POLYHEDRALITY IN ORLICZ SPACES

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Abstract. We present a construction of an Orlicz space admitting a C^∞-smooth bump which depends locally on finitely many coordinates, and which is not isomorphic to a subspace of any $C(K)$, K scattered. In view of the related results this space is possibly not isomorphic to a polyhedral space.

1. Introduction

In the present paper we investigate the properties of Orlicz sequence spaces admitting bump functions that depend locally on finitely many coordinates (LFC).

The first use of the LFC notion for a function was the construction of C^∞-smooth and LFC renorming of c_0, due to Kuiper, which appeared in [BF]. The LFC notion was explicitly introduced and investigated in the paper [PWZ] of Pechanec, Whitfield and Zizler. In their work the authors have proved that every Banach space admitting a LFC bump is saturated with copies of c_0, providing in some sense a converse to Kuiper's result. Not surprisingly, it turns out that the LFC notion is closely related to the class of polyhedral spaces, introduced by Klee [K] and thoroughly investigated by many authors (see [JL, Chapter 15] for results and references). (We note that polyhedrality is understood in the isomorphic sense in this paper.) Indeed, prior to [PWZ], Fonf [F1] has proved that every polyhedral Banach space is saturated with copies of c_0. Later, it was independently proved in [F2] and [Haj1] that every separable polyhedral Banach space admits an equivalent LFC norm. Using the last result Fonf's result is a corollary of [PWZ]. The notion of LFC has been exploited (at least implicitly) in a number of papers, in order to obtain very smooth bump functions, norms and partitions of unity on non-separable Banach spaces, see e.g. [To], [Ta], [DGZ1], [GPWZ], [GTWZ], [FZ], [Hay1], [Hay2], [Hay3], [S1], [S2], [Haj1], [Haj2], [Haj3], and the book [DGZ]. In fact, it seems to be the only general approach to these problems. The reason is simple; it is relatively easy to check the (higher) differentiability properties of functions of several variables, while for functions defined on a Banach space it is very hard.

For separable spaces, one of the main known results is that a separable Banach space is polyhedral if and only if it admits a LFC renorming (resp. C^∞-smooth and LFC renorming) ([Haj1]). This smoothing up result is however obtained by using the boundary of a Banach space, rather than through some direct smoothing procedure. Another recent result ([HJ1]) is that a separable Banach space with a (shrinking) Schauder basis has a C^∞-smooth and LFC bump function whenever it has a continuous LFC bump. This seems to be the first relatively general result in this direction.

The main result of the paper, contained in Section 4, is a certain rather subtle construction of an Orlicz sequence space having a C^∞-smooth and LFC bump function, which we suspect to be non-polyhedral. Such an example is of course needed to justify the whole theory, since in the polyhedral case the smoothing up (and structural) results are well known and easier. In fact, our paper, and in particular the example was motivated by the beautiful theory of polyhedrality for separable Banach spaces with Schauder basis, and especially Orlicz sequence spaces, developed by Leung in [L1] and [L2]. The key result of these works is the following theorem.

Theorem ([L2])**.** *The following statements are equivalent for every non-degenerate Orlicz function M:*

(i) There exists a constant $K > 0$ such that $\lim\limits_{t\to 0+} \frac{M(Kt)}{M(t)} = \infty$.

(ii) The Orlicz sequence space h_M is isomorphic to a subspace of $C(\omega^\omega)$.

(iii) The Orlicz sequence space h_M is isomorphic to a subspace of $C(K)$ for some scattered compact K.

All spaces satisfying (ii) are polyhedral, and Leung conjectured that conversely all polyhedral Orlicz sequence spaces fall under this description. There is a strong evidence supporting this idea. First, Theorem 8, part of which is also in Leung's paper, shows that the naturally defined LFC renormings exist precisely for those spaces. Second,

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negating the condition in (i) we obtain the following formula

$$(\forall K>0)(\exists\{t_n\}_{n=1}^{\infty}, t_n\searrow 0)\lim_{n\to\infty}\frac{M(Kt_n)}{M(t_n)}<\infty.$$

Reversing the order of the quantifiers we obtain the following stronger (less general) condition

$$(\exists\{t_n\}_{n=1}^{\infty}, t_n\searrow 0)(\forall K>0)\lim_{n\to\infty}\frac{M(Kt_n)}{M(t_n)}<\infty.$$

Leung proved that Orlicz sequence spaces satisfying the last condition are not polyhedral (although they may be c_0 saturated).

Thus Leung's theorem above is a near characterisation of polyhedrality for Orlicz sequence spaces, the gap lying in the exchange of quantifiers. Our example of an Orlicz sequence space with C^∞-smooth and LFC bump lies strictly in between the above conditions. Therefore, our space is either a non-polyhedral space admitting a LFC bump (we are inclined to believe this alternative), or Leung's polyhedral conjecture is false.

We refer to [FHHMPZ], [LT] and [JL] for background material and results.

2. Preliminaries

We use a standard Banach space notation. If $\{e_i\}$ is a Schauder basis of a Banach space, we denote by $\{e_i^*\}$ its biorthogonal functionals. P_n are the canonical projections associated with the basis $\{e_i\}$, P_n^* are the operators adjoint to P_n, i.e. the canonical projections associated with the basis $\{e_i^*\}$. Given a set $A\subset\mathbb{N}$ we denote by P_A the projection associated with the set A, i.e. $P_Ax=\sum_{i\in A}e_i^*(x)e_i$. By R_n we denote the projections $R_n=I-P_n$, where I is the identity operator. For a finite set B, $|B|$ denotes the number of elements of B. $U(x,\delta)$ is an open ball centered at x with radius δ.

The notion of a function, defined on a Banach space with a Schauder basis, which is locally dependent on finitely many coordinates was introduced in [PWZ]. The following definition is a slight generalisation which was used by many authors.

Definition 1. *Let X be a topological vector space, $\Omega\subset X$ an open subset, E be an arbitrary set, $M\subset X^*$ and $g\colon\Omega\to E$. We say that g* depends only on M *on a set $U\subset\Omega$ if $g(x)=g(y)$ whenever $x,y\in U$ are such that $f(x)=f(y)$ for all $f\in M$. We say that g* depends locally on finitely many coordinates from M *(LFC-M for short) if for each $x\in\Omega$ there are a neighbourhood $U\subset\Omega$ of x and a finite subset $F\subset M$ such that g depends only on F on U. We say that g* depends locally on finitely many coordinates *(LFC for short) if it is LFC-X^*.*

We may equivalently say that g depends only on $\{f_1,\dots,f_n\}\subset X^*$ on $U\subset\Omega$ if there exist a mapping $G\colon\mathbb{R}^n\to E$ such that $g(x)=G(f_1(x),\dots,f_n(x))$ for all $x\in U$. Notice, that if $g\colon\Omega\to E$ is LFC and $h\colon E\to F$ is any mapping, then also $h\circ g$ is LFC.

The canonical example of a non-trivial LFC function is the sup norm on c_0, which is LFC-$\{e_i^*\}$ away from the origin. Indeed, take any $x=(x_i)\in c_0$, $x\neq 0$. Let $n\in\mathbb{N}$ be such that $|x_i|<\|x\|_\infty/2$ for $i>n$. Then $\|\cdot\|_\infty$ depends only on $\{e_1^*,\dots,e_n^*\}$ on $U(x,\|x\|_\infty/4)$.

A norm on a normed space is said to be LFC, if it is LFC away from the origin. Recall that a bump function (or bump) on a topological vector space X is a function $b\colon X\to\mathbb{R}$ with a bounded non-empty support.

Let X be a Banach lattice. We say that a function $f\colon X\to\mathbb{R}$ is a lattice function if it satisfies either $f(x)\le f(y)$ whenever $|x|\le|y|$, or $f(x)\ge f(y)$ whenever $|x|\le|y|$. Recall that a Banach space X with an unconditional basis $\{e_i\}$ has a natural lattice structure defined by $\sum a_ie_i\ge 0$ if and only if $a_i\ge 0$ for all $i\in\mathbb{N}$.

The word "coordinate" in the term LFC originates of course from spaces with bases, where LFC was first defined using the coordinate functionals. In order to apply the LFC techniques to spaces without a Schauder basis, the notion had to be obviously generalised using arbitrary functionals from the dual. However, as shown in [HJ1], the generalisation does not substantially increase the supply of LFC functions on Banach spaces with a Schauder basis, and we can always in addition assume that the given LFC function in fact depends on the coordinate functionals. This fact is not only interesting in itself; it is the main tool for smoothing up LFC bumps on separable spaces with basis.

The following results from [HJ1] will be needed in the sequel:

Lemma 2. *Let X be a Banach space with a Schauder basis $\{e_i\}$ and E be an arbitrary set. Then $f\colon X\to E$ is LFC-$\{e_i^*\}$ if and only if for each $x\in X$ there is $\delta>0$ and $n_0\in\mathbb{N}$ such that $f(y)=f(P_ny)$ whenever $\|x-y\|<\delta$ and $n\ge n_0$.*

Theorem 3. *Let E be a set, X be a Banach space with a shrinking Schauder basis $\{e_i\}$, $g\colon X\to E$ be a LFC mapping and $\varepsilon>0$. Then there is a (shrinking) Schauder basis $\{x_i\}$ of X, $(1+\varepsilon)$-equivalent to $\{e_i\}$, such that g is LFC-$\{x_i^*\}$.*

Theorem 4. *Let X be a Banach space with an unconditional Schauder basis $\{e_i\}$, which admits a continuous LFC bump. Then X admits a C^∞-smooth LFC-$\{e_i^*\}$ lattice bump.*

3. Spaces with symmetric Schauder bases

Let X be a Banach space with a symmetric Schauder basis. In such spaces it is possible to define a notion of the *non-increasing reordering*, which will be one of the main tools in the sequel. For any $x \in X$, $x = (x_i)$, let us denote by $\widehat{x}$ a vector in X with its coordinates formed by the non-increasing reordering of the sequence $(|x_i|)$. Notice that we can view X as a linear subspace of c_0 through the natural "coordinate" embedding. In the following lemma we gather some simple properties of this reordering which will be used later.

Lemma 5. *Let X be a Banach space with a symmetric Schauder basis, $x, y \in X$ be arbitrary.*

(a) Let $\|\cdot\|$ be a symmetric lattice norm on X. Then $\big|\|P_k\widehat{x}\| - \|P_k\widehat{y}\|\big| \le \|x-y\|$ for any $k \in \mathbb{N}$.
(b) $\widehat{R_n\widehat{x}} \le \widehat{R_n x}$ in the lattice sense for any $n \in \mathbb{N}$.
(c) $\|\widehat{x}-\widehat{y}\|_\infty \le \|x-y\|_\infty$.
(d) Let $\|\cdot\|$ be a lattice norm on X such that the basis is normalised. Then the mapping $x \mapsto P_n\widehat{x}$ is n-Lipschitz for any $n \in \mathbb{N}$.

Proof. (a): Consider a set $A \subset \mathbb{N}$, $|A| = k$, such that $\widehat{P_A x} = P_k\widehat{x}$. Since $\|\cdot\|$ is symmetric and lattice, $\|P_k\widehat{x}\| = \|P_A x\|$ and $\|P_k\widehat{y}\| \ge \|P_A y\|$. Therefore $\|P_k\widehat{x}\| - \|P_k\widehat{y}\| \le \|P_A x\| - \|P_A y\| \le \|P_A(x-y)\| \le \|x-y\|$.

(b): Let $A \subset \mathbb{N}$, $|A| \le n$ be such that $\widehat{R_n x} = \widehat{w}$, where $w = \widehat{x} - P_A\widehat{x}$. We put $z = R_n\widehat{x}$. Then $\widehat{z}_i = \widehat{x}_{i+n}$ for $i \in \mathbb{N}$. Let $\pi\colon \mathbb{N} \to \mathbb{N}$ be a one to one mapping such that $\widehat{w}_i = w_{\pi(i)}$. Then $\widehat{w}_i = \widehat{x}_{\pi(i)}$ for $i \in \mathbb{N}$. As $i \le \pi(i) \le i+n$, it follows that $\widehat{z}_i = \widehat{x}_{i+n} \le \widehat{x}_{\pi(i)} = \widehat{w}_i$.

(c): Let $\pi\colon \mathbb{N} \to \mathbb{N}$ and $\sigma\colon \mathbb{N} \to \mathbb{N}$ be one to one mappings such that $\widehat{x}_i = |x_{\pi(i)}|$ and $\widehat{y}_i = |y_{\sigma(i)}|$. Pick any $n \in \mathbb{N}$. There is $k \le n$ such that $|y_{\pi(k)}| \le |y_{\sigma(n)}|$. (Otherwise there would be at least n coordinates of y for which their absolute value is greater than $|y_{\sigma(n)}|$ which is impossible.) Consequently, $\widehat{x}_n - \widehat{y}_n = |x_{\pi(n)}| - |y_{\sigma(n)}| \le |x_{\pi(k)}| - |y_{\pi(k)}| \le |x_{\pi(k)} - y_{\pi(k)}| \le \|x-y\|_\infty$.

(d): Using the fact that the basis is normalised, then (c) and then the fact that $\|\cdot\|$ is lattice we obtain $\|P_n\widehat{x} - P_n\widehat{y}\| = \|P_n(\widehat{x}-\widehat{y})\| \le \sum_{i=1}^n |(\widehat{x}-\widehat{y})_i| \le n\|\widehat{x}-\widehat{y}\|_\infty \le n\|x-y\|_\infty \le n\|x-y\|$. □

This is the key lemma:

Lemma 6. *Let X be a Banach space with a symmetric Schauder basis $\{e_i\}$, $\Phi\colon X \to \mathbb{R}$ be a continuous function such that $\Phi(x) > 0$ if $x \ne 0$ and $\{\gamma_n\} \subset \mathbb{R}$ be a sequence decreasing to 1. For any $N \in \mathbb{N}$, define*

$$\Psi_N(x) = \max_{1 \le n \le N} \gamma_n \Phi(P_n\widehat{x}).$$

Then each function Ψ_N is LFC-$\{e_i^*\}$ on $X \setminus \{0\}$.

Proof. Without loss of generality we may assume that $\|\cdot\|$ is symmetric and lattice. Let $N \in \mathbb{N}$ and $x \in X\setminus\{0\}$ be given. We claim that there exist a neighbourhood V of x and $N_1 \in \mathbb{N}$ such that $\widehat{x}_{N_1} > \widehat{x}_{N_1+1}$ and $\Psi_N(y) = \Psi_{\min\{N,N_1\}}(y)$ for all $y \in U$. If $|\operatorname{supp} x| \ge N$, then there exists $N_1 \ge N$ such that $\widehat{x}_{N_1} > \widehat{x}_{N_1+1}$ and the claim follows. Otherwise, find $N_1 < N$ such that $\widehat{x}_{N_1} > \widehat{x}_{N_1+1} = 0$. Then choose $0 < \delta < \widehat{x}_{N_1}/2$ such that

$$|\Phi(z) - \Phi(\widehat{x})| < \frac{\gamma_{N_1} - \gamma_{N_1+1}}{2\gamma_1}\Phi(\widehat{x})$$

if $\|z - \widehat{x}\| < (N_1+1)\delta$. Denote $B = \operatorname{supp} x$ and notice that $|B| = N_1$. If $\|x-y\| < \delta$, $i \in B$ and $j \notin B$, then

$$|y_i| \ge |x_i| - \delta \ge \widehat{x}_{N_1} - \delta > 2\delta - \delta = \delta = |x_j| + \delta \ge |y_j|$$

and hence

$$\|R_{N_1}\widehat{y}\| = \|P_{\mathbb{N}\setminus B}\, y\| = \|P_{\mathbb{N}\setminus B}(y-x)\| \le \|y-x\| < \delta.$$

Thus, for any $n \ge N_1$,

$$\|P_n\widehat{y} - \widehat{x}\| = \|P_n\widehat{y} - P_{N_1}\widehat{x}\| \le \|R_{N_1}\widehat{y}\| + \|P_{N_1}\widehat{y} - P_{N_1}\widehat{x}\| < \delta + N_1\|\widehat{y}-\widehat{x}\|_\infty \le \delta + N_1\|\widehat{y}-\widehat{x}\| < (N_1+1)\delta.$$

(For the last but one inequality use Lemma 5(c).) It follows from the choice of δ that for $n > N_1$ we have

$$\gamma_n\Phi(P_n\widehat{y}) < \gamma_n\left(1 + \frac{\gamma_{N_1} - \gamma_{N_1+1}}{2\gamma_1}\right)\Phi(\widehat{x}) \le \gamma_{N_1+1}\left(1 + \frac{\gamma_{N_1} - \gamma_{N_1+1}}{2\gamma_{N_1+1}}\right)\Phi(\widehat{x}) = \frac{\gamma_{N_1} + \gamma_{N_1+1}}{2}\Phi(\widehat{x}).$$

On the other hand,

$$\gamma_{N_1}\Phi(P_{N_1}\widehat{y}) > \gamma_{N_1}\left(1 - \frac{\gamma_{N_1} - \gamma_{N_1+1}}{2\gamma_1}\right)\Phi(\widehat{x}) \ge \gamma_{N_1}\left(1 - \frac{\gamma_{N_1} - \gamma_{N_1+1}}{2\gamma_{N_1}}\right)\Phi(\widehat{x}) = \frac{\gamma_{N_1} + \gamma_{N_1+1}}{2}\Phi(\widehat{x}).$$

This means that $\Psi_N(y) = \max_{1\le n\le N_1} \gamma_n \Phi(P_n\widehat{y})$ for $\|x-y\| < \delta$, which proves the claim.

Using N_1 and V from the claim, let $\varepsilon = (\widehat{x}_{N_1} - \widehat{x}_{N_1+1})/2$. Choose $A \subset \mathbb{N}$, $|A| = N_1$, such that $P_{N_1}\widehat{x} = \widehat{P_A x}$. If $\|x-y\| < \varepsilon$, then $|y_i| > |y_j|$ whenever $i \in A$ and $j \notin A$. Hence for $1 \le n \le N_1$ the mappings $y \mapsto P_n\widehat{y}$ depend only on $\{e_i^*\}_{i\in A}$ on $U(x,\varepsilon)$. By the choice of N_1, it follows that Ψ_N depends only on $\{e_i^*\}_{i\in A}$ on $V \cap U(x,\varepsilon)$.

□

4. Orlicz Sequence Spaces

This section contains the main result of the paper, namely a construction of an Orlicz sequence space h_M with a C^∞-smooth and LFC bump, which does not embed into any $C(K)$ space, K scattered compact. As explained in the introduction, our space is possibly non-polyhedral. If so, it would be the first separable example of a Banach space for which the best smoothness (in the wider sense) of its bumps exceeds the best smoothness of its renormings. Indeed, our space has C^∞-smooth renormings, but, if non-polyhedral, it would have no LFC renormings. Up to now, the only examples (due to Haydon [Hay3], see also [DGZ]) with a similar property are non-separable. Recall that Haydon's space has a C^∞-smooth bump, but no equivalent Gâteaux smooth norm (and in fact using basically the same proof one can conclude that it neither has an equivalent LFC renorming).

For the basic properties of Orlicz sequence spaces we refer e.g. to [LT].

Let M be a non-degenerate Orlicz function and h_M be the respective Orlicz sequence space. We define a function $\nu\colon h_M \to [0,\infty)$ by $\nu(x) = \sum_{i=1}^\infty M(|x_i|)$. It is easily checked that this function is convex, symmetric and lattice, $\nu(0) = 0$, $\nu(x) > 0$ for $x \ne 0$, and, by the definition of the norm in h_M, $\|x\| = 1$ if and only if $\nu(x) = 1$. It follows from the convexity that $\nu(x) \le \|x\|$ for $x \in B_{h_M}$, while $\nu(x) \ge \|x\|$ if $\|x\| \ge 1$.

Lemma 7. *The mapping $\mu\colon h_M \to \ell_1$ defined by $\mu(x) = \big(M(|x_i|)\big)$ is continuous. Thus the function $\nu(x) = \|\mu(x)\|_{\ell_1}$ is continuous.*

Proof. Suppose $x \in h_M$ and $0 < \varepsilon < 1$. Choose $N \in \mathbb{N}$ such that $\|R_N x\| < \varepsilon/2$. Then, by the continuity of M, we can choose $0 < \delta < \varepsilon/2$ such that $\|P_N(\mu(x) - \mu(y))\|_{\ell_1} = \sum_{i=1}^N |M(|x_i|) - M(|y_i|)| < \varepsilon$ if $\|x-y\| < \delta$. Further, if $\|x-y\| < \delta$, then $\|R_N y\| \le \|R_N x\| + \|R_N(x-y)\| \le \|R_N x\| + \|x-y\| < \varepsilon$ and hence

$$\begin{aligned}\|\mu(x) - \mu(y)\|_{\ell_1} &\le \|P_N(\mu(x) - \mu(y))\|_{\ell_1} + \|R_N\mu(x)\|_{\ell_1} + \|R_N\mu(y)\|_{\ell_1}\\ &\le \varepsilon + \nu(R_N x) + \nu(R_N y) \le \varepsilon + \|R_N x\| + \|R_N y\| < 3\varepsilon.\end{aligned}$$

□

Let M be a non-degenerate Orlicz function such that there is a $K > 1$ for which $\lim_{t\to 0+} M(Kt)/M(t) = \infty$. Leung in [L1] constructs a sequence $\{\eta_k\}$ of real numbers decreasing to 1 such that the norm on h_M defined by $|||x|||_1 = \sup_k \eta_k \|P_k\widehat{x}\|$ has the property that for each $x \in h_M$ there is $j \in \mathbb{N}$ such that $|||x|||_1 = |||P_j x|||_1$ and the supremum is attained at some $n \in \mathbb{N}$. An immediate consequence of this is that the norm $|||x||| = \sup_k \eta_k^2 \|P_k\widehat{x}\|$ is LFC-$\{e_i^*\}$. To see this, fix $x \in h_M \setminus \{0\}$ and let $n \in \mathbb{N}$ be such that $\eta_n \|P_n\widehat{x}\| = \sup_k \eta_k \|P_k\widehat{x}\|$. Let $\varepsilon = \eta_n \|P_n\widehat{x}\| (\eta_n - \eta_{n+1})/(\eta_n^2 + \eta_{n+1}^2)$ and take $y \in h_M$ satisfying $\|x-y\| < \varepsilon$. Then, by Lemma 5(a), $\big|\|P_k\widehat{x}\| - \|P_k\widehat{y}\|\big| < \varepsilon$ for any $k \in \mathbb{N}$. Thus, for $k > n$,

$$\eta_n^2 \|P_n\widehat{y}\| > \eta_n^2 \|P_n\widehat{x}\| - \eta_n^2\varepsilon = \eta_{n+1}\eta_n \|P_n\widehat{x}\| + \eta_{n+1}^2\varepsilon \ge \eta_k\eta_n \|P_n\widehat{x}\| + \eta_k^2\varepsilon \ge \eta_k^2 \|P_k\widehat{x}\| + \eta_k^2\varepsilon > \eta_k^2 \|P_k\widehat{y}\|,$$

which implies that $|||y||| = \sup_{k\le n} \eta_k^2 \|P_k\widehat{y}\|$. Combining this with Lemma 6 we obtain that $|||\cdot|||$ is LFC-$\{e_i^*\}$.

Theorem 8 (Leung)**.** *Let M be a non-degenerate Orlicz function. There is a sequence $\{\eta_k\}$ of real numbers decreasing to 1 such that the norm on h_M defined by*

$$|||x||| = \sup_k \eta_k \|P_k\widehat{x}\|$$

is LFC-$\{e_i^*\}$ if and only if there is a $K > 1$ such that

$$\lim_{t\to 0+} \frac{M(Kt)}{M(t)} = \infty. \tag{1}$$

Proof. For the "if" part see the remark preceding the theorem. To show the "only if" part (which also appeared in [L1], but not precisely formulated and without proof), suppose that (1) doesn't hold and let $\{\eta_k\}$ be any sequence decreasing to 1. We will construct a vector $x \in S_{h_M}$ such that its coordinates form a positive non-increasing sequence and $\eta_k \|P_k x\| < 1$ for each $k \in \mathbb{N}$. Then obviously $|||x||| = 1$, but $|||P_n x||| = \max_{k\le n} \eta_k \|P_k x\| < 1$ for any $n \in \mathbb{N}$ and so $|||\cdot|||$ is not LFC-$\{e_i^*\}$ by Lemma 2.

Let $\{K_n\}$ be an increasing sequence of real numbers, $K_n > 1$ and $K_n \to \infty$. For each $n \in \mathbb{N}$ let $C_n > 2$ and $\{t_k^n\}_{k=1}^\infty$ be such that $\lim_{k\to\infty} t_k^n = 0$ and $M(K_n t_k^n) < C_n M(t_k^n)$ for all $k \in \mathbb{N}$. Let $\{\varepsilon_n\}$ be a sequence of real numbers such that $0 < \varepsilon_n < \frac{1}{2}$ and $\sum_{n=1}^\infty \varepsilon_n C_n < \infty$. Put $m_0 = 1$ and find $A > 0$ such that $M(1/A) = 1$ (which means $\|e_i\| = A$ for any $i \in \mathbb{N}$).

We choose $t_1 \in \{t_k^1\}$ this way: Define

$$m_1 = \min\left\{k : \eta_k \left\|\sum_{i=1}^{k} t_1 e_i\right\| \geq 1\right\},$$

and choose $t_1 \in \{t_k^1\}$ small enough such that

$$M(t_1) < \varepsilon_2 \quad \text{and} \tag{2}$$

$$\eta_{m_1} < 1 + \frac{\varepsilon_2}{1-\varepsilon_2}\frac{K_1 - 1}{C_1 - 1}. \tag{3}$$

By the convexity of M we have

$$\begin{aligned} M(\eta_{m_1} t_1) &\leq \left(1 - \frac{\eta_{m_1} - 1}{K_1 - 1}\right) M(t_1) + \frac{\eta_{m_1} - 1}{K_1 - 1} M(K_1 t_1) < \left(1 - \frac{\eta_{m_1} - 1}{K_1 - 1}\right) M(t_1) + \frac{\eta_{m_1} - 1}{K_1 - 1} C_1 M(t_1) \\ &= \left(1 + (\eta_{m_1} - 1)\frac{C_1 - 1}{K_1 - 1}\right) M(t_1) < \left(1 + \frac{\varepsilon_2}{1 - \varepsilon_2}\right) M(t_1) = \frac{1}{1-\varepsilon_2} M(t_1), \end{aligned} \tag{4}$$

where the last inequality follows from (3). By the definition of m_1 we have $m_1 M(\eta_{m_1} t_1) \geq 1$. Consequently, using this inequality together with (4), $m_1 M(t_1) > m_1(1-\varepsilon_2)M(\eta_{m_1} t_1) \geq 1 - \varepsilon_2$. Hence, by (2),

$$(m_1 - 1)M(t_1) > 1 - 2\varepsilon_2.$$

We put $x_1 = \sum_{i=1}^{m_1 - 1} t_1 e_i$. Notice that by the definition of m_1 we have $1/\eta_{m_1 - 1} > \|x_1\| \geq 1/\eta_{m_1} - At_1$.

Let us continue by induction. Fix any $j > 1$. Suppose we have $t_i \in \{t_k^i\}$, $m_i \in \mathbb{N}$ and $x_i \in h_M$ already defined for all $i < j$ such that $\sum_{k=1}^{i}(m_k - m_{k-1})M(t_k) > 1 - 2\varepsilon_{i+1}$, $1/\eta_{m_i - 1} > \|x_i\| \geq 1/\eta_{m_i} - At_i$ and

$$x_i = \sum_{l=1}^{i} \sum_{k=m_{l-1}}^{m_l - 1} t_l e_k.$$

We choose $t_j \in \{t_k^j\}$ this way: Define

$$m_j = \min\left\{k \geq m_{j-1} : \eta_k \left\|x_{j-1} + \sum_{i=m_{j-1}}^{k} t_j e_i\right\| \geq 1\right\},$$

and choose $t_j \in \{t_k^j\}$ small enough such that

$$M(t_j) < \varepsilon_{j+1} \quad \text{and} \tag{5}$$

$$\eta_{m_j} < 1 + \frac{\varepsilon_{j+1}}{1 - \varepsilon_{j+1}} \min_{1 \leq i \leq j}\left\{\frac{K_i - 1}{C_i - 1}\right\}. \tag{6}$$

Notice that this is possible since $\|x_{j-1}\| < 1/\eta_{m_{j-1} - 1}$. Using again the convexity of M, the fact that $t_i \in \{t_k^i\}$ and (6), for any $1 \leq i \leq j$ we obtain

$$\begin{aligned} M(\eta_{m_j} t_i) &\leq \left(1 - \frac{\eta_{m_j} - 1}{K_i - 1}\right) M(t_i) + \frac{\eta_{m_j} - 1}{K_i - 1} M(K_i t_i) < \left(1 - \frac{\eta_{m_j} - 1}{K_i - 1}\right) M(t_i) + \frac{\eta_{m_j} - 1}{K_i - 1} C_i M(t_i) \\ &= \left(1 + (\eta_{m_j} - 1)\frac{C_i - 1}{K_i - 1}\right) M(t_i) < \left(1 + \frac{\varepsilon_{j+1}}{1 - \varepsilon_{j+1}}\right) M(t_i) = \frac{1}{1 - \varepsilon_{j+1}} M(t_i). \end{aligned}$$

These estimates together with the definition of m_j and x_{j-1} give

$$\begin{aligned} &\sum_{i=1}^{j-1}(m_i - m_{i-1})M(t_i) + (m_j - m_{j-1} + 1)M(t_j) \\ &\qquad > (1 - \varepsilon_{j+1})\left(\sum_{i=1}^{j-1}(m_i - m_{i-1})M(\eta_{m_j} t_i) + (m_j - m_{j-1} + 1)M(\eta_{m_j} t_j)\right) \geq 1 - \varepsilon_{j+1}, \end{aligned}$$

so the use of (5) yields

$$\sum_{i=1}^{j}(m_i - m_{i-1})M(t_i) > 1 - 2\varepsilon_{j+1}. \tag{7}$$

We put

$$x_j = \sum_{i=1}^{j} \sum_{k=m_{i-1}}^{m_i-1} t_i e_k$$

and notice that, by the definition of m_j,

$$1/\eta_{m_j-1} > \|x_j\| \geq 1/\eta_{m_j} - At_j. \tag{8}$$

We have inductively constructed a sequence $\{x_j\} \subset h_M$ given by the formula above, such that $\|x_j\| < 1$ and (7) holds for any $j \in \mathbb{N}$. Choose any $j > 1$. Since $\|x_j\| < 1$, it follows that $\sum_{i=1}^{j}(m_i - m_{i-1})M(t_i) < 1$ and comparing this with (7) for $j-1$ we obtain

$$(m_j - m_{j-1})M(t_j) < 2\varepsilon_j.$$

This implies that $x_j \to x \in h_M$. Indeed, suppose $K > 0$. Let $n \in \mathbb{N}$ be such that $K_n \geq K$. Then

$$\sum_{i=n}^{\infty}(m_i - m_{i-1})M(Kt_i) \leq \sum_{i=n}^{\infty}(m_i - m_{i-1})M(K_i t_i) \leq \sum_{i=n}^{\infty}(m_i - m_{i-1})C_i M(t_i) < 2\sum_{i=n}^{\infty}\varepsilon_i C_i < \infty$$

and so by the basic properties of h_M the vector $x = \sum_{i=1}^{\infty}\sum_{k=m_{i-1}}^{m_i-1} t_i e_k$ belongs to h_M. This means also that $t_j \to 0$ and thus from (8) we can conclude that $\|x\| = \lim \|x_j\| = 1$. Moreover, the construction of x_j (namely the choice of m_j) guarantees that $\eta_k \|P_k x\| < 1$ for each $k \in \mathbb{N}$.

□

The following theorem is a strengthening of a theorem from [L1]. Leung's statement is that the Orlicz sequence space h_M does not admit a LFC norm if M satisfies the condition below.

Theorem 9. *Let M be a non-degenerate Orlicz function for which there exists a sequence $\{t_n\}$ decreasing to 0 such that*

$$\sup_n \frac{M(Kt_n)}{M(t_n)} < \infty \quad \text{for all } 0 < K < \infty.$$

Then the Orlicz sequence space h_M does not admit any (even non-continuous) LFC bump function.

Proof. Suppose that h_M admits some LFC bump b. Without loss of generality we may assume that $b = \chi_A$ for some set $0 \in A \subset B_X$ (by shifting, scaling and composing with a suitable function) and that b is LFC-$\{e_i^*\}$. (Since h_M is c_0-saturated by [J, Theorem 15] (see also [PWZ]), it does not contain ℓ_1. As $\{e_i\}$ is unconditional, it is shrinking by James's theorem. Now consider $b \circ T$, where $T\colon X \to X$ is an equivalence isomorphism of the bases $\{x_i\}$ and $\{e_i\}$ from Theorem 3.)

Notice, that the vectors with coordinates in the set $\{t_n\} \cup \{0\}$ have the property of "boundedly completeness": If $\left\|\sum_{i=1}^{k} t_{m_i} e_i\right\| \leq 1$ for all $k \in \mathbb{N}$, where $m_i \in \mathbb{N} \cup \{0\}$ are not necessarily distinct (we put $t_0 = 0$), then $\sum_{i=1}^{\infty} t_{m_i} e_i$ converges in h_M. Indeed, it follows that $\sum_{i=1}^{k} M(t_{m_i}) \leq 1$ for all $k \in \mathbb{N}$. For all $0 < K < \infty$ and all $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} M(Kt_{m_i}) \leq \sup_n \frac{M(Kt_n)}{M(t_n)} \sum_{i=1}^{k} M(t_{m_i}) \leq \sup_n \frac{M(Kt_n)}{M(t_n)}.$$

Consequently, $\sum_{i=1}^{\infty} M(Kt_{m_i}) < \infty$ for all $0 < K < \infty$, and the sum $\sum_{i=1}^{\infty} t_{m_i} e_i$ converges in h_M.

We construct a sequence $\{x_k\} \subset A$ by induction. Put $x_0 = 0 \in A$ and define natural numbers $m_0 = n_0 = 1$. If $m_{k-1} \in \mathbb{N}$, $n_{k-1} \in \mathbb{N}$ and $x_{k-1} \in A$ are already defined, we put

$$M_k = \{(m,n) \in \mathbb{N}^2;\ m \geq m_{k-1}, n > n_{k-1} \text{ and } x_{k-1} + t_m e_n \in A\}.$$

As b depends only on some finite subset of $\{e_i^*\}$ on a neighbourhood of x_{k-1}, and $t_m \to 0$, we can see that $M_k \neq \emptyset$. Let $(m_k, n_k) = \min M_k$ in the lexicographic ordering of $\mathbb{N}^2$ and put $x_k = x_{k-1} + t_{m_k} e_{n_k}$.

Since $\{x_k\} \subset A \subset B_X$ and $x_k = \sum_{i=1}^{k} t_{m_i} e_{n_i}$, by the above argument $x_k \to x \in h_M$. We can find $\delta > 0$ and $N \in \mathbb{N}$ so that b depends only on $\{e_i^*\}_{i<N}$ on $U(x,\delta)$. Because x_k converges, we have $m_k \to \infty$ and so there is $j \in \mathbb{N}$ such that $x_j \in U(x, \delta/2)$, $\|t_{m_j} e_1\| < \delta/2$, $m_j < m_{j+1}$ and $n_j > N$. Then $x_j + t_{m_j} e_{n_j+1} \in A$ and therefore $(m_j, n_j + 1) \in M_{j+1}$. But $(m_j, n_j + 1) < (m_{j+1}, n_{j+1})$, which is a contradiction.

□

In [L1], Leung constructed a c_0-saturated Orlicz sequence space satisfying the condition in Theorem 9. Therefore, we have the following corollary:

Corollary 10. *Leung's space is a separable c_0-saturated Asplund space that does not admit any (even non-continuous) LFC bump function.*

The main construction of this paper is contained in the next theorem.

Theorem 11. *Let M be a non-degenerate Orlicz function for which there exist sequence $F_k \subset (0,1]$ such that*

(i) $\lim_{k\to\infty}(\sup F_k) = 0$,

(ii) there is a sequence $K_k > 1$ such that

$$\lim_{\substack{t\to 0+\\ t\notin F_k}} \frac{M(K_k t)}{M(t)} = \infty,$$

(iii) there is a $K > 1$ and a sequence $C_k \to \infty$ such that $M(Kt) \geq C_k M(t)$ for all $t \in F_k$.

Then there exists a C^∞-smooth LFC-$\{e_i^*\}$ lattice bump function on the Orlicz sequence space h_M.

Proof. Without loss of generality we may and do assume that $M(1) = 1$ (i.e. $\|e_1\| = 1$) and $C_k \geq C_1 > 0$ for any $k \in \mathbb{N}$.

For each $t \in \overline{F_k} \setminus \{0\}$ choose $\varepsilon_t^k > 0$ such that $M(s) < 2M(t)$ and $t/2 < s < 2t$ if $|s-t| < \varepsilon_t^k$. Define $G_k = \bigcup_{t\in\overline{F_k}\setminus\{0\}}(t-\varepsilon_t^k, t+\varepsilon_t^k)$. Then each G_k is open, $G_k \supset (\overline{F_k} \setminus \{0\})$ and $\sup G_k \leq 2\sup F_k$. Moreover, for any $s \in G_k$ the choice of an appropriate $t \in \overline{F_k} \setminus \{0\}$ from the definition of G_k yields $M(2Ks) > M(Kt) \geq C_k M(t) > C_k M(s)/2$ (using (iii) and the continuity of M). So, if we multiply K by 2 and each C_k by $\frac{1}{2}$ and denote these new constants K and C_k again to avoid carrying unnecessary factors, we have

$$\lim_{k\to\infty}(\sup G_k) = 0, \tag{9}$$

$$M(Kt) \geq C_k M(t) \quad \text{for all } t \in G_k. \tag{10}$$

Let us define a sequence of continuous functions φ_k on $[0,+\infty)$ such that $0 \leq \varphi_k(t) \leq t$, $\varphi_k(t) = 0$ for $t \in F_k$ and $\varphi_k(t) = t$ for $t \notin G_k$, and a mapping $\phi_k\colon h_M \to h_M$ by $\phi_k(x) = \big(\varphi_k(|x_i|)\big)$ for $x = (x_i) \in h_M$. (We can take for example $\varphi_k(t) = t\,\mathrm{dist}(t,F_k)/\big(\mathrm{dist}(t,F_k) + \mathrm{dist}(t,\mathbb{R}\setminus G_k)\big)$ for $t > 0$ and $\varphi_k(0) = 0$.)

Fix $k \in \mathbb{N}$.

First, observe that the mapping $\phi_k\colon h_M \to h_M$ is continuous: Choose $x \in h_M$ and $\varepsilon > 0$ and find $n \in \mathbb{N}$ such that $\|R_n x\| < \frac{\varepsilon}{8}$. As φ_k is continuous, there is $\delta > 0$ such that $\big||x_i| - |y_i|\big| < \delta$ implies $\big|\varphi_k(|x_i|) - \varphi_k(|y_i|)\big| < \frac{\varepsilon}{2n}$ for all $1 \leq i \leq n$. We have $\big||x_i| - |y_i|\big| \leq |x_i - y_i| = \|(x-y)_i e_i\| \leq \|x-y\|$. (The last inequality uses the fact that the norm $\|\cdot\|$ is a lattice norm.) Thus, whenever $\|x-y\| < \min\{\delta, \frac{\varepsilon}{4}\}$,

$$\begin{aligned}
\|\phi_k(x) - \phi_k(y)\| &\leq \|P_n(\phi_k(x) - \phi_k(y))\| + \|R_n(\phi_k(x) - \phi_k(y))\| \\
&\leq \sum_{i=1}^n \big|\varphi_k(|x_i|) - \varphi_k(|y_i|)\big| + \|R_n\phi_k(x)\| + \|R_n\phi_k(y)\| < \frac{\varepsilon}{2} + \|R_n x\| + \|R_n y\| \\
&\leq \frac{\varepsilon}{2} + \|R_n x\| + \|R_n x\| + \|R_n(x-y)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

The third and the fifth inequalities follows again from the fact that the norm $\|\cdot\|$ is lattice.

Claim 1. *There is a non-increasing sequence $\{\eta_n^k\} \subset \mathbb{R}$ satisfying $\eta_n^k \leq 2$ and $\lim_{n\to\infty}\eta_n^k = 1$, such that for each $x \in h_M$ for which $\phi_k(x) \neq 0$ there is $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for any $y \in U(x,\delta)$ we have*

$$\eta_n^k \nu(P_n\widehat{\phi_k(y)}) > \nu(\widehat{\phi_k(y)}) \quad \textit{for all } n \geq n_0.$$

We will construct the sequence η_n^k as follows: If $(0,a) \subset F_k$ for some $a > 0$, then any non-increasing sequence $\eta_n^k \to 1$ such that $1 < \eta_n^k \leq 2$ for all $n \in \mathbb{N}$ will do. Indeed, then there is $n_0 \in \mathbb{N}$ such that $|x_i| < a/2$ for $i \geq n_0$ and hence $\widehat{\phi_k(y)} = P_{n_0}\widehat{\phi_k(y)}$ whenever $\|x-y\| < a/2$.

Otherwise, put $b_n = \inf\Big\{\frac{M(K_k t)}{M(t)};\ 0 < t \leq \sqrt{M^{-1}(\frac{1}{n})},\ t \notin F_k\Big\}$. By our assumption, $b_n < \infty$ for all $n \in \mathbb{N}$. Notice, that b_n is non-decreasing and, by (ii), $b_n \to \infty$. Define $\eta_n^k = \min\{2, (1 - b_n^{-1/2})^{-1}\}$. It is trivial to check that η_n^k is non-increasing and $\eta_n^k \to 1$.

Define a mapping $Q_k\colon h_M \to h_M$ by $Q_k(x)_i = |x_i|$ if $|x_i| \notin F_k$, $Q_k(x)_i = 0$ otherwise.

Now choose $x \in h_M$ for which $\phi_k(x) \neq 0$. By Lemma 7 there is $0 < \delta < \frac{1}{2K_k}$ such that $\nu(\phi_k(y)) > \frac{1}{2}\nu(\phi_k(x))$ if $\|x-y\| < \delta$. Find $n_0 \in \mathbb{N}$ such that $\eta_n^k = (1 - b_n^{-1/2})^{-1}$, $b_n^{-1/2} < \frac{1}{2}\nu(\phi_k(x))$, $\|R_n x\| < \frac{1}{2K_k}$ and $M^{-1}(\frac{1}{n}) \leq 1/(\|x\| + \delta)^2$ for $n \geq n_0$. Fix $n \geq n_0$ and $y \in h_M$ such that $\|x-y\| < \delta$. Using Lemma 5(b) and the fact that the canonical norm on h_M is a symmetric lattice norm, we have

$$\Big\|R_n\widehat{Q_k(y)}\Big\| \leq \|R_n Q_k(y)\| \leq \|R_n y\| \leq \|R_n x\| + \|R_n(x-y)\| < \frac{1}{K_k}. \tag{11}$$

As $\sum_{i=1}^{\infty} M(\widehat{Q_k(y)}_i/\|y\|) \leq \sum_{i=1}^{\infty} M(|y_i|/\|y\|) = \nu(y/\|y\|) = 1$ and the sequence $\widehat{Q_k(y)}_i$ is non-increasing, it follows that $\widehat{Q_k(y)}_i/\|y\| \leq M^{-1}(\frac{1}{i})$ for any $i \in \mathbb{N}$. From the definition of n_0 we obtain $\widehat{Q_k(y)}_i \leq \|y\| M^{-1}(\frac{1}{i}) \leq (\|x\|+\delta)M^{-1}(\frac{1}{i}) \leq \sqrt{M^{-1}(\frac{1}{i})}$ for $i \geq n_0$. Notice further that $\widehat{Q_k(y)}_i \notin F_k$ for any $i \in \mathbb{N}$, thus by the definition of b_n and (11) we have

$$1 > \nu\left(K_k R_n \widehat{Q_k(y)}\right) = \sum_{i>n} M\left(K_k \widehat{Q_k(y)}_i\right) \geq \sum_{i>n} b_i M\left(\widehat{Q_k(y)}_i\right) \geq b_n \sum_{i>n} M\left(\widehat{Q_k(y)}_i\right),$$

which together with the easily checked inequality $\widehat{\phi_k(y)}_i \leq \widehat{Q_k(y)}_i$ for any $i \in \mathbb{N}$ implies

$$\sum_{i>n} M\left(\widehat{\phi_k(y)}_i\right) \leq \sum_{i>n} M\left(\widehat{Q_k(y)}_i\right) \leq \frac{1}{b_n}.$$

Notice that by the definition of δ and n_0 and by the symmetry of ν we have $\nu(\widehat{\phi_k(y)}) > b_n^{-1/2}$ and therefore (use this fact for the second inequality)

$$\nu(P_n\widehat{\phi_k(y)}) = \sum_{i=1}^{n} M(\widehat{\phi_k(y)}_i) \geq \sum_{i=1}^{\infty} M(\widehat{\phi_k(y)}_i) - \frac{1}{b_n} = \nu(\widehat{\phi_k(y)}) - \frac{1}{b_n} > (1-b_n^{-1/2})\nu(\widehat{\phi_k(y)}) = \frac{1}{\eta_n^k}\nu(\widehat{\phi_k(y)}),$$

which proves the claim.

Choose and arbitrary sequence $\{\gamma_k\} \subset \mathbb{R}$ decreasing to 1. Let us define a sequence of functions $g_k\colon h_M \to \mathbb{R}$ by

$$g_k(x) = \frac{1}{C_k} + \sup_n \gamma_{k+n}\eta_n^k \nu(P_n\widehat{\phi_k(x)}).$$

Claim 2. *Each g_k is a LFC-$\{e_i^*\}$ function on $\{x \in h_M,\ \phi_k(x) \neq 0\}$ and continuous on h_M.*

Indeed, for a fixed $k \in \mathbb{N}$ and $x \in h_M$, $\phi_k(x) \neq 0$, choose an appropriate δ and n_0 from Claim 1. Let $N \geq n_0$ be such that $\gamma_{k+n}\eta_n^k < \gamma_{k+n_0}$ whenever $n > N$. Then for $y \in U(x,\delta)$ and $n > N$ we have

$$\gamma_{k+n_0}\eta_{n_0}^k \nu(P_{n_0}\widehat{\phi_k(y)}) > \gamma_{k+n}\eta_n^k \nu(\widehat{\phi_k(y)}) \geq \gamma_{k+n}\eta_n^k \nu(P_n\widehat{\phi_k(y)})$$

and hence

$$g_k(y) = \frac{1}{C_k} + \max_{1\leq n\leq N} \gamma_{k+n}\eta_n^k \nu(P_n\widehat{\phi_k(y)}). \tag{12}$$

By Lemma 6 there is a neighbourhood V of $\phi_k(x)$ and a finite $A \subset \mathbb{N}$ such that the function $\max_{1\leq n\leq N} \gamma_{k+n}\eta_n^k \nu(P_n\widehat{z})$ depends only on $\{e_i^*\}_{i\in A}$ on V. But since ϕ_k is continuous, there is a neighbourhood U of x, $U \subset U(x,\delta)$, such that $\phi_k(U) \subset V$. Further, as $\phi_k(y)_i = \phi_k(z)_i$ whenever $y_i = z_i$ for any $i \in \mathbb{N}$, the function g_k depends only on $\{e_i^*\}_{i\in A}$ on U.

Moreover, each g_k is continuous on h_M: Using the continuity of ϕ_k, Lemma 5(d) and (12) we can see that g_k is continuous on $\{x \in h_M,\ \phi_k(x) \neq 0\}$. On the other hand,

$$\frac{1}{C_k} \leq g_k(x) \leq \frac{1}{C_k} + \gamma_k\eta_1^k \nu(\phi_k(x)),$$

and the continuity of g_k at any x with $\phi_k(x) = 0$ follows from the continuity of ϕ_k and the properties of ν.

Notice further that, since ν is lattice,

$$g_k(x) \leq \frac{1}{C_k} + \gamma_k\eta_1^k \nu(x), \tag{13}$$

and as $g_k(x) \geq \frac{1}{C_k} + \gamma_{k+n}\eta_n^k \nu(P_n\widehat{\Phi_k(x)})$ for each $n \in \mathbb{N}$, the continuity of ν implies

$$g_k(x) \geq \frac{1}{C_k} + \nu(\phi_k(x)), \tag{14}$$

for any $x \in h_M$ and any $k \in \mathbb{N}$.

Claim 3. *For each $x \in h_M$ there is $\delta > 0$ and $k_0 \in \mathbb{N}$ such that for any $y \in U(x,\delta)$ and $k \geq k_0$ we have*

$$\nu(y) < \frac{1}{C_k} + \nu(\phi_k(y)).$$

Indeed, choose $x \in h_M$. Let $n \in \mathbb{N}$ be such that $\|R_n x\| < \frac{1}{3K}$ and $0 < \delta < \frac{1}{3K}$ such that moreover $\delta \le \frac{1}{2}\min\{|x_i|;\ x_i \ne 0, i \le n\}$ if $P_n x \ne 0$. Pick any $y \in h_M$ for which $\|x - y\| < \delta$. Notice that if $|y_i| < \delta$ then either $x_i = 0$ or $i > n$. Let $A_1 = \{i;\ x_i = 0\}$, $A_2 = \{i;\ i > n\}$. Then

$$\|P_{A_1 \cup A_2} y\| \le \|P_{A_1} y\| + \|R_n y\| \le \|P_{A_1}(y - x)\| + \|R_n x\| + \|R_n(x - y)\| < \frac{1}{K}.$$

Therefore we have $\sum_{|y_i|<\delta} M\big(K\,|y_i|\big) < 1$. By (9) we can find $k_0 \in \mathbb{N}$ such that $G_k \subset (0, \delta)$ for all $k \ge k_0$ and hence, using (10),

$$\sum_{|y_i| \in G_k} M\big(|y_i|\big) < \frac{1}{C_k} \quad \text{for all } k \ge k_0.$$

It follows that, for any $y \in U(x, \delta)$ and $k \ge k_0$,

$$\begin{aligned} \nu(y) &= \sum_{i=1}^{\infty} M(|y_i|) = \sum_{|y_i| \in G_k} M(|y_i|) + \sum_{|y_i| \notin G_k} M(|y_i|) \\ &= \sum_{|y_i| \in G_k} M(|y_i|) + \sum_{|y_i| \notin G_k} M(\phi_k(y)_i) < \frac{1}{C_k} + \sum_{i=1}^{\infty} M(\phi_k(y)_i) = \frac{1}{C_k} + \nu(\phi_k(y)). \end{aligned}$$

Finally let us define a function $g\colon h_M \to \mathbb{R}$ by

$$g(x) = \sup_k \gamma_k g_k(x).$$

Choose $0 \ne x \in h_M$ and find δ and k_0 from Claim 3. Since ν is continuous, we may also assume that $\nu(y) \ge \nu(x)/2$ if $\|x - y\| < \delta$. There is $N \in \mathbb{N}$ such that $2\gamma_k/(\nu(x)C_k) + \gamma_k^2 \eta_1^k < \gamma_{k_0}$ for $k > N$. Then for any $y \in U(x, \delta)$ and $k > N$ we have (using first (13), then the definition of N, Claim 3 and finally (14))

$$\gamma_k g_k(y) \le \frac{\gamma_k}{C_k} + \gamma_k^2 \eta_1^k \nu(y) < \gamma_{k_0} \nu(y) < \frac{\gamma_{k_0}}{C_{k_0}} + \gamma_{k_0} \nu(\phi_{k_0}(y)) \le \gamma_{k_0} g_{k_0}(y). \tag{15}$$

This means that

$$g(y) = \sup_k \gamma_k g_k(y) = \max_{k \le N} \gamma_k g_k(y) \tag{16}$$

for $y \in U(x, \delta)$. In particular, since each g_k is continuous on h_M, it follows that g is continuous on $h_M \setminus \{0\}$. On the other hand, for any $y \in h_M$,

$$\frac{\gamma_1}{C_1} \le \gamma_1 g_1(y) \le g(y) \le \frac{\gamma_1}{C_1} + 2\gamma_1^2 \nu(y),$$

(the last inequality follows from (13)) and the continuity of ν implies that g is continuous at 0 and hence on the whole of h_M.

Let us define a set $D = \{x \in h_M,\ g(x) > \frac{\gamma_1}{C_1}\}$. Choose any $x \in D$ and find an appropriate N and δ for this x as above. Let $A = \{k : 1 \le k \le N,\ \phi_k(x) \ne 0\}$. If $k \in \{1, \dots, N\} \setminus A$, then

$$\gamma_k g_k(x) = \frac{\gamma_k}{C_k} \le \frac{\gamma_1}{C_1} < g(x).$$

By the continuity of all ϕ_k, g_k and g, there is a neighbourhood U of x, $U \subset U(x, \delta)$, such that $\phi_k(y) \ne 0$ for $k \in A$ and $\gamma_k g_k(y) < g(y)$ for $k \in \{1, \dots, N\} \setminus A$ whenever $y \in U$. Thus, by (16), $g(y) = \max_{k \in A} \gamma_k g_k(y)$ for $y \in U$. Since each g_k, $k \in A$, is LFC on U, so is g. Therefore g is LFC on D.

From the last two inequalities in (15) we can see that $g(x) > \nu(x)$ for any $x \in h_M$. Therefore $g(x) > \|x\|$ on $\{x \in h_M;\ \|x\| \ge 1\}$ and we can compose g with a suitable real continuous function to obtain a desired continuous LFC bump. To finish the proof, it remains to apply Theorem 4.

□

Theorem 12. *There is a non-degenerate Orlicz function M such that $\liminf_{t \to 0+} \frac{M(Kt)}{M(t)} < \infty$ for any $K > 1$, yet the corresponding Orlicz sequence space h_M admits a C^∞-smooth LFC-$\{e_i^*\}$ lattice bump.*

Proof. Suppose we have a sequence $b_n \ge 1$, $n \ge 0$. For $n = 0, 1, 2, \dots$, put $a_n = \prod_{m=0}^{n} b_m^{-1}$ and let $M(t)$ be a piecewise linear continuous function on $[0, \infty)$, such that $M(0) = 0$, $M'(t) = a_n$ for $2^{-(n+1)} < t < 2^{-n}$ and $M'(t) = 1$ for $t > 1$. Clearly, this is a non-degenerate Orlicz function and the constants b_n determine the ratio of the slopes of M on the two consecutive dyadic intervals. Suppose that $j \in \mathbb{N} \cup \{0\}$ and $2^{-(n+1)} \le t \le 2^{-n}$ for some $n \ge j$. Then

$$2^{j-n-2} a_{n-j+1} \le M(2^{j-n-1}) \le M(2^j t) \le M(2^{j-n}) \le 2^{j-n} a_{n-j}.$$

Hence, for $n \geq j \geq 2$,

$$2^{j-2} \prod_{m=n-j+2}^{n} b_m \leq \frac{M(2^j t)}{M(t)} \leq 2^{j+2} \prod_{m=n-j+1}^{n+1} b_m. \tag{17}$$

If F_k is chosen to be $\bigcup_{n \in I_k} [2^{-(n+1)}, 2^{-n})$ for some $I_k \subset \mathbb{N}$, then for conditions (i) to (iii) in Theorem 11 to hold, it is sufficient to require

(a) $\lim_{k\to\infty} \min I_k = \infty$,

(b) For each $k \in \mathbb{N}$, there exists $j_k \in \mathbb{N}$ such that $\lim_{\substack{n\to\infty \\ n \notin I_k}} \max\{b_{n-j_k}, \dots, b_n\} = \infty$,

(c) $\lim_{k\to\infty} \min_{n \in I_k} b_n = \infty$.

Indeed, (a) implies (i). If $t \in (0,1) \setminus F_k$, then there is $n \notin I_k$ such that $t \in [2^{-(n+1)}, 2^{-n})$ and thus (17) together with (b) implies (ii) for $K_k = 2^{j_k+2}$. Finally, (17) together with (c) implies (iii) for $K = 4$ and $C_k = \min_{n \in I_k} b_n$. On the other hand, condition

(d) $\liminf_{n\to\infty} \max\{b_{n-j}, \dots, b_n\} < \infty$ for all $j \in \mathbb{N}$

with (17) ensures that $\liminf_{t\to 0+} \frac{M(Kt)}{M(t)} < \infty$ for any $K > 1$.

Now we construct a sequence $b_n \geq 1$, $n \geq 0$ and a sequence $I_k \subset \mathbb{N}$ satisfying conditions (a) to (d). Choose a non-decreasing sequence $\{c_n\} \subset \mathbb{R}$ such that $c_n \geq 1$ and $c_n \to \infty$. For $i = 0, 1, 2, \dots$, $j = 0, \dots, i$ and $k = 0, \dots, j+1$, let

$$n(i,j,k) = \sum_{l=0}^{i-1} \sum_{m=1}^{l+1} (m+1) + \sum_{m=1}^{j} (m+1) + k$$

and define $\{b_n\}_{n=0}^{\infty}$ by $b_{n(i,j,0)} = c_i$ and $b_{n(i,j,k)} = c_j$ for $k = 1, \dots, j+1$. The sequence $\{b_n\}$ fills a triangular table, where the index $n = n(i,j,k)$ is interpreted as follows: i counts the rows, by j we index groups of columns, where the j-th group consists of $j+2$ columns, and k is an index of a column in the j-th group. So we have the following table

b_0	b_1												
b_2	b_3	b_4	b_5	b_6									
b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}					
b_{16}	b_{17}	b_{18}	b_{19}	b_{20}	b_{21}	b_{22}	b_{23}	b_{24}	b_{25}	b_{26}	b_{27}	b_{28}	b_{29}
$\dots$	$\dots$												

with the values

c_0	c_0												
c_1	c_0	c_1	c_1	c_1									
c_2	c_0	c_2	c_1	c_1	c_2	c_2	c_2	c_2					
c_3	c_0	c_3	c_1	c_1	c_3	c_2	c_2	c_2	c_3	c_3	c_3	c_3	c_3
$\dots$	$\dots$												

For any $j \in \mathbb{N}$ we have $\max\{b_{n(i,j,1)}, \dots, b_{n(i,j,j+1)}\} = c_j$ for all $i \geq j$ and (d) is clearly satisfied.

Now let $I_k = \bigcup_{m=k-1}^{\infty} \bigcup_{i=m}^{\infty} \{n(i,m,1), \dots, n(i,m,m+1)\}$ for $k \in \mathbb{N}$, i.e. I_k consists of all the columns in the table starting with the $(k-1)$-th group but without the first column in each group. Condition (a) obviously holds. If $n(i,j,l) \notin I_k$, then $l \leq j+1 < k$ or $l = 0$ but in both cases $\max\{b_{n(i,j,l)-k+1}, \dots, b_{n(i,j,l)}\} \geq b_{n(i,j,0)} = c_i$ and hence (b) is satisfied. Finally, $\min_{n \in I_k} b_n = c_{k-1}$ implies (c).

□

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