On the range of the derivative of Gâteaux-smooth functions on separable Banach spaces

ROBERT DEVILLE
Mathematiques Pures de Bordeaux, Université de Bordeaux
351, cours de la libération, 33400, Talence, France
email : deville@math.u-bordeaux.fr

PETR HÁJEK[1]
Mathematical Institute, Czech Academy of Science
Žitná 25, Prague, Czech republic
email : hajek@math.cas.cz

Abstract : We prove that there exists a Lipschitz function from ℓ^1 into $I\!R^2$ which is Gâteaux-differentiable at every point and such that for every $x, y \in \ell^1$, the norm of $f'(x) - f'(y)$ is bigger than 1. On the other hand, for every Lipschitz and Gâteaux-differentiable function from an arbitrary Banach space X into $I\!R$ and for every $\varepsilon > 0$, there always exists two points $x, y \in X$ such that $\|f'(x) - f'(y)\|$ is less than ε. We also construct, in every infinite dimensional separable Banach space, a real valued function f on X, which is Gâteaux-differentiable at every point, has bounded non-empty support, and with the properties that f' is norm to weak* continuous and $f'(X)$ has an isolated point a, and that necessarily $a \neq 0$.

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1) Introduction.

Let f be a mapping from a Banach space X into a Banach space Y which is Gâteaux-differentiable at every point. Our purpose is the study of the range of the derivative of f. We denote this range $f'(X)$. Let us recall that sufficient conditions on a subset A of a dual Banach space X^* so that it is the range of a real valued function on X which is Frechet-differentiable at each point have been obtained in [BFKL], [BFL], [AFJ] and [G1]. In this case, it was noticed in [AD] that whenever X is an infinite dimensional Banach space with separable dual, there exists a $\mathcal{C}^1$-smooth real valued function on X with bounded support and such that $f'(X) = X^*$. On the other hand, it follows from [H] that if f is a function on c_0 with locally uniformly continuous derivative, then $f'(c_0)$ is included in a countable union of norm compact subsets of ℓ^1. The structure of the range of f' whenever f' satisfies a Holder condition has been investigated in [G2]. In the case of functions or mappings which are Gâteaux-differentiable at each point, it was observed in [ADJ] that $f'(X)$ can coincide with $\mathcal{L}(X,Y)$. We shall investigate here phenomena which can occur when f is Gâteaux-differentiable, but not when f is Frechet-differentiable. In particular, for each infinite dimensional separable Bananach space X, we shall construct in section 2 a Gâteaux-differentiable function f on X, with bounded support, and such that for all $x \neq 0$, $\|f'(x) - f'(0)\| \geq 1$. In section 3, we shall consider the following question : let X, Y be two Banach spaces. Is it possible to construct a Lipschitz continuous mapping $f : X \to Y$, Gâteaux-differentiable at each point, and such that, for all $x, y \in X$, $x \neq y$, we have $\|f'(x) - f'(y)\| \geq 1$? Clearly, this is not possible whenever $\mathcal{L}(X,Y)$ is separable. We shall prove that this is not possible either whenever $Y = I\!R$, but such a construction will be carried out whenever $(X,Y) = (\ell^1, I\!R^2)$ and whenever $(X,Y) = (\ell^p, \ell^q)$ with $1 \leq p \leq q < +\infty$.

2) Isolated points in the range of the derivative of a function.

Let X be a Banach space, and f be a real valued function defined on X. If f is Frechet-differentiable at every point, then Maly's Theorem asserts that the range of f', denoted $f'(X)$, is connected. If f is Gâteaux-differentiable at every point of X and if f' is norm to weak* continuous, then $f'(X)$ is weak* connected. Therefore, if f is not affine, no point of $f'(X)$ is isolated in $f'(X)$ endowed with the weak*-topology. This result remains true even if f' is not assumed to be norm to weak* continuous, as shown by the following proposition. We shall see later that in this case $f'(X)$ is not necessarily norm connected.

Proposition : *Let X be an infinite dimensional Banach space, and let f be a real valued locally Lipschitz and Gâteaux-differentiable function on X. Then either f is affine, or, for every $x \in X$, $f'(x)$ lies in the weak* closure of $f'(X) \backslash \{f'(x)\}$.*

Remark : J. Saint Raymond constructed a mapping f from $I\!R^2$ into $I\!R^2$, Frechet-differentiable at each point, and so that $\{det(f'(x)); x \in I\!R^2\} = \{0,1\}$. Therefore $f'(I\!R^2)$ is not connected and for every $x \in I\!R^2$, $f'(x) \notin f'(X)\backslash\{f'(x)\}$. Consequently, there is no analog of Maly's theorem and of the above proposition for vector valued mappings.

Proof : Let f be a real valued locally Lipschitz and Gâteaux-differentiable function on X which is not affine. Therefore, $Card\big(f'(X)\big) \geq 2$. In order to get a contradiction, assume moreover that $f'(X) = A \cup \{y\}$, where $A \neq \emptyset$ and $y \notin \overline{A}^{w^*}$. Since $y \in f'(X)$, there exists $x \in X$ such that $y = f'(x)$. Replacing f by $f(x+.)$, we can assume that $x = 0$. Fix also $x_0 \in X$ such that $f'(x_0) \in A$. Since $y \notin \overline{A}^{w^*}$, there exists $x_1, x_2, ..., x_n \in X$ and $\varepsilon > 0$ such that, if we denote

$$\widetilde{y} = \big(y(x_1), y(x_2), ..., y(x_n)\big) \in I\!R^n$$

and

$$\widetilde{A} = \big\{\big(z(x_1), z(x_2), ..., z(x_n)\big); z \in A\big\} \subset I\!R^n$$

then, for every $\widetilde{z} \in \widetilde{A}$, $\|\widetilde{z} - \widetilde{y}\| > \varepsilon$. If we denote $\widetilde{\widetilde{y}} = \big(y(x_0), \widetilde{y}\big) \in I\!R^{n+1}$ and $\widetilde{\widetilde{A}} = \big(z(x_0), z(x_1), z(x_2), ..., z(x_n)\big); z \in A\big\} \subset I\!R^{n+1}$, then we also have that, for every $\widetilde{\widetilde{z}} \in \widetilde{\widetilde{A}}$, $\|\widetilde{\widetilde{z}} - \widetilde{\widetilde{y}}\| > \varepsilon$. Define $F : I\!R^{n+1} \to I\!R$ by

$$F(t_0, t_1, t_2, ..., t_n) = f\big(\sum_{i=0}^{n} t_i x_i\big)$$

Since F is Lipschitz continuous and Gâteaux-differentiable on $I\!R^{n+1}$, it is Fréchet-differentiable on $I\!R^{n+1}$ and

$$F'(t_0, t_1, t_2, ..., t_n) = \big(f'\big(\sum_{i=0}^{n} t_i x_i\big)(x_j)\big)_{j=0}^{n} \in \widetilde{\widetilde{A}} \cup \{\widetilde{\widetilde{y}}\}$$

Moreover $F'(0, 0, ..., 0) = \widetilde{\widetilde{y}}$, $F'(1, 0, ..., 0) \in \widetilde{\widetilde{A}}$. Therefore $F'(I\!R^{n+1})$ is not connected and this contradicts the Theorem of Maly.

¿From now on, we say that a real valued function on an infinite dimensional Banach space X is a *bump* function if it has bounded non empty support. We shall denote $B(r)$ the set of all $x^* \in X^*$ such that $\|x^*\| < r$. If E is a Banach space, $x \in E$ and $r > 0$, we denote $B_E(x, r)$ (resp. $\overline{B}_E(x, r)$) the open ball (resp. closed ball) in E of center x and radius r. If f is a continuous and Gâteaux-differentiable bump function on X, then, according to the Ekeland variational principle, the *norm* closure of $f'(X)$ contains a ball $B(r)$ for some $r > 0$. A natural conjecture would be that the norm closure of $f'(X)$ is norm connected, or at least that $f'(X)$ does not contain an isolated point. This is not so as shown by the following construction.

Theorem 1 : *Let X be an infinite dimensional separable Banach space. Then, there exists a bump function f on X such that f is Gâteaux-differentiable at every point, f' is norm to weak* continuous and $\|f'(0)–f'(x)\| \geq 1$ whenever $x \neq 0$. If X^* is separable, we can assume moreover that f is $\mathcal{C}^1$ on $X\backslash\{0\}$.*

Remark : According to the above discussion, 0 is not an isolated point of $f'(X)$, so necessarily $f'(0) \neq 0$.

Proof : We shall use two lemmas.

Lemma 1 : *Let X be a Banach space, U be an open connected subset of X^* such that $0 \in U$ and $x^* \in U$. Assume there exists on X a Lipschitz continuous bump function which is Gâteaux-differentiable (resp. Frechet-differentiable) at every point. Then there exists a Lipschitz continuous bump function β on X which is Gâteaux-differentiable (resp. Frechet-differentiable) at every point, such that $\beta'(X) \subset U$ and $\beta'(x) = x^*$ for all x in a neighbourhood of 0.*

Proof of lemma 1 : Since U is connected, there exists finitely many points $x_0^*, x_1^*, ..., x_n^* \in U$ such that $x_0^* = 0$, $x_n^* = x^*$, and the segments $[x_i^*, x_{i+1}^*]$ are included in U. The polygonal line $R = \bigcup_{i=0}^{n-1} [x_i^*, x_{i+1}^*]$ is compact, therefore there exists $\varepsilon > 0$ such that $R + B(\varepsilon) \subset U$. Let b be a Lipschitz bump function on X which is Gâteaux-differentiable (resp. Frechet-differentiable) at every point of X. By translation, we can assume that $b(0) \neq 0$. Replacing $b(x)$ by $\lambda_1 b(\lambda_2 x)$, we can also assume that there exists $0 < \delta < 1$ such that $b(x) \geq 1$ whenever $\|x\| \leq \delta$ and that the support of b is included in the unit ball. Composing b with a suitable $\mathcal{C}^\infty$-smooth function from $\mathbb{R}$ into $\mathbb{R}$, we can assume moreover that $b(x) = 1$ whenever $\|x\| \leq \delta$, and that $0 \leq b(x) \leq 1$ for all $x \in X$. By adding if necessary points on the polygonal line R, we can assume that for all $i \in \{1, 2, ..., n\}$, $\|x_i^* - x_{i-1}^*\| < \varepsilon/\|b'\|_\infty$. Define

$$b_i(x) = b(x).(x_i^* - x_{i-1}^*)(x)$$

We have $b_i'(x) = (x_i^* – x_{i-1}^*)(x).b'(x) + b(x).(x_i^* – x_{i-1}^*)$, with $b(x).(x_i^* - x_{i-1}^*) \in [0, x_i^* - x_{i-1}^*]$ and $\|(x_i^* - x_{i-1}^*)(x).b'(x)\| < \varepsilon$ for all $x \in X$, therefore $b_i'(X) \subset [0, x_i - x_{i-1}] + B(\varepsilon)$. Finally, set

$$\beta(x) = \sum_{i=1}^{n} \delta^{i-1} b_i\big(x/\delta^{i-1}\big)$$

β is a Lipschitz continuous bump function on X which is Gâteaux-differentiable (resp. Frechet-differentiable) at every point. Let $x \in X$ and assume that $\delta^i < \|x\| \le \delta^{i-1}$ for $1 \le i \le n$. If $j > i$, $\|x/\delta^{j-1}\| > 1$, so $b_j\big(y/\delta^{j-1}\big) = 0$ for all y in a neighbourhood of x and $b_j'\big(x/\delta^{j-1}\big) = 0$. If $j < i$, $\|x/\delta^{j-1}\| \le \delta$, so $b_j'\big(x/\delta^{j-1}\big) = x_j^* - x_{j-1}^*$. Therefore

$$\beta'(x) = \sum_{j=1}^{i-1}(x_j^* - x_{j-1}^*) + b_i'\big(x/\delta^i\big) = x_{i-1}^* + b_i'\big(x/\delta^i\big) \in [x_{i-1}, x_i] + B(\varepsilon)$$

Moreover, if $\|x\| \le \delta^n$, then $\beta'(x) = x_n^* = x^*$. Thus $\beta'(x) = x^*$ for all x in a neighbourhood of 0 and $\beta'(X) \subset R + B(\varepsilon) \subset U$.

Lemma 2 : *Let X, Y be two Banach spaces, $a \in X$, V be an open neighbourhood of a, and $f : V \to Y$ be continuous on V and Gâteaux-differentiable at every point of $V\backslash\{a\}$. If $f'(x)$ has a weak* limit ℓ as x tends to a, then f is Gâteaux-differentiable at a and $f'(a) = \ell$.*

Proof of lemma 2 : Fix $h \in X$. The mapping ϕ_h defined on the real line by $\phi_h(t) = f(a + th)$ whenever $t \neq 0$, $\phi_h'(t) = f'(a + th).h$ tends to $\ell.h$ as t tends to 0. Therefore f is differentiable at a in the direction h and $f'(a).h = \ell.h$. This proves that f is Gâteaux-differentiable at a and $f'(a) = \ell$.

In order to prove the theorem, let $a \in X^*$ such that $1 < \|a\| < 2$. Let (u_n) be a dense sequence in X and

$$V_n = \big\{x^* \in X^*; |x^*(u_i) - a(u_i)| < 1/2^n \text{ for all } i \in \{1, ..., n\}\big\}$$

$(V_n)_{n \ge 0}$ be a decreasing sequence of weak* open subsets contaning a so that, if $y_n \in V_n$ and if (y_n) is bounded, then (y_n) converges to a for the weak*-topology. Moreover, $W_n = V_n \cap \{x^* \in X^*; 1 < \|x^* - a\| < 2\}$ is connected for each n. Let $(x_n) \subset X^*$ be a sequence such that $x_1 = 0$ and for every n, $x_n \in W_n$. For each n, $1 < \|x_n - a\| < 2$ and (x_n) converges to a for the weak* topology. $W_n - x_n = \{x - x_n; x \in W_n\}$ is a norm open connected subset of X^* containing 0. Since $x_{n+1} \in W_{n+1} \subset W_n$, we also have $x_{n+1} - x_n \in W_n - x_n$. Since X is separable (resp. X^* is separable) there exists on X a Lipschitz continuous bump function which is Gâteaux-differentiable (resp. Frechet-differentiable) at each point. According to lemma 1, there exists a Lipschitz continuous bump b_n which is Gâteaux-differentiable (resp. Frechet-differentiable) at every point, such that $b_n'(X) \subset W_n - x_n$, with support in the unit ball and such that $b_n'(x) = x_{n+1} - x_n$ for all x satisfying $\|x\| < \delta_n$. Denote $c_1 = 1$ and, for $n \ge 2$, $c_n = \prod_{i=1}^{n-1} \delta_n$. Define

$$b(x) = \sum_{n=1}^{+\infty} c_n b_n\big(x/c_n\big)$$

b has bounded support since $b(x) = 0$ whenever $\|x\| \geq 1$. On $X\backslash\{0\}$ this sum is locally finite, so b is Gâteaux-differentiable (resp. Frechet-differentiable) at each point of $X\backslash\{0\}$. If $\delta_n \leq \|x\| < \delta_{n+1}$, then we have $b'(x) = x_n + b_n'(x) \in W_n$, so $\|b'(x)\|$ is uniformly bounded in x, $b'(X\backslash\{0\}) \subset X^*\backslash B(a,1)$, and $b'(x) \overset{w^*}{\longrightarrow} a$ as $x \to 0$. Lemma 2 then shows that b is Gâteaux-differentiable at 0 and that $b'(0) = a$.

3) Can all the derivatives be far away from each other?

We first notice that, under mild regularity assumptions, the answer to the above question is negative for functions.

Proposition : *Let X be a Banach space and $f : X \to I\!R$ be a Lipschitz continuous, everywhere Gâteaux-differentiable function. Then, for every $x \in X$ and every $\varepsilon > 0$, there exists $y, z \in B_X(x, \varepsilon)$ such that $\|f'(y) - f'(z)\| \leq \varepsilon$.*

Proof : We shall actually show that if $f : X \to I\!R$ is locally uniformly continuous and everywhere Gâteaux-differentiable, then, for every $x \in X$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $h \in X$, $\|h\| \leq \delta$, there exists $y \in B_X(x, \varepsilon)$ such that $\|f'(y+h) - f'(y)\| \leq \varepsilon$.
Fix $x \in X$ and $\varepsilon_0 > 0$ such that f is uniformly continuous on $B_X(x, 2\varepsilon_0)$. Fix also $0 < \varepsilon < \varepsilon_0$. By uniform continuity, there exists $\delta > 0$ such that $|f(z) - f(y)| < \varepsilon^2/4$ whenever $y, z \in B_X(x, 2\varepsilon_0)$ and $\|z - y\| \leq \delta$. Without loss of generality, we can assume that $\delta < \varepsilon/2$. Take any $h \in X$ such that $\|h\| \leq \delta$. Define $\varphi : X \to I\!R$ by $\varphi(y) = f(y+h) - f(y)$ if $\|y - x\| \leq \varepsilon_0$ and $\varphi(y) = +\infty$ otherwise. The function φ is lower semi-continuous on X and, for all $y \in B_X(x, \varepsilon_0)$, $-\varepsilon^2/4 < \varphi(y) < \varepsilon^2/4$. In particular, $\varphi(x) < \inf_{y \in X} \varphi(y) + \varepsilon^2/2$. The Ekeland variational principle then tells us the existence of $y \in X$ such that $\|y - x\| \leq \varepsilon/2$ and for all $u \in X$, $\varphi(u) \geq \varphi(y) - \varepsilon\|u - y\|$. Since $\|y - x\| \leq \varepsilon/2 < \varepsilon_0$, the function φ is Gâteaux differentiable at y and we obtain $\|\varphi'(y)\| \leq \varepsilon$. Hence, if we denote $z = y + h$, $\|f'(y) - f'(z)\| \leq \varepsilon$, and we have $\|z - x\| \leq \|h\| + \|y - x\| < \varepsilon$.

The derivatives of a Frechet differentiable mapping cannot be far away from each other for mappings which are everywhere Frechet-differentiable.

Proposition : *Let X, Y be separable Banach spaces and $f : X \to Y$ be an everywhere Fréchet-differentiable locally uniformly continuous mapping. Then, for every $x \in X$ and every $\varepsilon > 0$, there exists $y, z \in B_X(x, \varepsilon)$, $y \neq z$, such that $\|f'(y) - f'(z)\| \leq \varepsilon$.*

Proof : Fix $\varepsilon > 0$ and $n_0 > 0$ such that f is uniformly continuous on $B_X(x, \varepsilon + 1/n_0)$. For each $n \geq 1$, define

$$A_n = \left\{ y \in B_X(x, \varepsilon),\ \|f(y+h) - f(y) - f'(y).h\| \leq \varepsilon\|h\| \text{ whenever } \|h\| \leq 1/n \right\}$$

Since $B_X(x,\varepsilon) = \bigcup_{n\geq n_0} A_n$, there exists $n_1 \geq n_0$ and $u \in B_X(x,\varepsilon)$ such that u is an accumulation point of A_{n_1}. Pick $y, z \in A_{n_1}$ such that $y \neq z$ and $\|y-z\| < \alpha$, where α is chosen so that $\|f(u)-f(v)\| \leq \varepsilon/n_1$ whenever $u, v \in B(x, \varepsilon + 1/n_0)$ and $\|u-v\| < \alpha$. We have

$$\|f(y+h)-f(y)-f'(y).h\| \leq \varepsilon/n_1 \quad \text{and} \quad \|f(z+h)-f(z)-f'(z).h\| \leq \varepsilon/n_1$$

So, for all h such that $\|h\| \leq 1/n_1$,

$$\big\|\big(f(y+h)-f(z+h)\big)-\big(f(y)-f(z)\big)-\big(f'(y)-f'(z)\big).h\big\| \leq 2\varepsilon/n_1$$

Therefore,

$$\big\|\big(f'(y)-f'(z)\big).h\big\| \leq 4\varepsilon/n_1$$

Since this is satisfied for for all h such that $\|h\| \leq 1/n_1$, we obtain that $\|f'(y)-f'(z)\| \leq 4\varepsilon$.

In view of the above propositions, one could believe that whenever X, Y are Banach spaces (or vector normed spaces) and $f : X \to Y$ is a mapping Gâteaux-differentiable at each point of X, then for every $\varepsilon > 0$, there exists $y, z \in X$ such that $\|f'(y)-f'(z)\| \leq \varepsilon$. Our next result proves that this is not so.

Theorem 2 : *1) There exists a Lipschitz mapping $F : \ell^1 \to \mathbb{R}^2$, Gâteaux-differentiable at each point of ℓ^1, such that for every $x, y \in \ell^1$, $x \neq y$, then $\|F'(x)-F'(y)\|_{\mathcal{L}(\ell^1,\mathbb{R}^2)} \geq 1$. Moreover, for each $h \in \ell^1$, $x \to F'(x).h$ is continuous from ℓ^1 into $\mathbb{R}^2$.*

2) Let us denote D the vector normed space of elements of ℓ^1 with finite support. There exists a Lipschitz function $G : \ell^1 \to \mathbb{R}$, Gâteaux-differentiable at each point of ℓ^1, such that for every $x, y \in D$, $x \neq y$, then $\|G'(x)-G'(y)\|_{\ell^\infty} \geq 1$.

We shall construct F and G with the properties of theorem 2 using series. We were inspired by a construction from [DI]. We need an auxiliary construction.

Lemma 3 : *Given $\Delta = (a', a, b, b') \in \mathbb{R}^4$ such that $a' < a < b < b'$ and $\varepsilon > 0$, there exists a $\mathcal{C}^\infty$-function $\varphi = \varphi_{\Delta,\varepsilon} : \mathbb{R}^2 \to \mathbb{R}^2$ such that :*

(i) $|\varphi(x,y)| \leq \varepsilon$ *for all* $(x,y) \in \mathbb{R}^2$,

(ii) $\varphi(x,y) = 0$ *whenever* $x \notin [a', b']$,

(iii) $\|\frac{\partial \varphi}{\partial x}(x,y)\| \leq \varepsilon$ *for all* $(x,y) \in \mathbb{R}^2$,

(iv) $\|\frac{\partial \varphi}{\partial y}(x,y)\| = 1$ *whenever* $x \in [a,b]$,

(v) $\|\frac{\partial \varphi}{\partial y}(x,y)\| \leq 1$ *for all* $(x,y) \in \mathbb{R}^2$,

(vi) If we denote $\varphi(x,y) = \big(\varphi_1(x,y), \varphi_2(x,y)\big)$*, then* $\dfrac{\partial \varphi_1}{\partial y}(x,0) = 1$ *whenever* $x \in [a,b]$*.*

Proof of Lemma 3 : Let $b : I\!R \to I\!R$ be a $\mathcal{C}^\infty$-smooth function such that $0 \le b(x) \le 1$ for all x, $b(x) = 0$ whenever $x \notin [a',b']$ and $b(x) = 1$ whenever $x \in [a,b]$. If $n \ge 1$ is large enough, the function defined by $\varphi(x,y) = \dfrac{b(x)}{n}\big(\sin(ny), \cos(ny)\big)$ satisfies the desired properties.

We shall also use the following criterium of Gâteaux-differentiability of the sum of a series :

Lemma 4 : *Let X and Y be Banach spaces and, for all n, let $f_n : X \to Y$ be Gâteaux-differentiable mappings. Assume that $\big(\sum f_n\big)$ converges pointwise on X, and that there exists a constant $K > 0$ so that for all h,*

$$\sum_{n\ge1} \sup_{x\in X} \Big\| \frac{\partial f_n}{\partial h}(x) \Big\| \le K\|h\| \tag{1}$$

Then the mapping $f = \sum\limits_{n\ge1} f_n$ is Gâteaux-differentiable on X, for all x, $f'(x) = \sum\limits_{n\ge1} f_n'(x)$ (where the convergence of the series is in $\mathcal{L}(X,Y)$ for the strong operator topology), and f is K-Lipschitz. Moreover, if each f_n' is continuous from X endowed with the norm topology into $\mathcal{L}(X,Y)$ with the strong operator topology, then f' shares the same continuity property.

Proof of Lemma 4 : Fix $x \in X$. First observe that condition (1) implies that for all h, the series $\big(\sum \frac{\partial f_n}{\partial h}(x)\big) = \big(\sum f_n'(x).h\big)$ converges in Y. Therefore, the series $\big(\sum f_n'(x)\big)$ converges in $\mathcal{L}(X,Y)$ for the strong operator topology, to some operator $T \in \mathcal{L}(X,Y)$, and by (1), $\|T\| \le K$. For each $h \in X$, we define $g_n : I\!R \to Y$ by $g_n(t) = f_n(x + th)$. The function $g = \sum\limits_{n\ge1} g_n$ is well defined. Since

$$\sum_{n\ge1} \|g_n'\|_\infty \le \sum_{n\ge1} \sup_{x\in X} \Big\| \frac{\partial f_n}{\partial h}(x) \Big\| \le K\|h\|$$

the mapping g is differentiable and $g'(0) = \sum\limits_{n\ge1} g_n'(0) = \sum\limits_{n\ge1} \frac{\partial f_n}{\partial h}(x) = T(h)$. Thus we have proved that f is differentiable along every direction h and that $\frac{\partial f}{\partial h}(x) = T(h)$. In other words, f is Gâteaux-differentiable at x and $f'(x) = T$. Since for all x, $\|f'(x)\| \le K$, the mean value theorem implies that f is K-Lipschitz.

Proof of Theorem 2, part 1) : Fix an enumeration $\Delta_k = (a_k', a_k, b_k, b_k')$, $k \in N$, of all quadruples of dyadic numbers such that $a_k' < a_k < b_k < b_k'$.

Select integers m_k^n such that for each n, $n < m_k^n$ and $(m_k^n)_k$ is an increasing sequence, and satisfying

$$m_k^n = m_\ell^p \;\Rightarrow\; n = p \quad\text{and}\quad k = \ell \tag{2}$$

Fix $\varepsilon > 0$ and let ε_k^n be positive real numbers such that $\sum\limits_{n=1}^{\infty}\sum\limits_{k=1}^{\infty}\varepsilon_k^n = \varepsilon$. We shall notice $\varepsilon_k = \sum\limits_{n=1}^{\infty}\varepsilon_k^n$, so that $\sum\limits_{k=1}^{\infty}\varepsilon_k = \varepsilon$. Put $f_{n,k} : \ell^1 \to I\!R^2$ such that, if $x = (x_i) \in \ell^1$, then $f_{n,k}\big(x\big) = \varphi_{\Delta_k,\varepsilon_k^n}\big(x_n, x_{m_k^n}\big)$: $f_{n,k}$ is a $\mathcal{C}^\infty$ function on ℓ^1. The function $F : \ell^1 \to I\!R^2$ we are looking for is defined by :

$$F(x) = \sum_{n\in I\!N}\sum_{k\in I\!N} f_{n,k}(x)$$

Claim 1 : *F is well-defined.* Indeed, according to condition (i) of the lemma, $\|f_{n,k}\|_\infty = \|\varphi_{\Delta_k,\varepsilon_k^n}\|_\infty = \varepsilon_k^n$, so the series defining F converges uniformly.

Claim 2 : *F is Gâteaux-differentiable on ℓ^1 and F is $(1+\varepsilon)$-Lipschitz-continuous on ℓ^1.* To see this, we apply Lemma 4 : let $h = (h_1, ..., h_n, ...) \in \ell^1$. By (iii) and (v), we have for all n, k :

$$\sup_{x\in X}\Big\|\frac{\partial f_{n,k}}{\partial h}(x)\Big\| \le |h_{m_k^n}| + \varepsilon_k^n|h_n| \le |h_{m_k^n}| + \varepsilon_k^n\|h\|_1$$

So, because of condition (2),

$$\sum_{n,k}\sup_{x\in X}\Big\|\frac{\partial f_{n,k}}{\partial h}(x)\Big\| \le (1+\varepsilon)\|h\|_1$$

We have proved that condition (1) of Lemma 4 is satisfied with $K = 1+\varepsilon$, thus F is Gâteaux-differentiable on ℓ^1 and F is $(1+\varepsilon)$-Lipschitz-continuous on ℓ^1.

Claim 3 : *If $x \neq y \in \ell^1$, then $\|F'(x) - F'(y)\|_{\mathcal{L}(\ell^1, I\!R^2)} \ge 1 - 2\varepsilon$.*
Indeed, let $n \in I\!N$ such that $x_n \neq y_n$. Let k such that $x_n \in [a_k, b_k]$ and $y_n \notin [a'_k, b'_k]$. According to (ii) and (iv) of Lemma 3,

$$\Big\|\frac{\partial f_{n,k}}{\partial x_{m_k^n}}(x)\Big\| = 1 \qquad\text{and}\qquad \frac{\partial f_{n,k}}{\partial x_{m_k^n}}(y) = 0$$

On the other hand, for all r,

$$\Big\|\frac{\partial f_{m_k^n,r}}{\partial x_{m_k^n}}(x)\Big\| \le \varepsilon_r \qquad\text{and}\qquad \Big\|\frac{\partial f_{m_k^n,r}}{\partial x_{m_k^n}}(y)\Big\| \le \varepsilon_r$$

and, if $\ell \neq m_k^n$ and $(\ell, r) \neq (n, k)$,

$$\frac{\partial f_{\ell,r}}{\partial x_{m_k^n}}(x) = 0 \qquad\text{and}\qquad \frac{\partial f_{\ell,r}}{\partial x_{m_k^n}}(y) = 0$$

Therefore,

$$\begin{aligned}\|F'(x)-F'(y)\|_{\mathcal{L}(\ell^1,\mathbb{R}^2)} &\geq \Big\|\frac{\partial F}{\partial x_{m_k^n}}(x)-\frac{\partial F}{\partial x_{m_k^n}}(y)\Big\| \\ &\geq 1-\sum_{(\ell,r)\neq(n,k)}\|\frac{\partial f_{\ell,r}}{\partial x_{m_k^n}}(x)-\frac{\partial f_{\ell,r}}{\partial x_{m_k^n}}(y)\| \\ &\geq 1-2\varepsilon\end{aligned}$$

Let us now prove part 2) of Theorem 2. Since $F:\ell^1\to\mathbb{R}^2$, we can write $F=(G,H)$, where $G,H:\ell^1\to\mathbb{R}$. We shall also denote $f_{n,k}=(g_{n,k},h_{n,k})$. $G:\ell^1\to\mathbb{R}$ is Lipschitz continuous, Gâteaux-differentiable at each point of ℓ^1. Let $x=(x_i),y=(y_i)\in D$ and n such that $x_n\neq y_n$. Let k such that $x_n\in[a_k,b_k]$, $y_n\notin[a'_k,b'_k]$ and $x_{m_k^n}=0$. According to (vi) of Lemma 3, we have

$$\Big\|\frac{\partial g_{n,k}}{\partial x_{m_k^n}}(x)\Big\|=1 \qquad \text{and} \qquad \frac{\partial g_{n,k}}{\partial x_{m_k^n}}(y)=0$$

We conclude, as in the proof of Claim 3 of part 1), that

$$\|G'(x)-G'(y)\|_{\ell^\infty}\geq 1-2\varepsilon$$

Remark : 1) If we set $\Phi=\dfrac{f}{1-2\varepsilon}$, we have obtained for every $\alpha>0$, the construction of a function $\Phi:\ell^1\to\mathbb{R}^2$, Gâteaux-differentiable at every point of ℓ^1, satisfying :

(i) for all $x,y\in\ell^1$, $\|\Phi(x)-\Phi(y)\|\leq(1+\alpha)\|x-y\|_1$,

(ii) for all $x\neq y\in\ell^1$, $\|\Phi'(x)-\Phi'(y)\|_{\mathcal{L}(\ell^1,\mathbb{R}^2)}\geq 1$.

2) Fix $h\in\ell^1$. Since $x\to F'(x).h$ is continuous from ℓ^1 into $\mathbb{R}^2$, the set $\big\{F'(x).h\,;\,x\in\ell^1\big\}$ is connected. This is in contrast with the fact that $\big\{F'(x)\,;\,x\in\ell^1\big\}$ is discrete in $\mathcal{L}(\ell^1,\mathbb{R}^2)$.

3) A carefull look at the above construction shows that f is uniformly Gâteaux-differentiable.

4) Observe that for cardinality reasons, whenever $\mathcal{L}(X,Y)$ is separable, then for every Gâteaux-differentiable mapping from X into Y, and for every $\varepsilon>0$, there exists $y,z\in X$ such that $\|f'(y)-f'(z)\|\leq\varepsilon$. Therefore, it is not possible to replace ℓ^1 by ℓ^p $(p>1)$ in Theorem 2. However, there exists a Lipschitz function $H:\ell^2\to\ell^2$, Gâteaux-differentiable at each point of ℓ^2, such that for every $x,y\in\ell^2$, if $x\neq y$, then

$$\|H'(x)-H'(y)\|_{\mathcal{L}(\ell^2)}\geq 1$$

This will follow from the following more general result :

Theorem 3 : *Let $X_p = \ell^p$ if $1 \leq p < +\infty$ and $X_\infty = c_0$. Let us fix $1 \leq p, q \leq +\infty$. The following assertions are equivalent :*

(1) There exists a Lipschitz function $H : X_p \to X_q$, Gâteaux-differentiable at each point of X_p, such that for every $x, y \in X_p$, $x \neq y$, then $\|H'(x) - H'(y)\|_{\mathcal{L}(X_p,X_q)} \geq 1$.

(2) $p \leq q$.

(3) $\mathcal{L}(X_p, X_q)$ is not separable.

Proof of Theorem 3 : According to Remark 4) above, (1) implies (3). If $p > q$, then by Pitt's theorem, all operators from X_p to X_q are compact, hence $\mathcal{L}(X_p, X_q)$ is separable. Therefore (3) implies (2). So it remains to prove that (2) implies (1). Assume that $p \leq q$ and let (e_n) be the usual basis of X_p. Let $T_k \in \mathcal{L}(I\!R^2, X_q)$ defined by $T_k(x, y) = xe_{2k} + ye_{2k+1}$. Denote a_q the common norm of the operators T_k. Let Δ_k, ε_k^n, m_k^n and $\varphi_{\Delta_k,\varepsilon_k^n}$ defined as in the proof of Theorem 2. Put $f_{n,k} : X_p \to X_q$ such that, if $x = (x_i) \in X_p$, then $f_{n,k}\big(x\big) = T_{m_k^n} \circ \varphi_{\Delta_k,\varepsilon_k^n}\big(x_n, x_{m_k^n}\big)$: the functions $f_{n,k}$ is a $\mathcal{C}^\infty$ mapping from X_p into X_q. The function $H : X_p \to X_q$ we are looking for is defined by :

$$H(x) = \sum_{n \in I\!N} \sum_{k \in I\!N} f_{n,k}(x)$$

As in the proof of Theorem 2, H is well-defined. Lemma 4 is no longer applicable in order to show that H is Gâteaux-differentiable at each point of X_p. But lemma 4 remains true if the hypothesis (1) from lemma 4 is replaced by condition (2) below :

(2) for all h, $\big(\sum \frac{\partial f_n}{\partial h}(x)\big)$ converges uniformly with respect to x

So, fix $h = (h_1, ..., h_n, ...) \in X_p$. We have

$$\frac{\partial f_{n,k}}{\partial h}(x) = h_n u_{k,n}(x) + h_{m_k^n} v_{k,n}(x)$$

with $\|u_{k,n}(x)\|_q \leq \varepsilon_k^n a_q$, $v_{k,n}(x) \in span\{e_{2m_k^n}, e_{2m_k^n+1}\}$ and $\|v_{k,n}(x)\|_q \leq a_q$. We claim that both series $\big(\sum_{k,n} h_n u_{k,n}(x)\big)$ and $\big(\sum_n \sum_k h_{m_k^n} v_{k,n}(x)\big)$ are uniformly converging with respect to x. Indeed, for the first one, this follows from the fact that for each x, $\|h_n u_{k,m}(x)\|_q \leq \|h\|_p . a_q . \varepsilon_k^n$, and that $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \varepsilon_k^n < +\infty$. For the second one, $\big(\sum_k h_{m_k^n} v_{k,m}(x)\big)$ converges uniformly because it satisfies the uniform Cauchy condition. Indeed, fix $\delta > 0$ and a finite set $A \subset I\!N \times I\!N$ such that $\sum_n \sum_{(k,n) \notin A} h_{m_k^n}^p < \delta^p$. For fixed x, the $v_{k,n}(x)$ are elements of X_q with disjoint supports, so, for any finite subset F of $(I\!N \times I\!N) \backslash A$,

$$\Big\| \sum_{(n,k) \in F} h_{m_k^n} v_{k,m}(x) \Big\|_{X_q} = \Big(\sum_{(n,k) \in F} \big\| h_{m_k^n} v_{k,m}(x) \big\|_{X_q}^q \Big)^{1/q}$$

$$\leq a_q \Big(\sum_{(n,k) \in F} h_{m_k^n}^q \Big)^{1/q} \leq a_q \Big(\sum_{(n,k) \in F} h_{m_k^n}^p \Big)^{1/p} < a_q \delta$$

Notice that we used in the above chain of inequalities the fact that $p \leq q$. The above estimate is uniform in x, therefore the series $\big(\sum_k h_{m_k^n} v_{k,m}(x)\big)$ satisfies the uniform Cauchy condition. Applying the variant of lemma 4 mentioned above, we get that H is Lipschitz continuous and Gâteaux-differentiable at each point of ℓ^2. As in the proof of theorem 2, one sees that there exists $a > 0$ such that for every $x, y \in \ell^2$, if $x \neq y$, then $\|H'(x) - H'(y)\|_{\mathcal{L}(\ell^p,\ell^q)} \geq a$.

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