SMOOTH NORMS AND APPROXIMATION IN BANACH SPACES OF THE TYPE $\mathcal{C}(K)$

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1. Introduction

We prove two results about smoothness in Banach spaces of the type $\mathcal{C}(K)$. Both build upon earlier papers of the first named author [3,4]. We first establish a special case of a conjecture that remains open for general Banach spaces and concerns smooth approximation. We recall that a *bump function* on a Banach space X is a function $\beta : X \to \mathbb{R}$ which is not identically zero, but which vanishes outside some bounded set. The existence of bump function of class $\mathcal{C}^1$ implies that the Banach space X is an *Asplund space*, which, in the case where $X = \mathcal{C}(K)$, is the same as saying that K is *scattered*. It is a major unsolved problem to determine whether every Asplund space has a $\mathcal{C}^1$ bump function. Another open problem is whether the existence of just one bump function of some class $\mathcal{C}^m$ on a Banach space X implies that all continuous functions on X may be uniformly approximated by functions of class $\mathcal{C}^m$. It is to this question that we give a positive answer (Theorem 2) in the special case of $X = \mathcal{C}(K)$.

Our second result represents some mild progress with a conjecture made by the second author in [7]. The analysis in that paper of compact spaces constructed using trees suggested that for a compact space K the existence of an equivalent *norm* on $\mathcal{C}(K)$ which is of class $\mathcal{C}^1$ (except at 0 of course) might imply the existence of such a norm which is of class $\mathcal{C}^\infty$. Certainly, this is what happens with norms constructed using linear *Talagrand operators* as in [5,6,7]. The other important (and older) method of obtaining $\mathcal{C}^1$ norms is to construct a norm with locally uniformly rotund dual norm. What we show in Theorem 2 is that, whenever $\mathcal{C}(K)$ admits an equivalent norm with LUR dual norm, then there is also an infinitely differentiable equivalent norm on $\mathcal{C}(K)$.

For background on smoothness and renormings in Banach spaces, including an account of Asplund spaces, we refer the reader to [1]. In particular, an account is given there of the connection between smooth approximability of continuous functions and the existence of smooth *partitions of unity*. Following what seems to be standard practice in the literature, we have chosen to state the formal version of our first theorem (Theorem 1) in terms of partitions of unity, rather than approximation.

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Formalizing a definition which appears implicitly in [6,7], we shall say that a mapping $T : \mathcal{C}(K) \to c_0(K \times \Gamma)$ is a (non-linear) Talagrand operator of class $\mathcal{C}^m$ if

(1) for each non-zero $f \in \mathcal{C}(K)$ there exist $t \in K, u \in \Gamma$ such that $|f(t)| = \|f\|_\infty$ and $(Tf)(t,u) \neq 0$;
(2) each coordinate function $f \mapsto (Tf)(t,u)$ is of class $\mathcal{C}^m$ on the set where it is not zero.

It follows from Corollary 3 of [6] that if $\mathcal{C}(K)$ admits a Talagrand operator of class $\mathcal{C}^m$ then $\mathcal{C}(K)$ has a bump function of the same class. Corollary 2 of [6], or Theorem 2 of [5], shows that if $\mathcal{C}(K)$ admits a *linear* Talagrand operator then $\mathcal{C}(K)$ admits an equivalent $\mathcal{C}^\infty$ norm. Whilst there certainly exist examples of compact K such that $\mathcal{C}(K)$ has a $\mathcal{C}^\infty$ renorming but no linear Talagrand operator (for instance, the Cieselski-Pol space, [1, p260]), it is worth noting that, by the first of the theorems in this paper, a non-linear Talagrand operator exists whenever there is a bump function.

Admissible subsets of compact scattered spaces

In this short section, we establish some notation that will be used consistently later on, and prove one easy result. We consider a compact scattered space K. The derived set K' is defined as usual to be the set of points t in K which are not isolated in K. Successive derived sets $K^{\alpha)}$ are defined by the transfinite recursion

$$K^0 = K; \quad K^\beta = \bigcap_{\alpha<\beta} (K^\alpha)'.$$

There is an ordinal δ such that $K^{(\delta)}$ is non-empty and finite (so that $K^{(\delta+1)} = \emptyset$). For each $t \in K$ there is a unique ordinal $\alpha(t) \leq \delta$ such that $t \in K^{\alpha(t)} \setminus K^{\alpha(t)+1}$. Since t is an isolated point of $K^{\alpha(t)}$, there is a compact open subset V of K such that $V \cap K^{\alpha(t)} = \{t\}$; we choose such a V and call it V_t. For finite subsets B of K, we set $V_B = \bigcup_{t\in B} V_t$.

Lemma 1. *Let B be a nonempty finite subset of K and let $\alpha = \alpha(B)$ be maximal subject to $B \cap K^{(\alpha)} \neq \emptyset\}$. Then $V_B \cap K^\alpha = B \cap K^\alpha$ and hence $V_B \cap K^{\alpha+1} = \emptyset$.*

Proof. Let t be in B. If $\alpha(t) < \alpha$ then $V_t \cap K^{(\alpha)} = \emptyset$, whilst if $\alpha(t) = \alpha$ then $V_t \cap K^{(\alpha)} = \{t\}$. Thus $V_B \cap K^\alpha = B \cap K^\alpha$, as claimed.

We shall say that a finite subset A of K is *admissible* if $s \notin V(t)$ whenever s and t are distinct elements of A.

Lemma 2. *Let K be a compact scattered space and let H be a nonempty, closed subset of K. There is a unique admissible set A with the property that $A \subseteq H \subseteq V(A)$.*

Proof. We start by describing a recursive procedure which constructs one possible admissible A with the required property. Let $\alpha_0 = \max\{\alpha : H \cap K^{(\alpha)} \neq \emptyset\}$; thus, $H \cap K^{(\alpha_0)}$ is a nonempty finite set, which we shall call A_0. If $H \subseteq V_{A_0}$ we set $A = A_0$ and stop. Otherwise, we set $H_1 = H \setminus V_{A_0}$, $\alpha_1 = \max\{\alpha : H_1 \cap K^{(\alpha)} \neq \emptyset\}$, $A_1 = H_1 \cap K^{(\alpha_1)}$, and continue. In this way, we construct a decreasing (and so, necessarily finite) sequence $\alpha_0 > \alpha_1 > \cdots > \alpha_l$ of ordinals, and finite sets $A_j = H \cap K^{(\alpha_j)} \setminus V_{A_0\cup\cdots\cup A_{j-1}}$, in such a way that $H \subseteq V_{A_0\cup\cdots\cup A_l}$. By construction, the set $A = A_0 \cup \cdots \cup A_l$ is admissible.

We now show uniqueness. It will be convenient to proceed by transfinite induction on α_0. Let B be admissible and suppose that $B \subseteq H \subseteq V_B$. By Lemma 1, $\alpha(B) = \alpha_0$ and $B \cap K^{(\alpha_0)} \subseteq C \cap K^{(\alpha_0)} \subseteq V_B \cap K^{(\alpha_0)} = B \cap K^{(\alpha_0)}$. Thus $A_0 \subseteq B$. We now have a closed set $H_1 = H \setminus V_{A_0}$ and an admissible set $B_1 = B \setminus A_0$ with $B_1 \subseteq H_1 \subseteq V_{B_1}$. Since $\alpha_1 = \max\{\alpha : H_1 \cap K^{(\alpha)} \neq \emptyset\} < \alpha_0$, we may use our inductive hypothesis to deduce that $B_1 = A \setminus A_0$, whence $B = A$.

Improving bump functions on $\mathcal{C}(K)$

Let X be a Banach space that admits a bump function of class $\mathcal{C}^m$; so there is a function $\alpha \in \mathcal{C}^m(X)$ such that $\alpha(0) = 1$ while $\alpha(x) = 0$ for $\|x\| \geq 1$. By forming β, where $\beta(x) = \phi(\alpha(x/R))$, with $R > 0$ and $\phi : \mathbb{R} \to \mathbb{R}$ suitably chosen, we obtain a function of class $\mathcal{C}^m$, taking values in $[0, 1]$ satisfying

$$\begin{aligned} \beta(x) &= 1 \qquad \text{when } \|x\| \leq 1 \\ \beta(x) &= 0 \qquad \text{when } \|x\| \geq R. \end{aligned}$$

Of course, if X admits partitions of unity of class $\mathcal{C}^m$, then (starting with a partition of unity each of whose members has support of diameter at most ϵ) we easily obtain a function β satisfying the above conditions, with $R = 1 + \epsilon$ and ϵ an arbitrarily small positive real number. In general, we do not know whether a bump function can always be "improved" in this way. This section is devoted to showing how to achieve such an improvement in the case where X is a space $\mathcal{C}(K)$ equipped with the supremum norm. We start with an elementary and no doubt well known exercise in calculus.

Lemma 3. *Let K be a compact space and let $\theta : \mathbb{R} \to \mathbb{R}$ be of class $\mathcal{C}^m$. Then the mapping $\Theta : \mathcal{C}(K) \to \mathcal{C}(K)$ given by $\Theta(f) = \theta \circ f$ is of class $\mathcal{C}^m$.*

Proof. We proceed by induction on m. For $m = 0$ we are merely assuming θ to be continuous, and continuity of Θ follows from the uniform continuity of θ on bounded subsets of $\mathbb{R}$.

If $m \geq 1$ we consider f, h in $\mathcal{C}(K)$ and apply the mean value theorem point by point to obtain

$$\theta(f(t) + h(t)) - \theta(f(t)) = \theta'(f(t) + \zeta(t)h(t))h(t),$$

with $0 < \zeta(t) < 1$. The uniform continuity of θ' on bounded subsets of $\mathbb{R}$ now tells us that the right hand side of the above equality equals $\theta'(f(t)) + \mathrm{o}(\|h\|_\infty)$. So Θ is differentiable with

$$\mathrm{D}\Theta(f) \cdot h = (\theta' \circ f) \times h.$$

The linear mapping $\mathrm{D}\Theta(f) : \mathcal{C}(K) \to \mathcal{C}(K)$ is thus the operator $\mathrm{M}_{\theta'}$ of multiplication by $\theta' \circ f$. Thus the derivative $\mathrm{D}\Theta : \mathcal{C}(K) \to \mathcal{L}(\mathcal{C}(K))$ may be factored as follows

$$\mathcal{C}(K) \to \mathcal{C}(K) \to \mathcal{L}(\mathcal{C}(K)),$$

where the first factor is $f \mapsto \theta' \circ f$ and the second is the linear isometry $g \mapsto \mathrm{M}_g$. Our inductive hypothesis tells us that the first of these is of class $\mathcal{C}^{m-1}$. So $D\Theta$ is of class C^{m-1} and Θ of class C^m.

Proposition 1. *Let K be a compact space such that $\mathcal{C}(K)$ admits a bump function of class $\mathcal{C}^m$. Then, for all real numbers $\eta > \xi > 0$, there is a function $\beta_{\xi,\eta} : \mathcal{C}(K) \to [0,1]$ of class $\mathcal{C}^m$ such that*

$$\beta_{\xi,\eta}(f) = \begin{cases} 1 & \textit{when } \|f\|_\infty \le \xi \\ 0 & \textit{when } \|f\|_\infty \ge \eta \end{cases}$$

Proof. By hypothesis, there exists a function $\alpha : \mathcal{C}(K) \to \mathbb{R}$, of class $\mathcal{C}^m$, such that $\alpha(0) = 1$ while $\alpha(f) = 0$ for $\|f\|_\infty \ge 1$. As in our introductory remarks, we may assume that α takes values in $[0,1]$. We define $\beta_{\xi,\eta}$ by

$$\beta_{\xi,\eta}(f) = \alpha(\theta \circ f),$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is a function of class $\mathcal{C}^\infty$ chosen so that

$$\theta(x) = \begin{cases} 0 & \text{when } |x| \le \xi \\ 1 & \text{when } |x| \ge \eta \end{cases}$$

Construction of a Talagrand operator and of partitions of unity

This section is devoted to a proof of the following theorem.

Theorem 1. *Let K be a compact space and let m be a positive integer or ∞. The following are equivalent:*

(1) *$\mathcal{C}(K)$ admits a bump function of class $\mathcal{C}^m$;*
(2) *$\mathcal{C}(K)$ admits a Talagrand operator of class $\mathcal{C}^m$;*
(3) *$\mathcal{C}(K)$ admits a partitions of unity of class $\mathcal{C}^m$.*

It will be enough to prove that (1) implies both (2) and (3). We start by showing how to construct a Talagrand operator, starting with a bump function on $\mathcal{C}(K)$. As we remarked in the Introduction, the existence of a smooth bump function forces K to be scattered. So we can use the notion of admissible set as developed above. We let $\mathcal{Q}$ be the set of all triples (ξ, η, ζ) in $\mathbb{Q}^3$ with $0 < \xi < \eta < \zeta$, and write $\mathcal{A}$ for the set of all admissible subsets of K. Choose positive real numbers $c(\xi, \eta, \zeta)$ with $\sum_{(\xi,\eta,\zeta)\in\mathcal{Q}} c(\xi, \eta, \zeta) < \infty$. For $0 < \xi < \eta$ let $\beta_{\xi,\eta}$ be as in Proposition 1, and, finally, for $0 < \eta < \zeta$, let $\phi_{\eta,\zeta} : \mathbb{R} \to [0,1]$ be of class $\mathcal{C}^\infty$ with

$$\phi_{\eta,\zeta}(x) = \begin{cases} 0 & \text{when } |x| \le \eta \\ 1 & \text{when } |x| \ge \zeta \end{cases}$$

We define $T : \mathcal{C}(K) \to \ell^\infty(K \times \mathcal{Q} \times \mathcal{A})$ by

$$(Tf)(s, \xi, \eta, \zeta, A) = c(\xi, \eta, \zeta)\beta_{\xi,\eta}(f \times \chi_{K\setminus V_A}) \prod_{t\in A} \phi_{\eta,\zeta}(f(t))\chi_A(s).$$

We notice that, for this expression to be non-zero, we need $A \subseteq F \subseteq V_A$, where F is the closed set $\{t \in K : |f(t)| \ge \eta\}$. Now, we know by Lemma 2 that, for given η and f, there is just one A for this is true. It follows easily that T takes values in $c_0(K \times \mathcal{Q} \times \mathcal{A})$. It is also clear that each coordinate of T, that is to say, each mapping $f \mapsto (Tf)(s, \xi, \eta, \zeta, A)$, is of class $\mathcal{C}^m$.

To show that T has the Talagrand property, we consider $f \neq 0$ and set $F = \{t \in K : |f(t)| = \|f\|_\infty\}$. Let A be the admissible set for which $A \subseteq F \subseteq V_A$ and choose rationals $0 < \xi < \eta < \zeta$ such that $\zeta < \|f\|_\infty$ and $\xi > \|f \times \chi_{K \setminus V_A}\|_\infty$. For any $s \in A$ we have $|f(s)| = \|f\|_\infty$ and $(Tf)(s, \xi, \eta, \zeta) \neq 0$.

We now pass to the construction of partitions of unity. We shall proceed by transfinite recursion on the derived length of K. Recall from Section 2 that there is an ordinal δ such that $K^{(\delta)}$ is finite and non-empty, so that $K^{(\delta+1)}$ is the first empty derived set of K. We assume inductively that, if V is a compact space with $V^{(\delta)} = \emptyset$ and such that $\mathcal{C}(V)$ has a bump function of class $\mathcal{C}^m$, then $\mathcal{C}(V)$ admits $\mathcal{C}^m$ partitions of unity. We need to show that $\mathcal{C}(K)$ also admits $\mathcal{C}^m$ partitions of unity. To do this it will be enough to construct partitions of unity on the finite-codimensional subspace $X = \{f \in \mathcal{C}(K) : f(t) = 0$ for all $t \in K^{(\delta)}$. We shall use the following result from an earlier paper by the second author.

Proposition 2 (Theorem 2 of [6])**.** *Let X be a Banach space, let L be a set and let m be a positive integer or ∞. Let $T : X \to c_0(L)$ be a function such that each coordinate $x \mapsto T(x)_\gamma$ is of class $\mathcal{C}^m$ on the set where it is non-zero. For each finite subset F of L, let $R_F : X \to X$ be of class $\mathcal{C}^m$ and assume that the following hold:*

1. *for each F, the image $R_F[X]$ admits $\mathcal{C}^k$ partitions of unity;*
2. *X admits a $\mathcal{C}^k$ bump function;*
3. *for each $x \in X$ and each $\epsilon > 0$ there exists $\lambda > 0$ such that $\|x - R_F x\| < \epsilon$ if we set $F = \{u \in L : |(Tx)(u)| \geq \lambda\}$.*

Then X admits $\mathcal{C}^m$ partitions of unity.

In applying this result we shall take L to be $K \times \mathcal{Q} \times \mathcal{A}_0$, where $\mathcal{A}_0$ consists of the admissible subsets A such that $A \cap K^{(\delta)} = \emptyset$. The operator T is the Talagrand operator constructed above (though with the argument A restricted to lie in $\mathcal{A}_0$). We have already shown that T takes values in $c_0(L)$ and that the coordinates of T are of class $\mathcal{C}^m$. We define the reconstruction operators R_F as follows: if $F \subset L$ has elements $(s_i, \xi_i, \eta_i, \zeta_i, A_i)$ $(0 \leq i < n)$, we set $V(F) = \bigcup_{i<n} V_{A_i}$ and define $R_F(f) = f \times \chi_{V(F)}$. So $R_F : X \to X$ is a bounded linear operator and the image R_F may be identified with $\mathcal{C}(V(F))$, which, by our inductive hypothesis, admits partitions of unity of class $\mathcal{C}^m$.

It only remains to check that (3) holds, so let $f \in \mathcal{C}(K)$ and $\epsilon > 0$ be given. Let H be the set $\{t \in K : |f(t)| \geq \epsilon\}$ and let A be the admissible set such that $A \subseteq H \subseteq V_A$. For suitably chosen $0 < \xi < \eta < \zeta < \epsilon$ we have

$$(Tf)(s, \xi, \eta, \zeta, A) = c(\xi, \eta, \zeta) > 0$$

for all $s \in A$. We set $\lambda = c(\xi, \eta, \zeta)$ and note that $V(F) \supseteq V_A$. So

$$\begin{aligned} \|f - R_F f\|_\infty &= \|f \times \chi_{K \setminus V(F)}\|_\infty \\ &\leq \|f \times \chi_{K \setminus V_A}\|_\infty < \epsilon. \end{aligned}$$

Infinitely differentiable norms

We shall now prove the second theorem of this paper.

Theorem 2. *Let K be a compact space such that $\mathcal{C}(K)$ admits an equivalent norm with locally uniformly rotund dual norm. Then $\mathcal{C}(K)$ admits an equivalent which is of class $\mathcal{C}^\infty$ on $X \setminus \{0\}$.*

The norm which we construct will be a *generalized Orlicz norm*, defined on the whole of $\ell^\infty(K)$, which we shall show to be infinitely smooth on the subspace $\mathcal{C}(K)$. We recall some definitions. Suppose that, for each $t \in K$, we are given a convex function $\phi_t = \phi(t,\cdot) : [0,\infty) \to [0,\infty)$ satisfying $\phi(t,0) = 0$, $\lim_{x\to\infty} \phi(t,x) = \infty$ (that is to say an *Orlicz function*). The generalized Orlicz space $\ell_{\phi(\cdot)}(K)$ is defined to be the space of all functions $f : K \to \mathbb{R}$ such that $\sum_{t\in K} \phi(t, |f(t)|/\rho) < \infty$ for some $\rho \in (0,\infty)$. The generalized Orlicz norm of such a function is defined to be

$$\|f\|_{\phi(\cdot)} = \inf\{\rho > 0 : \sum_{t\in K} \phi(t, |f(t)|/\rho) \le 1\}.$$

The first of the following lemmas is elementary and the second uses the familiar idea of "local dependence on finitely many coordinates."

Lemma 4. *Suppose that there exist positive real numbers $R < S$ such that $\phi(t,R) = 0$ and $\phi(t,S) \ge 1$ for all $t \in K$. Then $\ell_{\phi(\cdot)}(K) = \ell^\infty(K)$ and*

$$R\|f\|_{\phi(\cdot)} \le \|f\|_\infty \le S\|f\|_{\phi(\cdot)}.$$

Lemma 5. *Let $\phi(\cdot)$, R and S be as in the preceding lemma, and let X be a linear subspace of $\ell^\infty(K)$. Suppose that, whenever $f \in X$ and $\|f\|_{\phi(\cdot)} = 1$, there exists a positive real number δ and a finite subset F of K such that $g(t) = 0$ whenever $g \in X$, $\|f - g\|_\infty < \delta$ and $t \notin F$. Assume further that each of the functions $\phi(t, cdot)$ is of class $\mathcal{C}^\infty$. Then the generalized Orlicz norm $\|\cdot\|_{\phi(\cdot)}$ is of class $\mathcal{C}^\infty$ on $X \setminus \{0\}$.*

Proof. If f, F and δ are as in the statement of the lemma, then the function Φ defined by

$$\Phi(g) = \sum_{t\in K} \phi(t, |g(t)|)$$

is of class $\mathcal{C}^\infty$ on $\{g \in X : \|f - g\|_\infty < \delta\}$, since it coincides with the finite sum $\sum_{t\in F} \phi(t, |g(t)|)$ of given $\mathcal{C}^\infty$ functions. Thus our hypothesis tells that there is an open subset U of X containing the set $\{f \in X : \|f\|_{\phi(\cdot)} = 1\}$ and such that Φ is $\mathcal{C}^\infty$ on U. We define $V = \{h, \rho) \in (X \setminus \{0\}) \times (0,\infty) : \rho^{-1}h \in U$, an open set in X. On V we define $\Psi(h,\rho) = \Phi(\rho^{-1}h)$, which is of class $\mathcal{C}^\infty$. For each $h \in X \setminus \{0\}$ there is a unique $\rho = \|h\|_{\phi(\cdot)}$ such that $(h,\rho) \in V$ and $\Psi(h,\rho) = 1$. Moreover, we may calculate the partial derivative

$$\mathrm{D}_2\Psi(h,\rho) = -\rho^{-2} \sum_t \phi_t'(\rho^{-1}|h(t)|),$$

and note that this is nonzero when $\rho = \|h\|_{\phi(\cdot)}$, since $\phi_t'(x) > 0$ whenever $\phi_t(x) > 0$. Thus the Implicit Function Theorem may be applied to conclude that $\|\cdot\|_{\phi(\cdot)}$ is of class $\mathcal{C}^\infty$ on $X \setminus \{0\}$.

To choose suitable Orlicz functions ϕ_t in our theorem, we shall need to use the special properties of K. The assumption that $\mathcal{C}(K)^*$ has an equivalent LUR dual norm implies (and, by a theorem of Raja [9], is actually equivalent to) the compact space K being σ-discrete. So we may assume that there are pairwise disjoint subsets D_i of K, each one discrete in its subspace topology, with $K = \bigcup_{i\in\omega} D_i$. We note in passing that we are not assuming the D_i to be closed, merely that D_i has empty intersection with its derived set D_i'. We fix positive real numbers $r_i < 1$ with $\prod_{i\in\omega} r_i > 0$ and, for $t \in K$, define two real numbers

$$\alpha(t) = \prod\{r_i : t \in \bar{D}_i\}$$
$$\beta(t) = \prod\{r_i : t \in D_i'\}.$$

We notice that $\beta(t) = \alpha(t) \times r_j$ where j is the (unique) natural number such that $t \in D_j$. (Note that it is here (and only here) that we use the discreteness hypothesis that $D_j \cap D_j' = \emptyset$.) In particular, therefore, $0 < \alpha(t) < \beta(t) < 1$. So we may choose an infinitely differentiable Orlicz function ϕ_t such that

$$\phi_t(x) = 0 \quad \text{when } x \le \alpha(t)$$
$$\phi_t(x) > 1 \quad \text{when } x \ge \beta(t).$$

We are going to show that Lemma 5 may be applied to these Orlicz functions and the subspace $\mathcal{C}(K)$ of ℓ^∞. It is convenient to state one of the ingredients of this proof as a property of the functions α and β.

Lemma 6. *Let t_n be a sequence of distinct elements of K which converges to some $t \in K$. Then $\beta(t) \le \liminf \alpha(t_n)$.*

Proof. By taking subsequences and diagonalizing, we may assume that $\alpha(t_n)$ tends to a limit as $n \to \infty$, and also that, for each $i \in \omega$, either all of $t_{i+1}, t_{i+2}, \dots$ are in $\bar{D}_i$, or else none is. Let M be the set of natural numbers i such that $t_{i+1}, \dots$ are in $\bar{D}_i$. Then, for each n we have

$$\prod_{i<n,\ i\in M} r_i \times \prod_{i\ge n} r_i \le \alpha(t_n) \le \prod_{i\in M} r_i,$$

whence $\alpha(t_n) \to \prod_{i\in M} r_i$ as $n \to \infty$. On the other hand, since the t_n are distinct and $t_n \in D_i$ whenever $n > i \in M$, it must be that the limit point t is in the derived set D_i' whenever $i \in M$. Thus

$$\beta(t) = \prod\{r_i : t \in D_i'\} \le \prod_{i\in M} r_i.$$

To complete the proof of the theorem, we consider $f \in \mathcal{C}(K)$ with $\|f\|_{\phi(\cdot)} = 1$. If no δ and F exist with the property of Lemma 5, there exist a sequence (f_n) in $\mathcal{C}(K)$ converging uniformly to f and a sequence of distinct elements (t_n) of K such that $\phi(t_n, |f_n(t_n)|) > 0$ for all n. For this to be the case, if must be that $|f_n(t_n)| \ge \alpha(t_n)$. Extracting a subsequence, we may suppose that the sequence (t_n) converges to some $t \in K$. Now by uniform convergence and the continuity of f we have $f(t) = \lim_n f_n(t_n)$, so that $|f(t)| \ge \beta(t)$ by Lemma 6. Thus $\phi(t, |f(t)|) > 1$ and $\|f\|_{\phi(\cdot)} > 1$, a contradiction, which ends the proof.

Remarks

It follows from Lemma 4 that the norm constructed in Theorem 2 satisfies

$$\|f\|_\infty \le \|f\|_{\phi(\cdot)} \le \alpha^{-1}\|f\|_\infty,$$

where $\alpha = \prod_{i\in\omega} r_i$. Since we may arrange for α to be arbitrarily close to 1, we have shown that the supremum norm may be uniformly approximated on bounded subsets of $\mathcal{C}(K)$ by infinitely differentiable norms. We do not know whether all equivalent norms on $\mathcal{C}(K)$ (with K σ-discrete) can be thus approximated. In particular, we do not know this for the locally uniformly rotund norm recently constructed by the second author [8]. Of course it follows from the results of the present paper that any equivalent norm on our $\mathcal{C}(K)$ (like any other continuous function) can be uniformly approximated on bounded subsets by infinitely differentiable *functions*.

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