A quantitative version of Krein's Theorem

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In memoriam
Professor Vlastimil Pták (1925-1999)

Abstract

Using double-limit techniques we proof that if a bounded set M of a Banach space X has the property that, for some $\varepsilon \geq 0$, $\overline{M}^{w^*} \subset X + \varepsilon B_{X^{**}}$ (where $\overline{(\cdot)}^{w^*}$ denotes the closure in (X^{**}, w^*)), then $conv(M)$, the convex hull of M, has the same property with constant 2ε. We give also some instances where the same constant works. Some applications to the characterization of subspaces of weakly compactly generated Banach spaces are also given.

1 Introduction

Krein's Theorem (see, for example, [Ko, §24.5]) says that the closed convex hull $\overline{conv}(K)$ of a compact subset K of a locally convex space X is itself compact if and only if $\overline{conv}(K)$ is complete in the Mackey topology (i.e., the topology on X of the uniform convergence on absolutely convex and weak-star compact subsets of X^*). In particular, if X is a Banach space and $K \subset X$ is weakly compact, so it is $\overline{conv}(K)$ (see, e.g., [FHHPMZ, Thm. 3.58]).

A Banach space X is *weakly compactly generated* (WCG, in short) if a linearly dense and weakly compact subset of X exists. Subspaces of WCG

*Supported by grants AV 1019003 and GAČR 201/01/1198.

†Supported by grants AV 1019003 and GAČR 201/01/1198 and by a grant from the Universidad Politécnica de Valencia. This author acknowledges the hospitality and working conditions provided by this University.

‡Supported in part by Project pb96-0758(Spain) and by the Universidad Politécnica de Valencia.

Banach spaces are not, in general, WCG (the first example of this pathology was provided by Rosenthal [Ro]). Recently [FMZ] a characterization of subspaces of WCG Banach spaces has been provided in terms of the existence of a countable family of subsets of the unit ball which are "somehow" weakly compact. More precisely: a subset $K \subset X$ is *weakly relatively compact* if its weak closure is weakly compact. This amounts to say that it is bounded and its w^*-closure in X^{**} remains in X. So, if K is bounded and *not* weakly relatively compact, $\overline{K}^{w^*} \not\subset X$. How far this closure is from X gives a certain "quantification" of the phenomenon:

Definition 1 *Let X be a Banach space and let M be a bounded subset of X. Given $\varepsilon \geq 0$, we say that M is $\varepsilon-$*weakly relatively compact *($\varepsilon-$WRK, in short) if $\overline{M}^{w^*} \subset X + \varepsilon B_{X^{**}}$.*

The case $\varepsilon = 0$ is the classical weakly relatively compactness.

The aforesaid characterization reads

Theorem 2 ([FMZ]) *A Banach space X is a subspace of a WCG Banach space if and only if it admits a family $\{M_{n,p};\ n,p \in \mathbb{N}\}$, of convex symmetric subsets of B_X such that $\bigcup_{n=1}^{\infty} M_{n,p}$ is dense in B_X for every $p \in \mathbb{N}$, and that $\overline{M_{n,p}}^{w^*} \subset X + \frac{1}{p} B_{X^{**}}$ for every $n,p \in \mathbb{N}$.*

Surprisingly, the following natural question related to Krein's Theorem has not been investigated (up to our knowledge): Assume $M \subset X$ is $\varepsilon-$WRK. Is $conv(M)$ also $\varepsilon-$WRK? (if $\varepsilon = 0$ this is the classical statement of Krein's Theorem). Apparently the answer to this question is much more difficult than expected. In this note we are able to proof , using techniques of double limits due to Grothendieck and Pták, that it is so in some cases (subspaces of WCG Banach spaces, or spaces such that the dual does not contain a copy of ℓ^1) and the answer is yes in the general case if a relaxation to 2ε of the constant is allowed. We do not know if the answer is yes in full generality. The following is the main result of this note.

Theorem 3 *Let $(X, \|\cdot\|)$ be a Banach space. Let $M \subset X$ be a bounded subset of X. Assume that M is $\varepsilon-$WRK for some $\varepsilon > 0$. Then $conv(M)$ is $2\varepsilon-$WRK. If X is a subspace of a WCG Banach space, or if X^* does not contain a copy of ℓ^1, then $conv(M)$ is $\varepsilon-$WRK.*

2 Proofs

Given a Banach space X and an element $x^{**} \in X^{**}$, the distance $d(x^{**}, X)$ in the norm from x^{**} to X is just $\|q(x^{**})\|$, where $q : X^{**} \to X^{**}/X$ is the canonical quotient mapping. It follows that

$$d(x^{**}, X) = \sup\langle x^{**}, B_{X^\perp}\rangle, \tag{1}$$

where $X^\perp \subset X^{***}$ is the subspace of X^{***} orthogonal to X.

Given a dual pair $\langle X, Y\rangle$, denote by $\mu(X, Y)$ the associated Mackey topology on X, i.e., the topology on X of the uniform convergence on absolutely convex and $w(Y, X)-$compact subsets of Y.

The following elementary proposition relates the distance $d(x^{**}, X)$ to the values of x^{**} on neighbourhoods of 0 in B_{X^*}:

Lemma 4 *Let X be a Banach space. Given $x^{**} \in X^{**}$, let $d := d(x^{**}, X)$. Then*
(i) For every $\varepsilon > 0$, there exists W, a neighbourhood of 0 in (B_{X^*}, w^*), such that $\sup\langle x^{**}, W\rangle < d + \varepsilon$.
(ii) For every absolutely convex and weakly compact subset M of X,

$$\sup\langle x^{**}, M^\circ \cap B_{X^*}\rangle \geq d,$$

where M° denotes the polar set of M in X^*.

Proof: (i) Given $\varepsilon > 0$, choose $x \in X$ such that $d \leq \|x^{**} - x\| < d + \frac{1}{2}\varepsilon$. Let $W := \{x^* \in B_{X^*} : \ |\langle x, x^*\rangle| < \frac{1}{2}\varepsilon\}$. Let $y^* \in W$. Then

$$\begin{aligned}\langle x^{**}, y^*\rangle &= \langle x^{**} - x, y^*\rangle + \langle x, y^*\rangle \leq \\ &\leq \|x^{**} - x\| + \langle x, y^*\rangle < d + \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = d + \varepsilon.\end{aligned}$$

(ii) Recall that $w(X^{***}, X^{**})$ and $\mu(X^{***}, X^{**})$ are compatible topologies on X^{***}, so

$$\overline{B_{X^*}}^{\mu(X^{***},X^{**})} = \overline{B_{X^*}}^{w(X^{***},X^{**})} = B_{X^{***}}. \tag{2}$$

Obviously, the topology $w(X^{**}, X^{***})$ on X^{**} (i.e., its weak topology) induces the topology $w(X, X^*)$ on X (i.e., its weak topology), and then M is also weakly compact in X^{**}, hence equicontinuous for the topology $\mu(X^{***}, X^{**})$ on X^{***}. Given $\delta > 0$, choose, by (1), $x^\perp \in B_{X^\perp}$ such that

$\langle x^{**}, x^\perp\rangle > d-\delta$. By (2) we can find $x^* \in B_{X^*}$ such that $x^\perp - x^* \in \{x^{***} \in X^{***} : \sup\langle M, x^{***}\rangle \le 1\}$ (then $\sup\langle M, x^\perp - x^*\rangle \le 1$ and so $x^* \in M^\circ \cap B_{X^*}$) and $|\langle x^{**}, x^* - x^\perp\rangle| < \delta$. It follows that

$$\langle x^{**}, x^*\rangle = \langle x^{**}, x^\perp\rangle + \langle x^{**}, x^* - x^\perp\rangle > d - \delta - \delta = d - 2\delta,$$

so, as $\delta > 0$ is arbitrary, $\sup\langle x^{**}, M^\circ \cap B_{X^*}\rangle \ge d$. ∎

Remark 1: The result in Lemma 4 is closely connected to [DGZ, III.2.3], where the behaviour of an element $x^{**} \in X^{**}$ as a function from (B_{X^*}, w^*) onto $\mathbb{R}$ is investigated.

The use of double limits in the study of compactness is implicit in the approach of Eberlein [Eb] and explicit in Grothendieck (see, for example, [Gr]). The following concept relaxes the usual double limit condition.

Definition 5 *Let M be a bounded set of a Banach space X, and let S be a bounded subset of X^*. We say that M $\varepsilon-$*interchanges limits *with S if for any two sequences (x_n) in M and (x^*_m) in S such that the following limits exist,*

$$\lim_n \lim_m \langle x_n, x^*_m\rangle, \quad \lim_m \lim_n \langle x_n, x^*_m\rangle,$$

then

$$|\lim_n \lim_m \langle x_n, x^*_m\rangle - \lim_m \lim_n \langle x_n, x^*_m\rangle| \le \varepsilon.$$

In this case we shall write

$$M\S\varepsilon\S S.$$

Proposition 6 *Let M be a bounded set and $\varepsilon \ge 0$ some number. Then we have*
(i) If M is $\varepsilon-WRK$ then $M\S 2\varepsilon\S B_{X^*}$.
(ii) If $M\S\varepsilon\S B_{X^*}$ then M is $\varepsilon-WRK$.

Proof: (i) Let (x_n) and (x^*_m) be sequences in M and B_{X^*}, respectively, such that both limits

$$\lim_n \lim_m \langle x_n, x^*_m\rangle, \quad \lim_m \lim_n \langle x_n, x^*_m\rangle$$

exist. Let $x^{**} \in \overline{M}^{w^*}$ be a w^*-cluster point of (x_n). Then

$$\lim_n \langle x_n, x^*_m\rangle = \langle x^{**}, x^*_m\rangle, \ \forall m.$$

Fix $\delta > 0$. We can find $x \in X$ such that $\|x^{**} - x\| \leq \varepsilon + \delta$. Choose a subsequence of (x_m^*) (denoted again by (x_m^*)) such that $\lim_m \langle x, x_m^* \rangle$ exists. Let $x^* \in X^*$ be a w^*-cluster point of (x_m^*). We get

$$\lim_m \langle x_n, x_m^* \rangle = \langle x_n, x^* \rangle, \ \forall n,$$
$$\lim_n \lim_m \langle x_n, x_m^* \rangle = \lim_n \langle x_n, x^* \rangle = \langle x^{**}, x^* \rangle,$$

and then

$$\begin{aligned} &|\lim_n \lim_m \langle x_n, x_m^* \rangle - \lim_m \lim_n \langle x_n, x_m^* \rangle| = |\lim_n \langle x_n, x^* \rangle - \lim_m \langle x^{**}, x_m^* \rangle| = \\ &= |\langle x^{**}, x^* \rangle - \lim_m \langle x^{**}, x_m^* \rangle| = |\lim_m \langle x^{**}, x^* - x_m^* \rangle| \leq \\ &\leq |\lim_m \langle x, x^* - x_m^* \rangle| + 2(\varepsilon + \delta) = 2(\varepsilon + \delta). \end{aligned}$$

As $\delta > 0$ is arbitrary, we get the conclusion.

(ii) Assume now $M \S \varepsilon \S B_{X^*}$. Let $x^{**} \in \overline{M}^{w^*}$ and let $d := d(x^{**}, X)$. We shall define inductively two sequences, (x_n) in M and (x_m^*) in B_{X^*}: choose $x_1 \in M$. Define $N(x_1; 1) := \{x^* \in B_{X^*};\ |\langle x_1, x^* \rangle| < 1\}$, a neighbourhood of 0 in (B_{X^*}, w^*). By Lemma 4 we can find $x_1^* \in N(x_1; 1)$ such that

$$d - 1 \leq \langle x^{**}, x_1^* \rangle < d + 1.$$

Choose now $x_2 \in M$ such that $|\langle x^{**} - x_2, x_1^* \rangle| < 1/2$. Define $N(x_1, x_2; 1/2) := \{x^* \in B_{X^*};\ |\langle x_i, x^* \rangle| < 1/2,\ i = 1, 2\}$, a neighbourhood of 0 in (B_{X^*}, w^*). Find $x_2^* \in N(x_1, x_2; 1/2)$ such that $d - 1/2 \leq \langle x^{**}, x_2^* \rangle < d + 1/2$. Continue in this way. We get (x_n) and (x_m^*) such that

$$x_n \in M, \quad x_m^* \in B_{X^*},\ \forall n, m,$$
$$|\langle x^{**} - x_n, x_m^* \rangle| < \frac{1}{n},\ m = 1, 2, \ldots, n,$$
$$|\langle x_n, x_m^* \rangle| < \frac{1}{m},\ n = 1, 2, \ldots, m,$$
$$d - \frac{1}{m} \leq \langle x^{**}, x_m^* \rangle < d + \frac{1}{m},\ m = 1, 2, \ldots$$

We get

$$\lim_n \langle x_n, x_m^* \rangle = \langle x^{**}, x_m^* \rangle,\ \forall m,$$
$$\lim_m \lim_n \langle x_n, x_m^* \rangle = \lim_m \langle x^{**}, x_m^* \rangle = d,$$
$$\lim_m \langle x_n, x_m^* \rangle = 0,\ \forall n,$$
$$\lim_n \lim_m \langle x_n, x_m^* \rangle = 0,$$

so

$$|\lim_m \lim_n \langle x_n, x_m^* \rangle - \lim_n \lim_m \langle x_n, x_m^* \rangle| = d \leq \varepsilon.$$

$\blacksquare$

Remark 2: The case $\varepsilon = 0$ gives the Grothendieck's characterization of relatively weak compatness (see [Gr]).

Remark 3: In Proposition 6, (i) cannot be improved, even for separable Banach spaces. In fact, the following is true:

Proposition 7 *In every separable Banach space X which contains an isomorphic copy of ℓ^1 there exists an equivalent norm such that, in this norm, B_X is (obviously) 1-WRK although $B_X \S\varepsilon\S B_{X^*}$ is false for every $0 < \varepsilon < 2$.*

In order to see this we need some preliminary facts:

Given $x^{**} \in X^{**}$, the following function on (B_{X^*}, w^*) is introduced in [DGZ, III.2, p.105]: $\hat{x}^{**} : B_{X^*} \to I\!R$ is the infimum of the real continuous functions on (B_{X^*}, w^*) which are greater or equal than x^{**}. The following proposition gives two alternative description of $\hat{x}^{**}$:

Proposition 8 *Let X be a Banach space. Then, given $x^{**} \in X^{**}$,*
(i)

$$\hat{x}^{**}(x^*) = \lim_{N \in \mathcal{N}(x_0^*)} \{\sup \langle x^{**}, N \rangle\}, \quad \forall x^* \in B_{X^*}, \tag{3}$$

where $\mathcal{N}(x_0^*)$ denotes the filter of neighborhoods of x^* in (B_{X^*}, w^*).
(ii) *[DGZ, III.2.3]*

$$\hat{x}^{**}(x^*) = \inf\{\langle x, x^* \rangle + \|x^{**} - x\|;\ x \in X\},\ \forall x^* \in B_{X^*}. \tag{4}$$

Proof of (i): Let $x_0^* \in B_{X^*}$. Choose $\varepsilon > 0$; there exists a continuous function $f : (B_{X^*}, w^*) \to I\!R$ such that $f \geq x^{**}$ on B_{X^*} and

$$\langle x^{**}, x_0^* \rangle \leq \hat{x}^{**}(x_0^*) \leq f(x_0^*) < \hat{x}^{**}(x_0^*) + \varepsilon.$$

Let $N(x_0^*)$ be an open neighbourhood of x_0^* in (B_{X^*}, w^*) such that

$$f(x_0^*) - \varepsilon < f(x^*) < f(x_0^*) + \varepsilon,\ \forall x^* \in N(x_0^*).$$

Then

$$\langle x^{**}, x^* \rangle \leq f(x^*) < f(x_0^*) + \varepsilon < \hat{x}^{**}(x_0^*) + 2\varepsilon,\ \forall x^* \in N(x_0^*),$$

hence

$$\sup\langle x^{**}, N(x_0^*)\rangle \leq \hat{x}^{**}(x_0^*) + 2\varepsilon.$$

The family $\{\sup\langle x^{**}, N\rangle;\ N \in \mathcal{N}(x_0^*)\}$ is a bounded decreasing net and then converges. We get

$$\lim_{N\in\mathcal{N}(x_0^*)} \sup\langle x^{**}, N\rangle \leq \hat{x}^{**}(x_0^*) + 2\varepsilon.$$

To get a lower bound, let's consider an arbitrary neighbourhood $N(x_0^*)$ of x_0^* in (B_{X^*}, w^*) and assume

$$\langle x^{**}, x^*\rangle \leq \hat{x}^{**}(x_0^*) - \varepsilon,\ \forall x^* \in N(x_0^*).$$

Tietze's Theorem allows us to define a continuous function $g : (B_{X^*}, w^*) \to \mathbb{R}$ such that

$$\begin{aligned} g(x_0^*) &= \hat{x}^{**}(x_0^*) - \varepsilon, \\ \hat{x}^{**}(x_0^*) - \varepsilon \leq g(x^*) &\leq \|x^{**}\|,\ \forall x^* \in B_{X^*} \\ g(x^*) &= \|x^{**}\|,\ \forall x^* \in B_{X^*} \setminus N(x_0^*). \end{aligned}$$

It follows that $g \geq x^{**}$ and $g(x_0^*) < \hat{x}_0^{**}(x_0^*)$, a contradiction. Therefore, we can find $x^* \in N(x_0^*)$ such that

$$\hat{x}_0^{**}(x_0^*) - \varepsilon < \langle x^{**}, x^*\rangle,$$

and then

$$\hat{x}_0^{**}(x_0^*) - \varepsilon < \sup\langle x^{**}, N(x_0^*)\rangle.$$

We finally get

$$\hat{x}_0^{**}(x_0^*) - \varepsilon < \lim_{N\in\mathcal{N}(x_0^*)} \{\sup\langle x^{**}, N\rangle\} \leq \hat{x}_0^{**}(x_0^*) + 2\varepsilon.$$

As ε was arbitrary we get the conclusion. ∎

Let X be a Banach space. A norm $\|\cdot\|$ on X is said to be *octahedral* (see, for example, [DGZ, III.2]) if for every finite dimensional subspace F of X and every $\eta > 0$, there exists $y \in S_X$ such that for every $x \in F$, we have

$$\|x + y\| \geq (1 - \eta)(\|x\| + 1).$$

By [DGZ, Lemma III.2.2], if there exists $x^{**} \in X^{**}\setminus\{0\}$ such that $\|x^{**}+x\| = \|x^{**}\|+\|x\|$ for every $x \in X$, then $\|\cdot\|$ is octahedral. The converse implication is true if X is separable ([GK]). The following proposition characterizes such elements x^{**} in X^{**}:

Proposition 9 *Let X be a Banach space, $x^{**} \in S_{X^{**}}$. The following assertions are equivalent:*
(i) $\|x^{**}+x\| = \|x^{**}\| + \|x\|$ *for every* $x \in X$.
(ii) $\hat{x}^{**}(x^*) = 1$, *for every* $x^* \in B_{X^*}$.
(iii) For every $0 < \delta < 1$, $S(x^{**};\delta)$ *is dense in* (B_{X^*}, w^*), *where*

$$S(x^{**};\delta) := \{x^* \in B_{X^*};\ \langle x^{**}, x^*\rangle > 1-\delta\}.$$

(iv) For every $\alpha \in]-1,1[$, $K(x^{**};\alpha)$ *is dense in* (B_{X^*}, w^*), *where*

$$K(x^{**};\delta) := \{x^* \in B_{X^*};\ \langle x^{**}, x^*\rangle = \alpha\}.$$

Proof: The equivalence between (i) and (ii) is proved in [DGZ, III.2.4].

$(ii) \Rightarrow (iii)$: Let $x_0^* \in B_{X^*}$. Let $N_1(x_0^*)$ be a neighbourhood of x_0^* in (B_{X^*}, w^*). By Proposition 8, given $0<\delta<1$ we can find $N_2(x_0^*) \subset N_1(x_0^*)$, a neighbourhood of x_0^* in (B_{X^*}, w^*), such that $\sup\langle x^{**}, N_2(x_0^*)\rangle \geq 1$. Choose $x^* \in N_2(x_0^*)$ such that $\langle x^{**}, x^*\rangle > 1-\delta$. Then $x^* \in S(x^{**};\delta) \cap N_1(x_0^*)$. It follows that $S(x^{**};\delta)$ is dense in (B_{X^*}, w^*).

$(iii) \Rightarrow (ii)$ follows from Proposition 8 and $(iv) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (iv)$: If (iii) is true, so it is (i). It follows that $-x^{**}$ also satisfies (i) and then (iii). Let O be a non-empty open convex subset of (B_{X^*}, w^*). Then, given $\alpha \in]-1,1[$, choose $\delta \in]0, 1-|\alpha|[$ and let $x_1^* \in S(x^{**};\delta) \cap O$, $x_2^* \in O$ such that $\langle x^{**}, x_2^*\rangle < -1+\delta$. Then there exists $x_3^* \in [x_1^*, x_2^*]$ (the linear segment connecting x_1^* and x_2^*) such that $\langle x^{**}, x_3^*\rangle = \alpha$ and $x_3^* \in O$. ∎

Proof of Proposition 7: Fix $0<\varepsilon<2$. By [DGZ, Thm. III.2.5], there exists an octahedral equivalent norm $|||\cdot|||$ on X (in the rest of the proof we shall refer only to this norm on X). Then, by [GK], there exists $x^{**} \in S_{X^{**}}$ such that $|||x^{**}+x||| = |||x^{**}||| + |||x|||$ for every $x \in X$. Choose $0 < \delta < (2-\varepsilon)/2$. By Proposition 9, given $x^* \in -S(x^{**};\delta)$ we can find a sequence (x_m^*) (as (B_{X^*}, w^*) is metrizable) in $S(x^{**};\delta)$ such that $x_m^* \to x^*$ in the w^*–topology. By a diagonal procedure we can choose a sequence (x_n) in B_X such that $x_n \to x^{**}$ on the set $\{x^*, x_m^*;\ m \in I\!N\}$. Then we have

$$\begin{aligned}|\lim_n \lim_m \langle x_n, x_m^*\rangle - \lim_m \lim_n \langle x_n, x_m^*\rangle| &= \\ |\lim_n\langle x_n, x^*\rangle - \lim_m\langle x^{**}, x_m^*\rangle| &= |\langle x^{**}, x^*\rangle - \lim_m\langle x^{**}, x_m^*\rangle| = \\ &= |\lim_m\langle x^{**}, (x^* - x_m^*)\rangle| > 2-2\delta > \varepsilon.\end{aligned}$$

and the assertion is proved. ∎

The proof of the following theorem is a quantitative modification of the proof of Krein's Theorem devised by Pták using his combinatorial lemma in conjunction with the Grothendieck's double limit criterion (see, for example, [Pt], [Ko, §24.5] or [BHO]). We need the following definitions: $C(I\!N) := \{\lambda : I\!N \to [0,1] :\ supp\ \lambda \text{ finite },\ \lambda(I\!N) = 1\}$, where $supp\ \lambda$ denotes the support of λ , i.e., the set $\{n \in I\!N :\ \lambda(n) \neq 0\}$, and $\lambda(B) := \sum_{n\in B} \lambda(n)$ for any $B \subset I\!N$. Let $\mathcal{G}$ be a family of finite subsets of $I\!N$. Given $B \subset I\!N$, let

$$C(B) := \{\lambda \in C(I\!N) :\ supp\ \lambda \subset B\}.$$

Given $\gamma > 0$, let $C(B, \mathcal{G}, \gamma) := \{\lambda \in C(B) :\ \lambda(G) < \gamma,\ \forall G \in \mathcal{G}\}$. Pták's Combinatorial Lemma reads:

Lemma 10 (Pták[Pt]) *The two following conditions on $\mathcal{G}$ are equivalent:*

1. *There exists a strictly increasing sequence $A_1 \subset A_2 \subset \ldots$ of finite subsets of $I\!N$ and a sequence (G_n) in $\mathcal{G}$ with $A_n \subset G_n$ for all n.*

2. *There exists an infinite subset $B \subset I\!N$ and an $\gamma > 0$ such that*

$$C(B, \mathcal{G}, \gamma) = \emptyset.$$

Theorem 11 *Let $(X, \|\cdot\|)$ be a Banach space. Let $M \subset X$ be a bounded subset of X. Assume that $M§\varepsilon§B_{X^*}$ for some $\varepsilon \geq 0$. Then $conv(M)§\varepsilon§B_{X^*}$.*

Proof: Assume $\|x\| \leq \mu$ for all $x \in M$ and some $\mu > 0$. Choose $\varepsilon > 0$ and $0 < \beta < \varepsilon$. Select now $\delta > 0$ and $\gamma > 0$ such that $\beta + 2\gamma\mu < \varepsilon - \delta$. Suppose that there exists a sequence (x_n) in $conv(M)$ and a sequence (x^*_m) in B_{X^*} such that

$$|\lim_n \lim_m \langle x_n, x^*_m\rangle - \lim_m \lim_n \langle x_n, x^*_m\rangle| = \varepsilon > 0.$$

Let $x^*_0 \in B_{X^*}$ be a cluster point of (x^*_m) in (B_{X^*}, w^*). Let $T \subset M$ be a countable set such that $\{x_n :\ n \in I\!N\} \subset conv(T)$ and choose a subsequence (denoted again by (x^*_m)) such that $x^*_m \to x^*_0$ on the set T. Then, for some $\sigma \in \{-1, 1\}$,

$$\sigma(\lim_n \langle x_n, x^*_0\rangle - \lim_m \lim_n \langle x_n, x^*_m\rangle) = \varepsilon.$$

By suppressing a finite number of indices, we may assume

$$\sigma(\lim_n \langle x_n, x^*_0\rangle - \lim_n \langle x_n, x^*_m\rangle) = \sigma \lim_n \langle x_n, x^*_0 - x^*_m\rangle > \varepsilon - \delta,\ \forall m.$$

Define

$$\Gamma(t) := \{m \in I\!N : \ |\langle t, x_0^* - x_m^*\rangle| \geq \beta\}, \ t \in T.$$

Those are finite subsets of $I\!N$. Let

$$\mathcal{G} := \{\Gamma(t) : \ t \in T\}.$$

Assume $C(I\!N, \mathcal{G}, \gamma) \neq \emptyset$ and choose $\lambda \in C(I\!N, \mathcal{G}, \gamma)$. It follows that

$$\lambda(\Gamma(t)) < \gamma, \ \forall t \in T.$$

Form

$$x^* := \sum_{k \in I\!N} \lambda(k)(x_0^* - x_k^*) \in 2B_{X^*}.$$

Given $t \in T$,

$$|\langle t, x^*\rangle| = \left|\sum_{k \in I\!N} \lambda(k)\langle t, x_0^* - x_k^*\rangle\right| \leq$$
$$\leq \textstyle\sum_{\Gamma(t)} \lambda(k)|\langle t, x_0^* - x_k^*\rangle| + \sum_{I\!N \setminus \Gamma(t)} \lambda(k)|\langle t, x_0^* - x_k^*\rangle| < 2\gamma\mu + \beta.$$

It follows that $|\langle x_n, x^*\rangle| \leq 2\gamma\mu + \beta, \ \forall n$. Then

$$2\gamma\mu + \beta \geq \lim_n |\langle x_n, x^*\rangle| =$$
$$= |\textstyle\sum_{k \in I\!N} \lambda(k) \lim_n \langle x_n, x_0^* - x_k^*\rangle| = \sigma \sum_{k \in I\!N} \lambda(k) \lim_n \langle x_n, x_0^* - x_k^*\rangle > \varepsilon - \delta,$$

a contradiction.

Assume then $C(I\!N, \mathcal{G}, \gamma) = \emptyset$. Then, by Lemma 10 we can find $A_p := \{m_1, m_2, \ldots, m_p\} \subset I\!N$ and $t_p \in T$ such that

$$A_p \subset \Gamma(t_p), \ \forall p \in I\!N,$$

i.e., $|\langle t_p, x_0^* - x_{m_k}^*\rangle| \geq \beta, \ k = 1, 2, \ldots, p$. Choose a subsequence of (t_n) (denoted again by (t_n)) such that there exists $\lim_n \langle t_n, x_0^* - x_{m_k}^*\rangle$, for any k. Then we get

$$\lim_n \lim_k \langle t_n, x_{m_k}^*\rangle = \lim_n \langle t_n, x_0^*\rangle,$$
$$|\lim_n \langle t_n, x_0^*\rangle - \lim_k \lim_n \langle t_n, x_{m_k}^*\rangle| = \lim_k \lim_n |\langle t_n, x_0^* - x_{m_k}^*\rangle| \geq \beta,$$

so

$$|\lim_n \lim_k \langle t_n, x_{m_k}^*\rangle - \lim_k \lim_n \langle t_n, x_{m_k}^*\rangle| \geq \beta. \tag{5}$$

As β satisfies $0 < \beta < \varepsilon$ and it is otherwise arbitrary, we get the conclusion. ∎

Proof of Theorem 3: The general case follows from Proposition 6 and Theorem 11. In order to prove the WCG case, the following modification of Proposition 6 is needed:

Proposition 12 *Let M be a bounded set and let $\varepsilon > 0$. Then we have*
(i) If M is $\varepsilon-$WRK then $M\S\varepsilon\S(x_n^*)$, where (x_n^*) is any w^*-null sequence in B_{X^*}.
(ii) If X is WCG and $M\S\varepsilon\S(x_n^*)$ for any w^*-null sequence in B_{X^*} then M is $\varepsilon-$WRK.

Proof: (i) follows directly from the proof of (i) in Proposition 6. To establish (ii) here modify the proof of (ii) in the same proposition in the following way: Let $M \subset X$ be a linearly dense absolutely convex and weakly compact subset of X; change

$$N_p := N(x_1, x_2, \ldots, x_p; 1/p) := \{x^* \in B_{X^*};\ |\langle x_i, x^*\rangle| < 1/p,\ i = 1, 2, \ldots, p\}$$

to

$$\begin{aligned} N_p &:= N(x_1, x_2, \ldots, x_p, K, 1/p) := \\ &:= \{x^* \in B_{X^*};\ |\langle x_i, x^*\rangle| < 1/p,\ i = 1, 2, \ldots, p,\ \sup|\langle K, x^*\rangle| < 1/p\}, \end{aligned}$$

where $K \subset X$ is a linearly dense absolutely convex and weakly compact subset of X. It is enough now to observe that the sequence (x_m^*) constructed there is w^*-null. This proves the WCG case.

If X is a subspace of a WCG Banach space Z, first observe that we can define a sequence $(M_{n,p})_{p,n=1}^{\infty}$ of subsets of B_X such that $B_X = \cup_{n=1}^{\infty} M_{n,p},\ p = 1, 2, \ldots$, and each $M_{n,p}$ is $(1/p)-$WRK: just write

$$M_{n,p} := \left(nK + \frac{1}{p} B_{Z^{**}}\right) \cap B_X,$$

where K is an absolutely convex weakly compact and linearly dense subset of Z (see [FMZ]) Now, the following modification of (ii) in Lemma 4 is easy to prove (we refer to the notations there): *Let M be an absolutely convex* *and weakly compact subset of X^{**}. Assume that $M \subset X + \varepsilon B_{X^{**}}$ for some $\varepsilon > 0$. Then, given $x^{**} \in X^{**}$,*

$$\sup\langle x^{**}, M^\circ \cap B_{X^*}\rangle \geq \frac{d}{1+3\varepsilon}$$

Now, enumerate the family $(M_{n,p})_{n,p=1}^{\infty}$ using an one-to-one and onto mapping $j : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that $i \to \|j(i)\|_1$ is increasing and use the resulting sequence in the inductive construction of (x_m^*) in Proposition 6, (ii) modified as in the WCG case above. ∎

Remark 4: Our conjecture is that, in general, the property of being $\varepsilon-$WRK is preserved when passing to convex hulls. Our approach does not produce this much. The following theorem gives another setting in which the constant is preserved.

Theorem 13 *Let X be a Banach space. If X^* does not contains a copy of ℓ^1 and $M \subset X$ is a $\varepsilon-$WRK for some $\varepsilon > 0$, then $\overline{conv(M)}$ is again $\varepsilon-$WRK.*

Proof: Let $C := \overline{conv(M)}^{w^*}$, a w^*-compact convex subset of X^{**}. It is well known (see, for example, [Di, p.215]) that

$$C = \overline{conv(Ext\ C)}^{\|\cdot\|},$$

where $Ext\ C$ denotes the set of extreme points of C. By Milman's Theorem (see, for example, [Ko, §25.1.7]), $Ext\ C \subset \overline{M}^{w^*}$. As $\{x^{**} \in X^{**} : d(x^{**}, X) \leq \varepsilon\}$ is $\|\cdot\|-$closed, where d denotes the distance in the norm, this proves that $\overline{conv(M)}$ is $\varepsilon-$WRK. ∎

Remark 5: After Theorem 3 the requirement that sets $M_{n,p}$ in Theorem 2 should be convex and symmetric can be avoided.

The following Theorem is, of course, more restrictive than Theorem 3, as it is stated only for separable Banach spaces and gives four times the constant. However, the technique of the proof is completely different, as it uses Simons inequality (see, for example, [FHHPMZ, Lemma 3.47]) instead of double limits. It has the slight advantage that it estimates the distance from $x^{**} \in \overline{conv(M)}^{w^*}$ to $\overline{conv(\overline{M}^{w^*})}^{\|\cdot\|}$.

Theorem 14 *Let X be a separable Banach space. Let $M \subset X$ be a bounded subset of X and assume that M is $\varepsilon-$WRK for some $\varepsilon \geq 0$. Then $conv(M)$ is $4\varepsilon-$WRK.*

Proof: Let $x_0^{**} \in \overline{conv}^{w^*}(M)$ and let d be the distance from x^{**} to $\overline{conv}^{\|\cdot\|}(\overline{M}^{w^*})$. Find $x_0^{***} \in S_{X^{***}}$ such that

$$s + d := \sup\langle \overline{conv}^{\|\cdot\|}(\overline{M}^{w^*}), x_0^{***}\rangle + d = \langle x_0^{**}, x_0^{***}\rangle.$$

Given $x^{**} \in \overline{M}^{w^*}$, find $p(x^{**}) \in X$ such that $\|x^{**} - p(x^{**})\| \leq \varepsilon$. We get $p[\overline{M}^{w^*}] \subset X$ and, as X is separable, there exists a countable $\|\cdot\|-$dense subset $N \subset p[\overline{M}^{w^*}]$. We can find a sequence (x_n^*) in B_{X^*} such that $x_n^* \to x_0^{***}$ on points in $N \cup \{x_0^{**}\}$. In particular,

$$\langle x_0^{**}, x_n^*\rangle \to \langle x_0^{**}, x_0^{***}\rangle = s + d.$$

Fix $\delta > 0$. We can assume $\langle x_0^{**}, x_n^*\rangle \geq s + d - \delta$, for all $n \in I\!N$. Let

$$S := \{x^* \in B_{X^*} :\ \langle x_0^{**}, x^*\rangle \geq s + d - \delta\},$$

a $\|\cdot\|-$closed non-empty convex section of B_{X^*}. $\overline{M}^{w^*}$ is a boundary of the set $\overline{conv}^{w^*}(M)$ for the Banach space X^*. From now on we shall work in the Banach space $(l^\infty(\overline{M}^{w^*}), \|\cdot\|_\infty)$. In this space S is a superconvex subset, and (x_n^*) a sequence in S. Let $u := \limsup x_n^*$. Given $x^{**} \in \overline{M}^{w^*}$,

$$\begin{aligned}\langle x^{**}, x_n^*\rangle &= \langle p(x^{**}), x_n^*\rangle + \langle x^{**} - p(x^{**}), x_n^*\rangle = \\ &= \langle v, x_n^*\rangle + \langle p(x^{**}) - v, x_n^*\rangle + \langle x^{**} - p(x^{**}), x_n^*\rangle,\end{aligned}$$

where $v \in N$ and $\|p(x^{**}) - v\| < \delta$. Moreover

$$\langle v, x_n^*\rangle \to \langle v, x_0^{***}\rangle = \langle x^{**}, x_0^{***}\rangle + \langle v - x^{**}, x_0^{***}\rangle,$$

and $\|v - x^{**}\| < \varepsilon + \delta$. It follows that

$$u(x^{**}) \leq s + 2\varepsilon + 2\delta.$$

As $\delta > 0$ is arbitrary, $\sup u(\overline{M}^{w^*}) \leq s + 2\varepsilon$. Apply now Simons inequality to (x_n) and $\overline{M}^{w^*}$. We get

$$\begin{aligned}&s + 2\varepsilon \geq \sup u(\overline{M}^{w^*}) \geq \inf\{\sup x^*(\overline{M}^{w^*}) :\ x^* \in S\} = \\ &\quad \inf\{\sup x^*[\overline{conv}^{w^*}(M)] :\ x^* \in S\} \geq \inf\{\langle x_0^{**}, x^*\rangle :\ x^* \in S\} \geq s + d - \delta.\end{aligned}$$

It follows that $2\varepsilon \geq d - \delta$. As $\delta > 0$ was arbitrary, $2\varepsilon \geq d$ and we get

$$\|\cdot\| - dist\left(x_0^{**}, \overline{conv}^{\|\cdot\|}(\overline{M}^{w^*})\right) \leq 2\varepsilon.$$

This implies that $conv(M)$ is $4\varepsilon-$WRK. ∎

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