ODD DEGREE POLYNOMIALS ON REAL BANACH SPACES

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Abstract. A classical result of Birch claims that for given k, n integers, n-odd there exists some $N = N(k, n)$ such that for arbitrary n-homogeneous polynomial P on $\mathrm{I\!R}^N$, there exists a linear subspace $Y \hookrightarrow \mathrm{I\!R}^N$ of dimension at least k, where the restriction of P is identically zero (we say that Y is a null space for P). Given $n > 1$ odd, and arbitrary real separable Banach space X (or more generally a space with w^*-separable dual X^*), we construct a n-homogeneous polynomial P with the property that for every point $0 \neq x \in X$ there exists some $k \in \mathrm{I\!N}$ such that every null space containing x has a dimension at most k. In particular, P has no infinite dimensional null space. For a given n odd and a cardinal τ, we obtain a cardinal $N = N(\tau, n) = \exp^{n+1}\tau$ such that every n-homogeneous polynomial on a real Banach space X of density N has a null space of density τ.

The main result of this note is a construction, in every real separable Banach space X (or more generally every real Banach space with w^* separable dual X^*), of a n-homogeneous polynomial P ($n > 1$ arbitrary odd integer) which has no infinite dimensional null space. In fact, we prove a stronger result, namely for every $0 \neq x \in X$ there exists some integer k such that every null space $x \in Y \hookrightarrow X$ has dimension at most k. This answers a question proposed by the first named author. Let us briefly explain the significance of this result. In the complex case, Plichko and Zagorodnyuk [8], have shown that a complex polynomial on a complex infinite dimensional Banach space has an infinite dimensional null space. This result is shown using a relatively simple inductive argument, building up the desired subspace from an increasing nested family of finite dimensional null spaces. It is clear that a real scalar version of the theorem fails, since every separable real space admits a positive (away from the origin, of course) $2k$-homogeneous polynomial for every $k \in \mathrm{I\!N}$. The remaining case when the degree is odd (and the polynomial is homogeneous) has been treated in the finite dimensional setting by many authors working in the field of algebraic geometry (e.g.[3], [9], [10], [11]). In fact, a classical theorem of Birch implies that for a finite dimensional space of high enough dimension $N(k, n)$, every n-homogeneous polynomial (n-odd) has a k-dimensional null space. Recently, [2] obtained a reasonable estimate on the values of $N(k, n)$, using a topological approach to the problem. Our present result in particular shows that it is impossible to continue the above constructions inductively, obtaining nested sequences of null spaces of increasing dimension.

Regarding the infinite dimensional situation, in [6] it is shown that for any given $\varepsilon > 0$, a homogeneous odd degree polynomial P on a separable real Banach space X can be restricted to an infinite dimensional subspace Y so that $sup_{B_Y}|P| < \varepsilon$.

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It is easy to repeat this result inductively to obtain an asymptotically null space Y with a Schauder basis $\{e_i\}_{i=1}^{\infty}$ where $sup_{B_{[e_i:i\geq n]}}|P| = \varepsilon_n \to 0$ as $n \to \infty$. Our present result thus in particular shows, that in general one cannot have $\varepsilon_n = 0$.

We then proceed investigating the nonseparable situation. Using the Erdos-Rado theorem we prove the natural intuitive conjecture that increasing the density of X leads to the increase of the density of the null spaces. In general, the dependence varies for different spaces. The slowest growth happens for the space $\ell_1(\Gamma)$, which is a significant space for polynomials also because all polynomials defined formally using a formula are continuous and their defining formula is convergent.

We present some results and examples related to $\ell_p(\Gamma)$ spaces.

It is clear, that every continuous polynomial P on $\ell_p(\Gamma)$, $c_0(\Gamma)$, can be uniquely coded by its "formula", i.e. a formal sum over the appropriate index set of its coefficients multiplied by the corresponding powers of the coordinates. This formal expression turns into a finite sum, whenever we evaluate at finitely supported vectors, and in this sense the formula indeed provides the values of P. It is straightforward to check that in $\ell_1(\Gamma)$ the formula converges absolutely for every vector. In general, however, the convergence of the sum cannot be expected. In the first part of our note, we will choose coordinates small enough so that the formulas will be in fact absolutely convergent for all points. In the second part of our note, we do not assume any convergence, which is fine since we will be working with finitely supported vectors.

We say that the dual X^* has w^* density character $w^* - \text{dens}(X^*) = \Gamma$, if there exists a set $S \subset X^*$ of cardinality Γ, such that $\overline{S}^{w^*} = X^*$, and moreover Γ is the minimal cardinal with this property. Recall the following well-known fact.

Fact 1. *Let X be a Banach space, then $w^* - dens(X^*) \leq \Gamma$ iff there exists a bounded linear injection $T : X \to \ell_\infty(\Gamma)$.*

Proof. If $\{f_\gamma\}_{\gamma\in\Gamma} = S \subset X^*$ is the w^*-dense set, then $T(x) = (\frac{f_\gamma(x)}{\|f_\gamma\|})_{\gamma\in\Gamma}$ is an injection into $\ell_\infty(\Gamma)$. On the other hand, starting from the injection $T : X \to \ell_\infty(\Gamma)$, it suffices to put

$$S = \{T^*(f) : f \in \ell_1(\Gamma) \text{ is finitely supported and has rational coordinates}\}.$$

$\square$

Theorem 2. *Let X be an infinite dimensional real Banach space with $w^* - densX^* = \omega$, $n > 1$ an odd integer. Then there exists a n-homogeneous polynomial $P : X \to \mathbb{R}$ without any infinite dimensional null space. More precisely, given any $0 \neq x \in X$, $P(x) = 0$, there exists a $N \in \mathbb{N}$ such that every null space $x \in Y \hookrightarrow X$ has $dimY \leq N$.*

Proof. Suppose that we have already proven the statement of the theorem for $X = c_0$ and $n = 3$. Let $P : c_0 \to \mathbb{R}$ be the polynomial. Given any Banach space X with $w^* - \text{dens}X^* = \omega$, and $n = 3 + 2l$, we can construct the desired n-homogeneous polynomial $Q : X \to \mathbb{R}$ as follows. Fix any bounded linear injection $T : X \to c_0$ (put for example $T(x) = (\frac{f_i(x)}{i})_{i=1}^{\infty}$, where $\{f_i\}_{i=1}^{\infty} \subset B_{X^*}$ is a separating set of functionals), and put $Q(x) = P \circ T(X) \cdot (\sum_{i=1}^{\infty} \frac{1}{2^i} f_i(x)^{2l})$. It is easy to verify that a linear subspace of X where Q vanishes translates via T into a linear subspace (of the same dimension) of c_0 where P vanishes, which concludes the implication. It remains to produce P on c_0. We put

$$P((x_i)) = \sum_{k=1}^{\infty} x_k \sum_{i=k+1}^{\infty} \alpha_k^i x_i^2.$$

where $\alpha_k^i > 0$, together with the auxiliary system $\tau_{k,i}^j > 0$, are chosen satisfying conditions (0)-(3) below.

(0) $\sum\limits_{k=1}^{\infty} \sum\limits_{i=k+1}^{\infty} |\alpha_k^i| < \infty$.

(1) $\frac{1}{i}\alpha_k^i > \sum\limits_{j=k+1}^{\infty} \alpha_j^i$.

(2) $\frac{1}{2i}\alpha_k^i \geq \sum\limits_{j=1}^{\infty} \tau_{k,i}^j$.

(3) $(\alpha_p^r)^2 \leq \frac{1}{16}\tau_{r,p}^q \tau_{r,q}^p$ whenever $r < p < q$.

To construct such a system of coefficients α_k^i (and auxiliary system $\tau_{k,i}^j > 0$) is rather straightforward, proceeding inductively by the infinite rows of the matrix $\{\alpha_k^i\}$. Indeed, the additional conditions always require that elements of a certain row are small enough depending on the elements of the previous rows. Note that our choice guarantees that the formula for P converges absolutely for every $x \in c_0$.

Claim 1. *Given any $0 \neq x \in c_0$, $P(x) = 0$, there exists $N \in \mathbb{N}$ such that for every null space $x \in Y \hookrightarrow c_0$ we have that $dim Y \leq N$.*

We may WLOG assume that $\|x\|_\infty \leq 1$. Consider a (nonhomogeneous) 3rd degree polynomial $R(y) = P(x + y)$.

$$R((y_i)) = \sum_{k=1}^{\infty} (x_k + y_k) \sum_{i=k+1}^{\infty} \alpha_k^i (x_i + y_i)^2.$$

Writing $R = R_0 + R_1 + R_2 + R_3$, where R_m is the m-homogeneous part of R, we obtain in particular:

$$R_2((y_i)) = \sum_{k=1}^{\infty} x_k \sum_{i=k+1}^{\infty} \alpha_k^i y_i^2 + \sum_{k=1}^{\infty} y_k \sum_{i=k+1}^{\infty} 2\alpha_k^i x_i y_i.$$

Thus $R_2((y_i)) = \sum\limits_{s=1}^{\infty} \sum\limits_{l=s}^{\infty} \beta_s^l y_s y_l$, where $\beta_s^s = \sum\limits_{k=1}^{s-1} x_k \alpha_k^s$, $\beta_s^l = 2x_l \alpha_s^l$.

To prove the claim it suffices to find $N \in \mathbb{N}$, such that R_2, restricted to $Z = [e_i : i > N] \hookrightarrow c_0$ (Z has codimension N) is strictly positive outside the origin. Indeed, if so, then $R(\lambda z) = \sum\limits_{m=0}^{3} \lambda^m R_m(z)$ is a nontrivial 3rd degree polynomial in λ, for every $z \in Z$, and in particular for every $z \in Z$ there exists some $\lambda \in \mathbb{R}$ such that $P(x + \lambda z) = R(\lambda z) \neq 0$. Now if $x \in Y \hookrightarrow c_0$ is a null space, then $Z \cap Y = \{0\}$, and so $\dim Y \leq N$, as stated.

Let us WLOG assume that $x_r > 0$, where $r = min\{i : x_i \neq 0\}$. We choose $N > r$ large enough, so that the following are satisfied.

(i) $\beta_s^s = \sum\limits_{j=r}^{s-1} x_j \alpha_j^s \geq \frac{1}{2} x_r \alpha_r^s$ for every $s \geq N + 1$.

There exists a decomposition $\beta_s^s \geq \sum\limits_{i=N+1}^{\infty} \delta_s^i$, $\delta_s^i > 0$ such that

(ii) $(\alpha_p^q)^2 \leq \frac{1}{16}\delta_p^q\delta_q^p$ whenever $N < p < q$.

To see that such a choice on N is possible, we estimate using property (1), whenever $s > \frac{3}{x_r}$

$$\beta_s^s \geq x_r\alpha_r^s - \sum_{j=r+1}^{s-1} x_j\alpha_j^s \geq x_r\alpha_r^s - \sum_{j=r+1}^{s-1} \alpha_j^s > \frac{1}{2}x_r\alpha_r^s.$$

Thus $N > \frac{3}{x_r}$guarantees that (i) is satisfied. To see (ii), for N large enough, and $s > N$, $\frac{1}{2}x_r > \frac{1}{2N} > \frac{1}{2s}$, so we have $\beta_s^s \geq \frac{1}{2s}\alpha_r^s$. So putting $\delta_s^i = \tau_{r,s}^i$ suffices using properties (2) and (3).

The conditions are set up so that R_2 restricted to $Z = [e_i : i > N]$ satisfies

$$R_2((y_i)) \geq \sum_{p=N+1}^{\infty} \sum_{q=p+1}^{\infty} (\delta_p^q y_p^2 + \delta_q^p y_q^2 + 2\alpha_p^q x_q y_p y_q).$$

However, condition (ii) implies that

$$\delta_p^q y_p^2 + \delta_q^p y_q^2 + 2\alpha_p^q x_q y_p y_q \geq \delta_p^q y_p^2 + \delta_q^p y_q^2 - 2\alpha_p^q |y_p y_q| \geq$$
$$\frac{3}{4}(\delta_p^q y_p^2 + \delta_q^p y_q^2) + (\frac{1}{2}\sqrt{\delta_p^q}|y_p| - \frac{1}{2}\sqrt{\delta_q^p}|y_q|)^2 > \frac{1}{2}(\delta_p^q y_p^2 + \delta_q^p y_q^2).$$

The last expression is clearly a positive quadratic form in variables y_p, y_q, which concludes the claim that

$$R_2((y_i)) \geq \sum_{p=N+1}^{\infty} \sum_{q=p+1}^{\infty} (\frac{1}{2}\delta_p^q y_p^2 + \frac{1}{2}\delta_q^p y_q^2) > 0$$

for every $0 \neq (y_i) \in Z$.

$\square$

The statement of the theorem applies to all separable Banach spaces, ℓ_∞, $C(K)$, where K is separable (not necessarily metrizable). It is inherited by the subspaces, so since $\ell_1(c) \hookrightarrow \ell_\infty$, it applies also to $\ell_1(c)$. We are going to investigate further the size of maximal null sets in the nonseparable setting, and the spaces $\ell_1(\Gamma)$ will play the main role in this investigation. It is known in set theory, that the cardinality of continuum c is consistently "arbitrarily large" cardinal. On the other hand, one would intuitively expect that increasing the density of the space X should lead to the (uniform) increase of the null sets. We will prove that this intuition is correct.

Our objective now is to obtain some estimate on the size of card(Γ), such that every n-homogeneous odd polynomial on $\ell_1(\Gamma)$ has large null sets. Our approach is via subsymmetric polynomials ([6],[7]). We then need to produce a subset of $S \subset \Gamma$, where the restriction of P to $\ell_1(S)$ behaves subsymmetrically. In this step we are loosing on the cardinality.

Given and ordinal Γ, we say that a polynomial $P : \ell_1(\Gamma) \to \mathrm{I\!R}$ is subsymmetric if $P(\sum\limits_{i=1}^{l} x_i e_{\gamma_i}) = P(\sum\limits_{i=1}^{l} x_i e_{\beta_i})$ whenever we have $\gamma_1 < \gamma_2 < \cdots < \gamma_l$, $\beta_1 < \cdots < \beta_l$, for arbitrary $x_i \in \mathrm{I\!R}$.

Lemma 3. *Let $P : \ell_1(\Gamma) \to \mathrm{I\!R}$ be a subsymmetric n-homogeneous polynomial, n odd. Then P has a null set of density Γ.*

Proof. Consider all polynomials b_i, which can be represented using a multiindex $(\beta^i_j)^m_{j=1}$ $\beta^i_j \in \mathbb{N}$, $\sum\limits_{j=1}^m \beta^i_j \leq n$, so that $b_i(\sum x_\alpha e_\alpha) = \sum\limits_{\alpha_1<\dots<\alpha_m} x^{\beta^i_1}_{\alpha_1} \cdot \dots \cdot x^{\beta^i_m}_{\alpha_m}$. Formally, we may assume that the index i of b_i runs through the finite set of all distinct multiindexes. We say that m is the length of the multiindex (or the corresponding polynomial). We will call these polynomials standard ([7]), and write for simplicity $b_i = (\beta^i_j)^m_{j=1}$. Standard polynomials of degree n constitute the vector space basis of the finite dimensional space of all n-homogeneous and subsymmetric polynomials. The finite system of equations $b_i(v) = 0$ for all standard polynomials b_i of odd degree not exceeding n has a nonzero and finitely supported solution v, due to the proof in [3] (or [2]). Note that the homogeneity of b_i implies immediately that $b_i(zv) = 0$ for every $z \in \mathbb{R}$. Assume WLOG $supp(v) = \{1, \dots, d\}$. Since P is subsymmetric and homogeneous of odd degree, $P = \sum \xi_i b_i$ and consequently, $P(v) = 0$. Let us now consider a linear subspace of $\ell_1(\Gamma)$ generated by vectors $\{v_\alpha\}^\Gamma_{\alpha=1}$, such that v_α are translates of v with consecutive supports on Γ. It remains to show that $b_i(u) = 0$ for all standard polynomials of odd degree not exceeding n, and for all $u = \sum\limits_{j=1}^m z_j v_{\alpha_j}$. Let us verify this fact using an inductive argument on the length of the representing multiindex for b_i. If its length is one, i.e. $b_i = (l)$, for some odd $l \leq n$, then we have $(l)(\sum\limits_{j=1}^m z_j v_{\alpha_j}) = \sum\limits_{j=1}^m (l)(z_j v_{\alpha_j}) = 0$. Having proven the result for all odd multiindexes of length at most $d-1$, we proceed by writing the action of a standard polynomial with multiindex $\beta = (\beta_1, \dots, \beta_d)$. Denote by

$$D = \{\mathcal{A} = (A_1, \dots, A_l) : \cup A_i = \{1, \dots, d\} \text{ and } A_i \text{ consecutive and disjoint }\}.$$

For $\mathcal{A} = (A_1, \dots, A_l)$, and $1 \leq r \leq l$ we define $\beta^{\mathcal{A}}_r = (\beta_{\min A_r}, \dots, \beta_{\max A_r})$, to be a multiindex "contained within" β on the position of set A_r. Denote

$$\Phi_{\mathcal{A}} = \{\phi : \{1, \dots, l\} \to \{1, \dots, m\}, \phi \text{ is increasing}\}, \text{ and } \Phi = \cup \Phi_{\mathcal{A}}.$$

With these definitions, it is standard to check the following formula

$$(\beta_1, \dots, \beta_d)(\sum_{j=1}^m z_j v_{\alpha_j}) = \sum_{\mathcal{A}\in D, \phi\in\Phi_{\mathcal{A}}} (\prod_r \beta^{\mathcal{A}}_r (z_{\phi(r)} v_{\phi(r)}))$$

Each of the product terms on the right hand side either contains some multiplicative term $\beta^{\mathcal{A}}_r$ of odd degree less than n, or it is a single term $(\beta)(z_{\phi(r)} v_{\phi(r)})$. By the inductive assumption, in either case it is identically zero and the result follows. □

Denote by $\exp\alpha = 2^\alpha, \exp^{n+1}\alpha = \exp(\exp^n \alpha)$, where α is a cardinal. For a set S, let $[S]^n = \{X \subset S : \mathrm{card} X = n\}$. We will use the following result, which in the language of partition relations claims that $(\exp^{n-1}\alpha)^+ \to (\alpha^+)^n_\alpha$ ([4]).

Theorem 4. *(Erdos, Rado)*
Let α be an infinite cardinal, $n \in \mathbb{N}$, $\kappa = (exp^{n-1}\alpha)^+$ and $\{G_\gamma\}_{\gamma<\alpha}$ be a partition of $[\kappa]^n$. Then there exist $M \subset \kappa$, $cardM = \alpha^+$ and $[M]^n \subset G_\gamma$ for some $\gamma < \alpha$.

Theorem 5. *Suppose $card\Gamma \geq exp^n\alpha$, n odd. Then every n-homogeneous polynomial on $\ell_1(\Gamma)$ has a null space of density at least α^+.*

Proof. Let P be an n-homogeneous polynomial, suppose Γ is an ordinal. We partition the set $[\Gamma]^n$ using continuum many sets $\{G_{\{a_{i_1,\dots,i_n}:1\le i_1\le\dots\le i_n\le n\}} : a_{i_1,\dots,i_n}\in\mathbb{R}\}$ as follows. We put $[\gamma_1,\dots,\gamma_n]\in G_{\{a_{i_1,\dots,i_n}:1\le i_1\le\dots\le i_n\le n\}}$ iff $\{a_{i_1,\dots,i_n} : 1\le i_1\le\dots\le i_n\le n\}$ coincides with the set of coefficients of P, when restricted to the n-dimensional space with coordinate vectors $e_{\beta_1},\dots,e_{\beta_n}$ where $\{\beta_i\}$ is an increasingly reordered set $\{\gamma_i\}$ (in the order coming from Γ). Applying the Erdos-Rado theorem yields a subset $S\subset\Gamma$ of the desired cardinality, such that the restriction of P to $\ell_1(S)$ is a subsymmetric polynomial.

□

It is essentially impossible to improve the above result using combinatorial methods. By Lemma 39.1 of [4], we have the negative relation $\Gamma=\exp^{n-1}c\nrightarrow(n+1)^n_c$. Thus there exists a partition of $[\Gamma]^n$ into c subsets $\{G_t\}_{t\in(0,1)}$, for which there exists no subset $S\subset\Gamma$ with $n+1$ elements, such that all subsets of S with n elements belong to the same G_t. Define a polynomial on $\ell_1(\Gamma)$ using the formula $P((x_i))=\sum_{t,S=\{i_1,\dots,i_n\}\in\Gamma_t} t x_{i_1}\dots x_{i_n}$. This is a correct definition, and moreover the sum is absolutely convergent. The negative property of the partition $\{G_t\}$ translates directly into the fact that P restricted to any $n+1$ coordinates fails to be subsymmetric. Since null spaces are sensitive to any change of coefficients, it is hard to imagine a proof producing a large null space under these circumstances.

Theorem 6. *Let X be a real Banach space of dens$(X)\ge$ exp$^{n+1}\alpha$, where α is a cardinal, n odd integer. Then every n-homogeneous polynomial on X has a null space of density at least α^+.*

Proof. Let $\Gamma=\exp^n\alpha$. We construct a continuous injection $T:\ell_1(\Gamma)\to X$ inductively as follows. Having chosen $T(e_i)\in B_X$ for all $i<\beta<\Gamma$ together with functionals $f_i\in B_{X^*}$, $f_i(T(e_i))\ge\frac{1}{2}$, we choose $T(e_\beta)\in\bigcap_{i<\beta}\operatorname{Ker} f_i$. The last set is nonempty, since card$X\le 2^{w^*-\mathrm{dens}X^*}$, so $w^*-\mathrm{dens}X^*\ge\exp^n\alpha$ and we can continue the inductive process. Now it remains to note that $P\circ T$ is an n-homogeneous polynomial on $\ell_1(\Gamma)$, its null subspaces carry right into X, and the previous theorem applies.

□

Examples

Recall the classical fact that every continuous polynomial on c_0, (resp. every continuous polynomial on ℓ_p of degree less than p) is weakly sequentially continuous (wsc for short), in particular it maps weakly null sequences into sequences convergent to zero ([5]). This fact implies that in the formula for such a polynomial on the long space $c_0(\Gamma)$, resp. $\ell_p(\Gamma)$, the cardinality of nonzero coefficients is at most countable. We will exploit this property in the results below. We will say that a subspace of $\ell_p(\Gamma)$ is a block subspace, if it is generated by a set of disjointly supported vectors. Given a polynomial P on $\ell_p(\Gamma)$, and a disjoint decomposition of $\Gamma=\cup\Gamma_\alpha$, we say that P splits with respect to this decomposition, iff the formula for P contains no nonzero terms containing variables from distinct Γ_α. It is clear that this is exactly the case when we can write $P=\sum_\alpha P_\alpha$, where P_α are defined on Γ_α. So P has a "diagonal" form with respect to the given decomposition.

The existence of a splitting of cardinality τ for a n-homogeneous polynomial P, n odd, immediately implies the existence of a null set of density τ. Indeed, choose arbitrary vectors $0 \neq v_\alpha \in \ell_p(\Gamma_\alpha)$. The formula for P restricted to $[v_\alpha]$ is simply $P((x_\alpha)) = \sum_\alpha c_\alpha x_\alpha^n$. Assume WLOG that $c_\alpha > 0$, and fix a bijective correspondence $\alpha \leftrightarrow \beta$ between two disjoint subsets of the index set of cardinality τ. Now the space $[v_\alpha - (\frac{c_\alpha}{c_\beta})^{\frac{1}{n}} v_\beta]$ is easily checked to be the desired (block) null space.

Proposition 7. *Let Γ be an infinite cardinal, $P : c_0(\Gamma) \to \mathbb{R}$ be an arbitrary continuous polynomial. Then P has a null space of separable codimension in $c_0(\Gamma)$.*

Proof. Since P is wsc, it mapps in particular w-null sequences to sequences convergent to $0 \in \mathbb{R}$. Using a standard argument we see that P depends only on a countable set of coordinates $S \subset \Gamma$, and so P restricted to $\Gamma \setminus S$ is identically zero. $\square$

A similar proof based on wsc property for polynomials of degree less than p on ℓ_p spaces gives.

Proposition 8. *Let Γ be an infinite cardinal, $P : \ell_p(\Gamma) \to \mathbb{R}$ be an arbitrary continuous polynomial of degree less than p. Then P has a null space of separable codimension in $\ell_p(\Gamma)$.*

In order to investigate polynomials of degree higher than p on $\ell_p(\Gamma)$ spaces, we need the following lemma.

Lemma 9. *Let P be a polynomial of n-th degree on $\ell_p(\Gamma)$, $\Gamma > \omega$, $n < 2\lceil p \rceil$. Then there exists a subset $\Gamma' \subset \Gamma$, linearly ordered, such that the restriction of P to Γ' has the form*

$$P((x_i)) = \sum_{j \in \Gamma', \lceil p \rceil \le m \le n} \sum_{i_1 \le \cdots \le i_l \le j} a^m_{i_1,\ldots i_l,j} x_{i_1} \ldots x_{i_l} x_j^m$$

Proof. In the formula for $P((x_\gamma)) = \sum_{i_1 \le \cdots \le i_n} b_{i_1,\ldots,i_n} x_{i_1} \ldots x_{i_n}$, we claim that for a fixed $i_1, \ldots i_{\lceil p \rceil}$, the set of $\{i_{\lceil p \rceil + 1} \ldots i_n\}$ such that $b_{i_1,\ldots,i_n} \neq 0$ is countable. Indeed, the polynomial

$$R((y_i)) = \sum_{i_{\lceil p \rceil + 1} \le \cdots \le i_n} b_{i_1,\ldots,i_n} x_{i_1} \ldots x_{i_{\lceil p \rceil}} y_{\lceil p \rceil + 1} \ldots y_{i_n}$$

is a homogeneous polynomial of degree less than p, so it is wsc, and therefore depends on countably many coordinates. Discarding these coordinates from Γ, for all finite terms constructed so far, we can inductively define an ordinal Γ' of cardΓ, such that whenever $i_1 \le \cdots \le i_{\lceil p \rceil} \in \Gamma'$, there exists no $i_{\lceil p \rceil} < i_{\lceil p \rceil + 1} \le \cdots \le i_n\}$ such that $b_{i_1,\ldots,i_n} \neq 0$. This proves the claim. $\square$

The previous proposition may be further generalized to arbitrary degree polynomial. The resulting formula will contain only those mixed terms whose last power is of degree at least $\lceil p \rceil$.

Proposition 10. *Let P be a n-homogeneous polynomial on $\ell_p(\omega_1^+)$, $n < 2\lceil p \rceil$. Then P has an infinite dimensional (block) null space.*

Proof. Consider the P in the above form. Since for every j, the set of nonzero $a^m_{i_1,\dots i_l,j}$ is at most countable. We proceed inductively as follows. Pick the first ω_1 elements of $\Gamma = \omega_1^+$. It follows that there is some $k_0 \in \Gamma$, and a set Γ_1, $\min\Gamma_1 > k_0$, of cardinality ω_1^+ such that $a^m_{i_1,\dots i_l,j} = 0$, whenever $k_0 \in \{i_1, \dots , i_l\}$, for all $j \in \Gamma_1$. Since ω_1^+ is a regular cardinal, we can in the next step choose the initial ω_1-interval of Γ_1, and k_1 in there, such that for some $\Gamma_2 \subset \Gamma_1$, $\min\Gamma_2 > k_1$ of cardinality ω_1^+ we have that $a^m_{i_1,\dots i_l,j} = 0$, whenever $k_1 \in \{i_1, \dots , i_l\}$, for all $j \in \Gamma_2$.

We proceed inductively along ω. The final set $\{k_j\}_{j=0}^\infty$ clearly defines a splitting of P restricted to this index set.

□

Proposition 11. *Let P be a 3rd degree polynomial on $\ell_2(\omega_1)$. Then P has an infinite dimensional null (block) space.*

Proof. WLOG, P has the formula $P((x_i)) = \sum_{j<\omega_1}\sum_{i\le j} a_{i,j}x_ix_j^2$. We are going to construct a block sequence $\{u_k\}_{k=1}^\infty$ inductively as follows. First step. If there exists some i such that $\Gamma_1 = \{j : i < j, a_{i,j} = 0\}$ is uncountable, then we choose $u_1 = e_i$. Clearly, P restricted to $[u_1, e_i : i \in \Gamma_1]$ splits with respect to the decomposition $\{i\}, \Gamma_1$.

Otherwise, for every i there exists $\varepsilon_i > 0$ such that $\Delta_i = \{j : j > l, |a_{l,j}| > \varepsilon_i\}$ is uncountable. Fix $i = 1$ and still using the previous assumption, pick an $l > 1$ such that the set $\Gamma_1 = \{j : j \in \Delta_1, j > i, |a_{i,j}| < \frac{\varepsilon_1}{2}\}$ is uncountable. Here we are using the property of the ground space ℓ_2, namely if such a choice were not possible, we would have some j for which the set $\{i : i < j, |a_{i,j}| \ge \frac{\varepsilon_1}{2}\}$ is infinite. This is a contradiction with the continuity of the linear term in the shifted polynomial $Q(x) = P(e_j + x)$. Assume WLOG that there exists some $\delta > 0$, $a = \varepsilon_1 > a - 3\delta > \frac{\varepsilon_1}{2} > b > b - 3\delta > c \ge 0$, and a disjoint decomposition of Γ_1 into uncountable subsets Γ_1^1, Γ_1^2 such that $|a_{1,j} - a| < \delta$ for all $j \in \Gamma_1$, $|a_{l,j} - b| < \delta$ for all $j \in \Gamma_1^1$ and $|a_{l,j} - c| < \delta$ for all $j \in \Gamma_1^2$. Put $u_1 = e_l - \frac{b+c}{2a}e_1$. Consider now the polynomial P restricted to the subspace generated by the basic long sequence $\{e_i^1 : i < \omega_1\} = \{u_1, e_j : j \in \Gamma_1\}$. Its formula has the canonical form $P((x_i)) = \sum_{j<\omega_1}\sum_{i\le j} a^1_{i,j}x_ix_j^2$, where moreover $|a^1_{1,i}| > \delta$ for all $i > 1$, and both sets $A = \{i : i > 1, a^1_{1,i} > \delta\}$ and $B = \{i : i > 1, a^1_{1,i} < -\delta\}$ are uncountable. Blocking once more, this time using a bijection $\phi : A \to B$ and suitable coefficients $c_i, i \in A$ we obtain the disjoint blocks $v_i = e_i + c_ie_{\phi(i)}$, $i \in A$, such that in the restriction of P to $[e_1^1, v_i]$ splits with respect to e_1^1 and $[v_i]$. The inductive step consists of repeating the previous argument, for the polynomial P restricted to the last index set defining the previous splitting. This leads to a sequence $\{u_k\}_{k=1}^\infty$, where each u_k lies in the block subsequent to blocks containing $u_i, i < k$, and defining a splitting of P. Thus P splits with respect to disjoint block vectors $\{u_k\}_{k=1}^\infty$, and the result follows.

□

Fact 12. *Let P be a n-homogeneous polynomial on $\ell_p(\Gamma)$, n odd, Γ uncountable, $Z \hookrightarrow \ell_p(\Gamma)$ be a null space of density $\tau > \omega$. Then P has a null block space of density τ.*

Proof. Let $Z \hookrightarrow \ell_p(\Gamma)$ be a subspaces of density τ. Proceed by transfinite induction, constructing disjointly supported nonzero vectors $v_\alpha \in Z$. Suppose that $Y = [v_\alpha :$

$\alpha < \eta < \tau]$ is a maximal element with respect to inclusion, and $\eta < \tau$ a cardinal. Consider the union of supports $S = \cup \operatorname{supp} v_\alpha$, and the continuous projection $P : \ell_p(\Gamma) \to \ell_p(S)$. Clearly, $P^{-1}(0) \cap Z \neq \{0\}$, since otherwise Z would have a family of separating functionals of cardinality $\eta <$ dens$Z$, which is impossible as it is a reflexive space ([5]). So choosing an element $v_\eta \in P^{-1}(0) \cap Z$ leads to a contradiction with the maximality of Y, proving the claim.

□

Remark. The assumption that τ is uncountable cannot be dropped. Indeed, consider the subspace of ℓ_p generated by vectors $v_n = \sum_{i=k_n}^{\infty} a_i^n e_i$ for some fast decreasing sequence $a_i^n \searrow 0$, and fast increasing $k_n \to \infty$. We have $\{v_n\} \sim \{e_n\}$ the canonical basis. The coordinates of $v_j(i), j \leq n$ in the intervals $i \in [k_n, k_{n+1})$ are chosen so that for every pair of nonzero vectors $x = \sum_{j=1}^{n} b_j v_j$, $y = \sum_{j=1}^{n} c_j v_j$ there exists some $i \in [k_n, k_{n+1})$ for which $x(i), y(i) \neq 0$. This can be obtained by a simple compactness argument. It follows, that $[v_n : n \in \mathbb{N}]$ contains no two nonzero disjoint blocks.

Given $n - 2 < p \leq n$, where n is odd, we define a polynomial operator $Q_p : \ell_p(c) \to \ell_1(c)$ by $Q((x_i)) = (x_i^n)$. Clearly, Q is n-homogeneous and injective. Let P be the 3-homogeneous polynomial on $\ell_1(c)$ without any infinite dimensional null space.

Lemma 13. *In the notation above, $R = P \circ Q$ is a $3n$-homogeneous polynomial on $\ell_p(c)$, which has no infinite dimensional block null space. In particular, it has no nonseparable null space. Moreover, for every $l \geq 4n+1$ odd, there exists an l-homogeneous polynomial on $\ell_p(c)$ without a nonseparable null space.*

Proof. If $[v_i : i \in \mathbb{N}]$ were a block null space for R, $[Q(v_i) : i \in \mathbb{N}]$ would be a block null space for P, using the the form of Q. This is a contradiction. To get the moreover statement, it suffices to use the polynomials $R \cdot \sum_{i \in c} x_i^{n+1}$.

□

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