assume that $F = \emptyset$ (consider $\omega \setminus F$ instead of ω).

Define $f\colon C_\omega \to \mathrm{RO}(\mathbb{P})$ by setting

$$f(s) = \|\widehat{s} \subset r\|_{\mathbb{P}} = \sum\nolimits^{\mathrm{RO}(\mathbb{P})} \{p \in \mathbb{P} \colon p \Vdash \widehat{s} \subset r\}.$$

We will check that f is a complete embedding of posets. Then, by Corollary 5.2, f extends to a complete monomorphism of $\mathrm{RO}(C_\omega)$ into $\mathrm{RO}(\mathbb{P})$.

Clearly, f is order preserving and $\perp$-preserving. Let $A \subset C_\omega$ be a maximal antichain. We need to show that $f[A]$ is a maximal antichain in $\mathrm{RO}(\mathbb{P})$. Fix $p_0 \in \mathbb{P}$. By the fact that r is a name for a C_ω-generic real, we have

$$1_{\mathbb{P}} \Vdash (\exists\, s \in \widehat{A})\ s \subset r.$$

Hence, by Theorem 9.4, there exists $p \leqslant p_0$ and $s \in A$ such that $p \Vdash \widehat{s} \subset r$. Hence $p \leqslant f(s)$. It follows that every element of $\mathbb{P}$ is compatible with some element of $f[A]$, so $f[A]$ is a maximal antichain in $\mathrm{RO}(\mathbb{P})$. $\square$

The converse to the above theorem also holds, by Proposition 5.1(b).

References

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