Dual renormings of Banach spaces

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ABSTRACT: We prove that a Banach space admitting an equivalent WUR norm is an Asplund space. Some related dual renormings are also presented.

It is a well-known result that a Banach space whose dual norm is Fréchet differentiable is reflexive. Also if the the third dual norm is Gâteaux differentiable the space is reflexive. For these results see e.g. [2], p.33.
Similarly, by the result of [9], if the second dual norm is Gâteaux differentiable the space is an Asplund space.
The main result of this note answers a question posed by Troyanski. It claims that a space that admits an equivalent weakly uniformly rotund (WUR) norm is an Asplund space. By the well known dual characterization of WUR norms (see [1]) it is equivalent to the dual norm being uniformly Gâteaux differentiable (UG). In fact, the existence of an equivalent (not necessarily dual) UG norm on the dual space is sufficient. This follows from the dual characterization of UG norms as norms the dual of which is weak-star uniformly rotund (W*UR). The restriction of a W*UR norm of the second dual space to the original space is easily shown to be WUR. Let us point out that by our Theorem 4 merely dual Gâteaux norm does not imply, in general, that the space is Asplund.
In the remaining part of the paper we strive to improve our knowledge of the higher dual norms of separable spaces.
We show that the space J of James admits a dual norm that is WUR. In particular, spaces whose second dual norm is UG do not necessarily have to be reflexive.
Let us recall that by a classical result [8] duals of separable spaces containing an isomorphic copy of ℓ_1 contain an isomorphic copy of $\ell_1(\mathfrak{c})$. As a consequence, these duals do not have an equivalent Gâteaux smooth renorming. Therefore the second dual norm of these spaces cannot be rotund. In contrast, spaces with separable dual admit a WUR norm whose second dual is W*UR. We investigate the existence of the second dual rotund norm on the classical James tree space (JT) and Hagler space (JH). These spaces do not contain an isomorphic copy of ℓ_1, but their dual is nonseparable. In Theorem 4 we construct an equivalent norm on the James tree space whose second dual is uniformly rotund in every direction (URED). On the other hand we prove that the space JH of Hagler has no equivalent norm whose second dual is rotund.
Altogether, the class of separable Banach spaces that admit a norm whose second dual is rotund lies strictly between spaces with separable dual and spaces not containing an isomorphic copy of ℓ_1.
In line of these results it seems natural to ask whether duals of separable Banach

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spaces not containing a copy of ℓ_1 always admit a dual Gâteaux smooth norm, thus giving a renorming characterization of these spaces.

Let us start with the definitions of the less standard notions of rotundity.

Definition The norm $\|\cdot\|$ on a Banach space X is said to be weakly uniformly rotund (WUR) if $\lim\limits_{n\to\infty}(x_n-y_n)=0$ in the weak topology whenever $x_n, y_n \in S_X, n \in \mathbb{N}$ are such that $\lim\limits_{n\to\infty}\|x_n+y_n\|=2$.

The dual norm $\|\cdot\|^*$ on a dual Banach space X^* is said to be weak-star uniformly rotund (W*UR) if $\lim\limits_{n\to\infty}(x_n-y_n)=0$ in the weak-star topology whenever $x_n, y_n \in S_{X^*}$ are such that $\lim\limits_{n\to\infty}\|x_n+y_n\|^*=2$.

The norm $\|\cdot\|$ on a Banach space X is said to be uniformly rotund in every direction (URED) if $\lim\limits_{n\to\infty}\|x_n-y_n\|=0$ whenever $x_n, y_n \in S_X$ are such that $\lim\limits_{n\to\infty}\|x_n+y_n\|=2$ and there is $z\in X$ such that $x_n-y_n\in \operatorname{span}(z)$ for every $n\in\mathbb{N}$.

The norm $\|\cdot\|$ on a Banach space X is said to be uniformly Gâteaux if for every $h\in S_X$

$$\lim_{t\to 0}\frac{\|x+th\|-\|x\|}{t}$$

exists and is uniform in $x\in S_X$.

Note that the condition $\lim\|x_n+y_n\|=2$ in the above definitions may be replaced by an equivalent condition $2\|x_n\|^2+2\|y_n\|^2-\|x_n+y_n\|^2\to 0$, which is very useful for constructions of rotund norms.

It can be shown (see e.g.[1]) that the norm $\|\cdot\|$ is WUR (resp. UG) if and only if its dual norm $\|\cdot\|^*$ on X^* is UG (resp. W*UR).

Theorem 1

A Banach space that admits an equivalent WUR norm is an Asplund space.

Proof. It is well-known that a space is an Asplund space if and only if every separable subspace has a separable dual. Thus it is enough to prove the statement for separable spaces. Let us therefore assume by contradiction that $(X,\|\cdot\|)$ is separable, $\|\cdot\|$ is WUR and X^* is nonseparable. Let us first define a partially ordered set $(T,>)$, usually called a binary tree. Put $T=\{(n,i), n=0,1,\dots;0\le i<2^n\}$. We partially order T by putting $(m,j)>(n,i)$ if $m>n$ and there exist integers $i_0=i, i_1,\dots,i_k=j$ with $k=m-n$ and $i_1\in\{2i,2i+1\}, i_2\in\{2i_1,2i_1+1\},\dots,i_k\in\{2i_{k-1},2i_{k-1}+1\}$. By a segment (t_1,t_2) we mean a set of the form $\{t, t_1<t<t_2; t_1,t_2\in T\}$. By a branch starting at $t\in T$ we mean a maximal linearly ordered subset of T whose minimal element is t. If $t=(0,0)$ we speak simply of a branch. We denote by Γ the set of all branches (starting at (0,0)).

According to the result of Stegall from [10], as presented in [2], p.239, for every $\varepsilon>0$ there exists a weak* compact set $\Delta\subset B_{X^*}$ with a Haar system $\{C_t\}_{t\in T}$ of relatively clopen subsets of Δ and a sequence $\{x_t\}_{t\in T}\subset(1+\varepsilon)B_X$ such that the following hold:

$$\Delta = \bigcup_{i=0}^{2^n-1} C_{n,i} \text{ for every } n = 0, 1, \dots$$

$$C_{t_1} \cap C_{t_2} = C_{t_1} \text{ if } t_1 \geq t_2,$$

$$C_{t_1} \cap C_{t_2} = \emptyset \text{ otherwise}$$

and $|f(x_{(n,i)}) - \chi_{C_{(n,i)}}(f)| < \frac{\varepsilon}{2^n}$ for $f \in \Delta$.
Choose for every $b \in \Gamma$ an element $f_b \in \bigcap\limits_{t \in b} C_t \neq \emptyset$.
For the rest of the proof, let us fix $\varepsilon > 0$ and the corresponding $\{x_t\}_{t \in T}$ and $\{C_t\}_{t \in T}$ as above.

Claim 2 *Let $\delta > 0$. For every branch b starting at t_1 there exist $t_b < r_b \in b$ and two vectors $\|x_b\|, \|y_b\| < 1 + \varepsilon + \delta$:*

$$x_b = \sum_{t_1 < t < t_b} \alpha_t x_t, \; y_b = \sum_{t_b < t < r_b} \beta_t x_t$$

where $|\alpha_t|, |\beta_t| \leq 1$ and $\sum\limits_{t_1 < t < t_b} \alpha_t = \sum\limits_{t_b < t < r_b} \beta_t = 1$, such that:

$$2\|x_b\|^2 + 2\|y_b\|^2 - \|x_b + y_b\|^2 < \delta.$$

Proof of Claim 2. Define a real function M_b on b by formula:

$$M_b(t_0) = \inf\Big\{ \|x\|, \; x = \sum_{t_0 < t < t^0} \gamma_t x_t \text{ where } t^0 \in b, \; |\gamma_t| \leq 1, \; \sum_{t_0 < t < t^0} \gamma_t = 1 \Big\}, \text{ for } t_0 \in b.$$

It follows from the properties of vectors x_t that $M_b(t_0) \leq 1 + \varepsilon$. The function $M_b(t)$ is clearly nondecreasing on b. For every $\rho > 0$ there exists $t_\rho \in b$ satisfying $|M_b(t_\rho) - \sup\limits_{t \in b} M_b(t)| < \rho$. Choose $t_b, r_b \in b$, $t_b, r_b > t_\rho$ and vectors x_b, y_b in the required form and such that:

$$M_b(t_\rho) > \|x_b\| - \rho \,, \;\; M_b(t_\rho) > \|y_b\| - 2\rho.$$

Necessarily $\|\frac{x_b + y_b}{2}\| > M_b(t_\rho)$. Thus for ρ small enough we obtain

$$2\|x_b\|^2 + 2\|y_b\|^2 - \|x_b + y_b\|^2 < \delta.$$

The claim is proved.

Let $\delta_n \searrow 0$. For some fixed $b^1 \in \Gamma$ starting at $(0,0)$ by t_1, r_1, x_1, y_1 we denote the $t_{b^1}, r_{b^1}, x_{b^1}, y_{b^1}$ from Claim 2 corresponding to b^1 and δ_1.
We construct by induction a system of sequences $t_i, r_i \in T$, $x_i, y_i \in X$ and branches b^i starting at t_{i-1} as follows:

Suppose we have constructed t_i, r_i, x_i, y_i, b^i for $i \leq k$. We choose b^{k+1} that starts at t_k, $b^{k+1} \cap (t_k, r_k) = \emptyset$.
$t_{k+1}, r_{k+1}, x_{k+1}, y_{k+1}$ are $t_{b^{k+1}}, r_{b^{k+1}}, x_{b^{k+1}}, y_{b^{k+1}}$ corresponding to b^{k+1} and δ_{k+1}. Denote by b^0 the branch starting at $(0,0)$ and containing the sequence $\{t_k\}_{k\in I\!N}$. By claim 2:

$$2\|x_k\|^2 + 2\|y_k\|^2 - \|x_k + y_k\|^2 \to 0.$$

Yet,

$$f_{b^0}(x_k - y_k) = \sum_{t_{k-1}<t<t_k} \alpha_t f_{b^0}(x_t) - \sum_{t_k<t<r_k} \beta_t f_{b^0}(x_t) \geq$$

$$\geq \sum_{t_{k-1}<(n,i)<t_k} \alpha_{(n,i)}(1 - \frac{\varepsilon}{2^n}) - \sum_{t_k<(n,i)<r_k} \frac{\varepsilon}{2^n} \geq 1 - 4\varepsilon.$$

a contradiction.

We now proceed by renorming the James space J by a dual WUR norm. Let us recall that the James space J is a separable Banach space with a boundedly complete basis $\{e_n\}_{n\in I\!N}$, and its predual basis $\{f_n\}_{n\in I\!N}$ (that is, $e_n(f_m) = \delta_n^m$) in the unique (see [1, p.117]) predual J_* so that the canonical norm on J is given by:

$$\|x\| = \sup_{n_1<m_1<n_2<\dots} \Big(\sum_{i\in I\!N}(\sum_{j=n_i}^{m_i} x_j)^2\Big)^{\frac{1}{2}}. \tag{1}$$

This norm is easily seen to be a dual norm on $J = (J_*)^*$. Indeed, every $x_j = f_j(x)$ is a w^*-lower semicontinuous function on J. Thus $\|\cdot\|$ is a supremum of w^*-lower semicontinuous functions, so it is itself w^*-lower semicontinuous. That is equivalent to being a dual norm. It can be shown that $\dim(J^*/J_*) = 1$. For more details on the space J we refer to [4].

Proposition 3
The space J admits an equivalent dual WUR norm.

Proof. Let us define an equivalent norm $|\!|\!|\cdot|\!|\!|$ on J as follows:

$$|\!|\!|x|\!|\!|^2 = \|x\|^2 + \sum_{m=1}^{\infty} \frac{1}{2^m} \sup_{n_1<n_2<\dots<n_{m+1}} \Big(\sum_{i=1}^{m}(\sum_{j=n_i}^{n_{i+1}-1} x_j)^2\Big) + \sum_{n=1}^{\infty} \frac{1}{2^n} x_n^2, \tag{2}$$

where the suprema (for a fixed m) are taken over all choices of $n_1 < \dots < n_{m+1}$. We claim that $|\!|\!|\cdot|\!|\!|$ is a dual W*UR norm on J.
Indeed, $\|\cdot\|^2$, $\frac{1}{2^n}x_n^2 = \frac{1}{2^n}f_n^2(x)$ and $\sum_{i=1}^{m}(\sum_{j=n_i}^{n_{i+1}-1} x_j)^2 = \sum_{i+1}^{m}(\sum_{j=n_i}^{n_{i+1}-1} f_j(x))^2$ are w^*-lower semicontinuous on J, as $f_n \in J_*$. Since $|\!|\!|\cdot|\!|\!|^2$ results from taking uniform limits and suprema of w^*-lower semicontinuous functions, it is a w^*-lower semicontinuous function itself. The convexity and positive homogeneity of $|\!|\!|\cdot|\!|\!|$ together with the equivalence of $|\!|\!|\cdot|\!|\!|$ and $\|\cdot\|$ are obvious. Thus $|\!|\!|\cdot|\!|\!|$ is a dual equivalent norm on

J. To show that $|||\cdot|||$ is W*UR, recall that whenever $x^n, y^n \in S_{(J,|||\cdot|||)}$ are such that $2|||x^n|||^2 + 2|||y^n|||^2 - |||x^n + y^n|||^2 \to 0$, it follows that

$$2\|x^n\|^2 + 2\|y^n\|^2 - \|x^n + y^n\|^2 \to 0,$$

$$2 \sup_{n_1<\cdots<n_{m+1}} \Big(\sum_{i=1}^{m}\Big(\sum_{j=n_i}^{n_{i+1}-1} x_j^n\Big)^2\Big) + 2 \sup_{n_1<\cdots<n_{m+1}} \Big(\sum_{i=1}^{m}\Big(\sum_{j=n_i}^{n_{i+1}-1} y_j^n\Big)^2\Big) -$$
$$- \sup_{n_1<\cdots<n_{m+1}} \Big(\sum_{i=1}^{m}\Big(\sum_{j=n_i}^{n_{i+1}-1} x_j^n + y_j^n\Big)^2\Big) \to 0 \quad \text{as} \quad n \to \infty$$

for every $j \in I\!N$. This is a standard fact that follows from the convexity of the participating functions and may be found in [1]. In particular, from the last limit we obtain $\lim_{n\to\infty} f_j(x^n - y^n) = 0$ for every $j \in I\!N$. Because $\operatorname{span}\{f_j\}$ is dense in J_* and $\{x^n\}$, $\{y^n\}$ are bounded, $\mathrm{w}^* - \lim_{n\to\infty}(x^n - y^n) = 0$. The claim is established. As $\dim(J^* \setminus J_*) = 1$, in order to show that $|||\cdot|||$ is WUR it is enough to prove that whenever $x^n, y^n \in J$ satisfy $|||x^n|||, |||y^n||| \le 1$ and

$$(3) \qquad 2|||x^n|||^2 + 2|||y^n|||^2 - |||x^n + y^n|||^2 \to 0 \text{ as } n \to \infty$$

we have $f(x^n - y^n) \to 0$, where $f(\cdot) = \sum_{i=1}^{\infty} f_i(\cdot) \in J^* \setminus J_*$. Suppose contrary, i.e. (by passing to a subsequence etc.)

$$(4) \qquad |f(x^n - y^n)| > 5\varepsilon > 0 \text{ for every } n \in I\!N.$$

Choose an integer $K > \frac{16}{\varepsilon^4}$. From (2) and (3) it follows that:

$$(5) \qquad \begin{aligned} &2 \sup_{n_1<n_2<\cdots<n_{K+1}} \sum_{i=1}^{K}\Big(\sum_{j=n_i}^{n_{i+1}-1} x_j^n\Big)^2 + 2 \sup_{n_1<n_2<\cdots<n_{K+1}} \sum_{i=1}^{K}\Big(\sum_{j=n_i}^{n_{i+1}-1} y_j^n\Big)^2 - \\ &- \sup_{n_1<n_2<\cdots<n_{K+1}} \sum_{i=1}^{K}\Big(\sum_{j=n_i}^{n_{i+1}-1} x_j^n + y_j^n\Big)^2 \to 0 \text{ as } n \to \infty. \end{aligned}$$

By standard arguments from renorming theory, to be found in [1, p.42], this condition is equivalent to:

$$(6) \quad 4 \sup_{n_1<n_2<\cdots<n_{K+1}} \sum_{i=1}^{K}\Big(\sum_{j=n_i}^{n_{i+1}-1} x_j^n\Big)^2 - \sup_{n_1<n_2<\cdots<n_{K+1}} \sum_{i=1}^{K}\Big(\sum_{j=n_i}^{n_{i+1}-1} x_j^n + y_j^n\Big)^2 \to 0$$

as $n \to \infty$, and

$$(7) \quad 4 \sup_{n_1<n_2<\cdots<n_{K+1}} \sum_{i=1}^{K}\Big(\sum_{j=n_i}^{n_{i+1}-1} y_j^n\Big)^2 - \sup_{n_1<n_2<\cdots<n_{K+1}} \sum_{i=1}^{K}\Big(\sum_{j=n_i}^{n_{i+1}-1} x_j^n + y_j^n\Big)^2 \to 0$$

as $n \to \infty$.
Choose a system of finite sequences $\{m_1^n, \dots, m_{K+1}^n\}_{n\in I\!N}$ such that:

$$\sup_{n_1<n_2<\dots<n_{K+1}} \sum_{i=1}^{K} (\sum_{j=n_i}^{n_{i+1}-1} x_j^n + y_j^n)^2 - \sum_{i=1}^{K} (\sum_{j=m_i^n}^{m_{i+1}^n-1} x_j^n + y_j^n)^2 \to 0 \;\text{ as }\; n \to \infty.$$

It follows from (5) that:

$$(8) \quad \sum_{i=1}^{K} \Big(2(\sum_{j=m_i^n}^{m_{i+1}^n-1} x_j^n)^2 + 2(\sum_{j=m_i^n}^{m_{i+1}^n-1} y_j^n)^2 - (\sum_{j=m_i^n}^{m_{i+1}^n-1} x_j^n + y_j^n)^2\Big) \to 0 \;\text{ as }\; n \to \infty.$$

From the fact that the canonical norm on a finitely dimensional Hilbert space ℓ_2^K is uniformly convex we derive:

$$\sum_{j=m_i^n}^{m_{i+1}^n-1} x_j^n - y_j^n \to 0 \text{ as } n \to \infty$$

for every i, $1 \le i \le K$. Thus

$$\sum_{j=m_1^n}^{m_{K+1}^n-1} x_j^n - y_j^n \to 0 \;\text{ as }\; n \to \infty.$$

This together with (4) implies that for n large enough at least one of the following expressions:

$$\Big|\sum_{j=1}^{m_1^n-1} x_j^n\Big|, \Big|\sum_{j=1}^{m_1^n-1} y_j^n\Big|, \Big|\sum_{j=m_{K+1}^n}^{\infty} x_j^n\Big|, \Big|\sum_{j=m_{K+1}^n}^{\infty} y_j^n\Big|$$

is larger then ε. We may assume without loss of generality that it is the first one and moreover (by passing to a subsequence) that this occurs for every $n \in I\!N$. Among the numbers $\Big\{\Big|\sum_{j=m_i^n}^{m_{i+1}^n-1} x_j^n\Big|, 1 \le i \le K\Big\}$ at least one, corresponding to some i_0 is smaller than $\frac{\varepsilon^2}{4}$, otherwise from (1) and the choice of K we get $\|x^n\| > 1$. We have:

$$\sup_{n_1<n_2<\dots<n_{K+1}} \sum_{i=1}^{K} (\sum_{j=n_i}^{n_{i+1}-1} x_j^n)^2 \ge$$

$$\ge (\sum_{j=1}^{m_1^n-1} x_j^n)^2 + \sum_{i=1}^{i_0-1} (\sum_{j=m_i^n}^{m_{i+1}^n-1} x_j^n)^2 + (\sum_{j=m_{i_0}^n}^{m_{i_0+2}^n-1} x_j^n)^2 + \sum_{i=i_0+2}^{K} (\sum_{j=m_i^n}^{m_{i+1}^n-1} x_j^n)^2 \ge$$

$$\ge \varepsilon^2 + \sum_{i=1}^{K} (\sum_{j=m_i^n}^{m_{i+1}^n-1} x_j^n)^2 - 2(\frac{\varepsilon^2}{4}) \ge \sum_{i=1}^{K} (\sum_{j=m_i^n}^{m_{i+1}^n-1} x_j^n)^2 + \frac{\varepsilon^2}{2}.$$

This is a contradiction with (6) and (8).

As we mentioned in the introduction, separable Banach spaces with separable dual admit an equivalent WUR renorming. On the other hand, separable Banach spaces containing an isomorphic copy of ℓ_1 do not admit a renorming the second dual of which is rotund. It is therefore natural in this context to investigate the spaces not containing ℓ_1, but having nonseparable dual. The first example of a separable Banach space not containing ℓ_1 but with nonseparable dual was constructed by James in [5]. In [6], this space was denoted by JT (James tree) and thoroughly investigated. It is shown there that JT is a dual space to JT_*, $\mathrm{JT}^*/\mathrm{JT}_* \equiv \ell_2(\Gamma)$ and $\mathrm{JT}^{**} \equiv \mathrm{JT} \oplus \ell_2(\Gamma)$. We will use [6] as our main reference to JT.
In our next theorem we show that there exists an equivalent (dual) norm on the James tree space JT whose second dual norm is URED. Spaces JT_* and JT are defined as certain spaces of functions on an infinite tree $(T, <)$ introduced in the proof of Theorem 1. The space JT consists of all real functions on T such that

$$\|x\| = \sup \big(\sum_{j=1}^{k} (\sum_{t \in \mathcal{S}_j} x(t))^2 \big)^{\frac{1}{2}} < \infty,$$

where the supremum is taken over all choices of pairwise disjoint segments $\mathcal{S}_1$, $\mathcal{S}_2$, $\ldots$, $\mathcal{S}_k$.
It can be shown that the vectors $e_{(n,i)} = \chi_{(n,i)}$ in the lexicographic ordering form a boundedly complete basis of JT. The functions $f_{(n,i)} = \chi_{(n,i)}$ in the lexicographic ordering can also be viewed as the Schauder basis of JT_*. The notation and properties concerning the space JT used in the following come from [6].

Theorem 4
The space JT admits an equivalent dual norm whose second dual is URED.

Proof.
Let us extend the definition of $<$ to the set $T \cup \Gamma$ where Γ is the set of all branches of T. We put $t < \gamma$ for $t \in T$, $\gamma \in \Gamma$ iff $t \in \gamma$ and elements in Γ are incomparable. By Theorem 1 of [6] and its proof the spaces JT^* and JT^{**} are isomorphic to spaces of functions on $T \cup \Gamma$ as follows: Let $f \in JT^*$. If $t \in T$, then $f(t) = f(e_t)$ where e_t is a basis vector in JT corresponding to the index $t = (n, i)$. We know from the theory of Schauder basis (see [7]) that:

$$f = \mathrm{w}^* - \lim \sum_{n=0}^{\infty} \sum_{i=0}^{2^n - 1} f(e_{(n,i)}) f_{(n,i)}.$$

For $\gamma \in \Gamma$ we put $f(\gamma) = \lim\limits_{\substack{(n,i) \in \gamma \\ n \to \infty}} f(e_{(n,i)})$.
For elements $F \in JT^{**}$ there exists $F_T \in JT$ and $F_\Gamma \in \ell_2(\Gamma)$ such that $F = F_T + F_\Gamma$ (using our representation of F, F_T, F_Γ as functions defined on $T \cup \Gamma$). $\| \cdot \|^{**}$ is then equivalent to $|F| = (\|F_T\| + \|F_\Gamma\|_2^2)^{\frac{1}{2}}$. In this setting we have:

$$JT_* = \{ f \in JT^*, f(\gamma) = 0 \ \text{ for } \ \gamma \in \Gamma \},$$

$$JT = \{ F \in JT^{**}, F(\gamma) = 0 \ \text{ for } \ \gamma \in \Gamma \}.$$

For arbitrary $f \in JT^*$ and arbitrary finitely supported (on $T \cup \Gamma$) $F \in JT^{**}$ we have:

$$F(f) = \sum_{t \in T \cup \Gamma} F(t) f(t).$$

Let us recall that the canonical norm $\|\cdot\|$ on JT is defined as:

$$\|x\| = \sup \big(\sum_{j=1}^{k} (\sum_{t \in \mathcal{S}_j} x(t))^2 \big)^{\frac{1}{2}} \tag{9}$$

where the supremum is taken over all choices of pairwise disjoint segments $\mathcal{S}_1, \ldots, \mathcal{S}_k$ of T.
Since $x(t) = f_t(x)$, $\|\cdot\|$ is a dual norm on JT. We define an equivalent norm $|\!|\!|\cdot|\!|\!|$ on JT as follows:

$$|\!|\!| x |\!|\!|^2 = \|x\|^2 + \sum_{t \in T} \delta_t x(t)^2 \tag{10}$$

where ε_n, $\delta_t > 0$ and

$$\sum_{\substack{n \geq n_0 \\ 0 \leq i \leq 2^n - 1}} \delta_{(n,i)} < \varepsilon_{n_0} \searrow 0 \text{ as } n_0 \to \infty. \tag{11}$$

As in the proof of Proposition 3, it is easily observed that $|\!|\!|\cdot|\!|\!|$ is a dual norm that is W*UR (as $\delta_t > 0$). We claim that the definition of $|\!|\!|\cdot|\!|\!|$ extends naturally onto JT^{**} as follows:

$$|\!|\!| F |\!|\!|^{**} = \big(\sup \big(\sum_{j=1}^{k} (\sum_{t \in \mathcal{S}_j} F(t))^2 \big) + \sum_{t \in T} \delta_t F(t)^2 \big)^{\frac{1}{2}} \tag{12}$$

where the supremum is taken over all choices of pairwise disjoint segments $\mathcal{S}_1, \ldots, \mathcal{S}_k$ of $T \cup \Gamma$, that is $\mathcal{S}_j = \{t \in T \cup \Gamma, \ \ t_{j_1} \leq t \leq t_{j_2} \ \text{ where } \ t_{j_1}, t_{j_2} \in T \cup \Gamma\}$. Once this is established, the rest of the proof is a standard verification (in the spirit of [1, p.62]) that $(JT^{**}, |\!|\!|\cdot|\!|\!|^{**})$ is URED.
To prove this claim, we list some elementary observations without proof:

(i) Let $x \in JT$, $(n,i) = t_0 \in T$. Put $x^{t_0} \in JT$ to be

$$x^{t_0} = \begin{cases} 0 & t \geq t_0 \\ x(t) & \text{otherwise.} \end{cases}$$

Then $|\!|\!| x^{t_0} |\!|\!| \leq (1 + \sum_{t \geq t_0} \delta_t) |\!|\!| x |\!|\!|$.

(ii) Similarly to (i) we have: Let $x \in JT$, $S \subset \{t = (n,i), \ \ n \geq n_0\}$. Put

$$x^S(t) = \begin{cases} 0 & t \geq s \text{ for } s \in S \\ x(t) & \text{otherwise.} \end{cases}$$

Then $\|\!|x^S|\!\|\leq (1+\varepsilon_{n_0})\|\!|x|\!\|$.

(iii) Let $x\in JT$, $t_0\in T$ $t_0\in\gamma\in\Gamma$ be such that: $x(t)=0$ for $t\geq t_0$, $t\notin\gamma$. Put:

$$x_{t_0,\gamma}(t_0)=\sum_{t\in\gamma,t\geq t_0} x(t),$$

$$x_{t_0,\gamma}(t_0)=0 \text{ for } t>t_0,t\in\gamma,$$

$$x_{t_0,\gamma}(t_0)=x(t) \text{ otherwise.}$$

Then $\|\!|x_{t_0,\gamma}|\!\|\leq (1+\sum_{t\geq t_0}\delta_t)\|\!|x|\!\|$.

(iv) Applying the previous statements, we obtain the following:
Let $x\in JT$, $t_1=(n_0,i_1)$, $t_2=(n_0,i_2),\dots,t_k=(n_0,i_k)$ and $t_1\in\gamma_1,\dots,t_k\in\gamma_k$, where $\gamma_i\in\Gamma$. Put $S=\{t=(n,j),\quad n>n_0,t\notin\gamma_1,\dots,t\notin\gamma_k\}$. Then for $y=(\dots(((x^S)_{t_1,\gamma_1})_{t_2,\gamma_2})\dots)_{t_k,\gamma_k}$ we have $\|\!|y|\!\|\leq(1+2\varepsilon_{n_0})\|\!|x|\!\|$.

Now we prove the formula (12). We may assume that F is finitely supported on $T\cup\Gamma$ by $\{t_1=(n_1,i_1),t_2=(n_2,i_2),\dots,t_k=(n_k,i_k)\}\cup\{\gamma_1,\dots,\gamma_l\}$. For n large enough we define $F_n\in JT$ a finitely supported vector as:

$$F_n(t)=F(t) \text{ for } t=(m,i),m<n,$$

$$F_n((n,i))=F(\gamma_j) \text{ where } (n,i)\in\gamma_j,$$

$$F_n(t)=0 \text{ otherwise.}$$

Obviously $F_n\overset{w^*}{\to}F$ as $n\to\infty$ and

$$\lim_{n\to\infty}\|\!|F_n|\!\|^{**}=\big(\sup\big(\sum_{j=1}^{k}(\sum_{t\in\mathcal{S}_j}F(t))^2\big)+\sum_{t\in T}\delta_tF(t)^2\big)^{\frac{1}{2}}\geq\|\!|F|\!\|^{**}.$$

To prove the opposite inequality, find for every F_n an element $f_n\in JT^*$, $\|\!|f_n|\!\|^*<1+\varepsilon_n, f_n(F_n)\geq\|\!|F_n|\!\|$. It follows from (ii) that the finite dimensional projections $Q_n:(JT,\|\!|\cdot|\!\|)\to(JT,\|\!|\cdot|\!\|)$ onto span$\{e_{(m,i)},m\leq n\}$ have norm bounded by $1+\varepsilon_n$. Therefore we have for $g_n=Q_n^*\circ f_n$: $\|\!|g_n|\!\|^*<1+3\varepsilon_n$, $g_n(F_n)\geq\|\!|F_n|\!\|$. Define $h_n\in JT^*$ as follows:

$$h_n(t)=g_n(t) \text{ for } t=(m,i),m\leq n,$$

$$h_n(t)=g_n((n,i)) \text{ for } (n,i)\leq t\in\gamma_j,$$

$$h_n(t)=0 \text{ otherwise.}$$

An easy argument using (iv) yields:

$$\|\!|h_n|\!\|^*\leq 1+12\varepsilon_n.$$

However, $F(h_n)=F_n(h_n)$. This proves the opposite inequality.

To prove that $\|\!|\cdot|\!\|^{**}$ is URED, suppose $F^n, G^n \in S_{(JT^{**},\|\!|\cdot|\!\|)}$,

$$2\|\!|F^n|\!\|^{**2}+2\|\!|G^n|\!\|^{**2}-\|\!|F^n+G^n|\!\|^{**2}\to 0 \quad \text{where} \quad F^n-G^n=\lambda_n F.$$

For every $t\in T$, $\lim F^n(t)-G^n(t)=\lim \lambda_n F(t)=0$. So if $F(t)\neq 0$ for some $t\in T$, we are done. Suppose $F(t)=0$ for every $t\in T$. For every $\omega>0, n\in I\!N$, there exists a system $\{\tilde{S}_j\}_{j=1}^k$ of pairwise disjoint segments such that

$$\sum_{j=1}^{k}(\sum_{t\in\tilde{S}_j}F^n(t)+G^n(t))^2>\sup\sum_{l=1}^{r}(\sum_{t\in S_L}f^n(t)+G^n(t))^2-\omega,$$

where the supremum is taken over all pairwise disjoint systems of segments. Consequently,

$$2\|\!|F^n|\!\|^{**2}+2\|\!|G^n|\!\|^{**2}-\|\!|F^n+G^n|\!\|^{**2}\geq +2\sum_{j=1}^{k}(\sum_{t\in\tilde{S}_j}G^n(t))^2+$$

$$+\sum_{t\in\Gamma\setminus\bigcup_{j=1}^{k}\tilde{S}_j}F^{n2}(t)+G^{n2}(t)-\sum_{j=1}^{k}(\sum_{t\in\tilde{S}_j}2G^n(t)+\lambda_nF(t))^2-\omega\geq$$

$$\sum_{j=1}^{k}\sum_{t\in\tilde{S}_j\cap\Gamma}\lambda_n^2F(t)^2-\omega+\sum_{t\in\Gamma\setminus\cup_{j=1}^{k}\tilde{S}_j}(\frac{\lambda_n}{2}F(t))^2\geq\frac{\lambda_n^2}{4}\|\!|F|\!\|^{**2}-\omega.$$

Since ω is arbitrary, $\lambda_n\to 0$ and $\|\!|\cdot|\!\|^{**}$ is URED.

Remark 5 The space JT has an equivalent norm whose second dual norm is URED. However, it does not have a norm (necessarily WUR) whose second dual is W*UR. The space $JT^{**}=JT\oplus\ell_2(\Gamma)$ admits an equivalent UG (not necessarily dual) norm. Therefore JT^* has an equivalent WUR norm and JT^{**} has an equivalent dual UG norm. Yet, due to [9], there is no equivalent and dual WUR norm on JT^*.

In our next theorem we prove the impossibility of certain renorming of the space JH of Hagler.

Let us recall the definition of JH and some of its properties. JH consists of real functions $x(\cdot)$ on $T\cup\Gamma$ such that $x(\gamma)=0$ for $\gamma\in\Gamma$ and

$$\|x\|=\sup_{n<m}\Big(\sum_{i=1}^{k}(\sum_{t\in\mathcal{S}_i}x(t))^2\Big)^{\frac{1}{2}}$$

where the supremum is taken over all systems of pairwise disjoint segments $\mathcal{S}_i$ such that $\mathcal{S}_i=\{t\in T,(n,j_i)\leq t\leq(m,k_i)\}$. JH is a separable space not containing an isomorphic copy of ℓ_1. Its dual is nonseparable. The dual space JH^* consists of certain functions f on $T\cup\Gamma$ for which $f|_\Gamma\in c_0(\Gamma)$. In particular, the set $\{\gamma\in\Gamma, f(\gamma)\neq 0\}$ is countable. Another important thing about JH is the fact that for every $\gamma\in\Gamma$ the subspace of JH consisting of vectors supported by elements from

T belonging to γ is isomorphic to c_0, and the system $\{\chi_t, t \in \gamma\}$ with the ordering as in T forms its summing basis. Let us recall that the summing basis of c_0 consists of the vectors $u_n = \sum_{i=1}^{n} e_i = (1, \dots, 1, 0, \dots)$ where e_i are the canonical basic vectors. It is easy to verify that $\{u_n\}$ form a Schauder basis of c_0 and $\| \sum_{n=1}^{\infty} a_n z_n \|_s = \sup_{k_1 < k_2} | \sum_{n=k_1}^{k_2} a_n |$ is equivalent norm on c_0. Note that the summing basis of c_0 is not shrinking. In fact, the biorthogonal functionals $\{v^n\}_{n=1}^{\infty}$ are given by $V^n = f_n - f_{n+1}$ where $\{f_n\}$ is the canonical basis of ℓ_1. Thus

$$\overline{\text{span}}\{v^n\}_{n=1}^{\infty} = \Big\{ f \in \ell_1,\ \sum_{n=1}^{\infty} f(n) = 0 \Big\}.$$

For more details we refer the reader to [3].

Theorem 6
The space JH admits no equivalent norm whose second dual is rotund.

Proof. We proceed by contradiction. Let $|||\cdot|||$ be such that $|||\cdot|||^{**}$ is rotund. As JH is separable, there exists a 1-norming countable subset S of $B_{(JH^*, |||\cdot|||^*)}$. By the above remarks about JH^*, card$(\bigcup_{f \in S} supp(f)) = \omega_0$. Since Γ is uncountable, there exists $\gamma \in \Gamma$ such that $f(\gamma) = 0$ for every $f \in S$. Consider the subspace E of JH (isomorphic to c_0) consisting of functions supported by elements from γ (isomorphic to the summing basis). The restrictions $g_f = f|_\gamma$ for $f \in S$ form a 1-norming subset of E^*. By passing to the canonical basis of $E \cong c_0$ we finally arrive at the following: There exists an equivalent norm $|||\cdot|||$ on c_0 such that the set $M = B_{(\ell_1, |||\cdot|||^*)} \cap \{f \in \ell_1, \sum_{i=1}^{\infty} f(i) = 0\}$ is 1-norming and $|||\cdot|||^{**}$ on ℓ_∞ is rotund. We show that this leads to a contradiction.
Let $\varepsilon_k \searrow 0$, by I^k we denote an element of c_0 such that $I^k(i) = 1$ for $i \le k$, $I^k(i) = 0$ otherwise. We construct by induction a system of sequences of integers $\{n_k\}_{k \in \mathbb{N}}$, $\{m_k\}_{k \in \mathbb{N}}$, $\{n'_k\}_{k \in \mathbb{N}}$ and $\{m'_k\}_{k \in \mathbb{N}}$, of vectors $\{x^k\}_{k \in \mathbb{N}}$, $\{y^k\}_{k \in \mathbb{N}}$ from c_0 and $\{f^k\}_{k \in \mathbb{N}}$, $\{g^k\}_{k \in \mathbb{N}}$ from M such that:

$$n_k < n'_k < m_k < m'_k < n_{k+1} \text{ for } k \in \mathbb{N},$$

$$supp(x^k) \subset [1, n_k],\ supp(y^k) \subset [1, m_k],$$

$$0 \le x^k(i) \le 1 \text{ and } -1 \le y^k(i) \le 0 \text{ for } k, i \in \mathbb{N},$$

$$supp(f^k) \subset [1, n'_k],\ supp(g^k) \subset [1, m'_k],$$

$$f^k(x^k) \ge |||x^k||| - \varepsilon_k,\ g^k(y^k) \ge |||y^k||| - \varepsilon_k,$$

$$(x^k - I^{n'_k})(i) = y^k(i) \text{ for } i \le n'_k,$$

$$x^{k+1}(i) = (y^k + I^{m'_k})(i) \text{ for } i \le m'_k,$$

$$\|\!|y^k|\!\|\geq \sup\{\|\!|y|\!\|, y\in c_0, -1\leq y(i)\leq 0 \text{ for } i\in I\!N,$$
$$y(i)=(x^k-I^{n'_k})(i) \text{ for } i\leq n'_k\}-\varepsilon_k,$$

$$\|\!|x^{k+1}|\!\|\geq \sup\{\|\!|x|\!\|, x\in c_0, 0\leq x(i)\leq 1 \text{ for } i\in I\!N,$$
$$x(i)=(y^k-I^{m'_k})(i) \text{ for } i\leq m'_k\}-\varepsilon_k.$$

It follows easily that $f^k(x^k)=f^k(x^i)=f^k(y^i)$ for $i\geq k$ and analogously $g^k(x^k)=g^k(x^i)=g^k(y^i)$ for $i\geq k+1$.
We put $x,y\in\ell_\infty$ to be

$$x=w^*-\lim x^k$$

$$y=w^*-\lim y^k.$$

From the above observation one gets

$$\lim_{k\to\infty} f^k(x)=\lim_{k\to\infty} f^k(y)=\|\!|x|\!\|=\|\!|y|\!\|=\|\!|\frac{x+y}{2}|\!\|$$

a contradiction.

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