Cluster expansion and the boxdot conjecture

Emil Jeřábek*

Institute of Mathematics of the Academy of Sciences
Žitná 25, 115 67 Praha 1, Czech Republic, email: jerabek@math.cas.cz

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Abstract

The boxdot conjecture asserts that every normal modal logic that faithfully interprets **T** by the well-known boxdot translation is in fact included in **T**. We confirm that the conjecture is true. More generally, we present a simple semantic condition on modal logics L_0 which ensures that the largest logic where L_0 embeds faithfully by the boxdot translation is L_0 itself. In particular, this natural generalization of the boxdot conjecture holds for **S4**, **S5**, and **KTB** in place of **T**.

1 The boxdot translation

The *boxdot translation* is the mapping $\varphi \mapsto \varphi^{\boxdot}$ from the language of monomodal logic into itself that preserves propositional variables, commutes with Boolean connectives, and satisfies

$$(\Box\varphi)^{\boxdot} = \boxdot\varphi^{\boxdot},$$

where $\boxdot\varphi$ is an abbreviation for $\varphi \land \Box\varphi$. It is easy to see that for any normal modal logic L, the set of formulas interpreted in L by the boxdot translation,

$$L^{\boxdot^{-1}} = \{\varphi : \ \vdash_L \varphi^{\boxdot}\},$$

is also a normal modal logic (nml), and contains the logic $\mathbf{T} = \mathbf{K} \oplus \Box p \to p$. The boxdot translation is a faithful interpretation of **T** in the smallest nml **K** (i.e., $\mathbf{K}^{\boxdot^{-1}} = \mathbf{T}$), and more generally, in any logic between **K** and **T**. The *boxdot conjecture*, formulated by French and Humberstone [4], states that the converse also holds:

$$L^{\boxdot^{-1}} = \mathbf{T} \implies L \subseteq \mathbf{T}.$$

French and Humberstone proved the conjecture for logics L axiomatized by formulas of modal degree 1, and Steinsvold [5] has shown it for logics of the form $L = \mathbf{K} \oplus \Diamond^h\Box^i p \to \Box^j\Diamond^k p$, but the full conjecture remained unsettled.

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In this paper we will establish the boxdot conjecture. The argument actually uses only one particular property of **T**, namely that Kripke frames for **T** can be blown up by duplicating each node in a frame in a certain way, and this allows us to state an analogue of the conjecture for a large class of reflexive logics.

Given an arbitrary nml $L_0 \supseteq \mathbf{T}$, we may ask about the structure of logics L in which L_0 faithfully embeds by the boxdot translation ($L^{\boxdot^{-1}} = L_0$). First, there is always at least one such logic, for example L_0 itself: this follows from the observation that $\vdash_{\mathbf{T}} \varphi \leftrightarrow \varphi^{\boxdot}$ for every formula φ. If $L_0 = \mathbf{K} \oplus X$ for some set of axioms X, then the logic $L_0^{\boxdot} = \mathbf{K} \oplus \{\varphi^{\boxdot} : \varphi \in X\}$ has the property

$$L_0 \subseteq L^{\boxdot^{-1}} \quad \text{iff} \quad L_0^{\boxdot} \subseteq L,$$

in particular $L_0^{\boxdot} \subseteq L_0$ is the *smallest* logic in which L_0 faithfully embeds. Clearly, the set of logics such that $L^{\boxdot^{-1}} = L_0$ is *convex*: if $L_1 \subseteq L_2 \subseteq L_3$ and $L_1^{\boxdot^{-1}} = L_3^{\boxdot^{-1}} = L_0$, then $L_2^{\boxdot^{-1}} = L_0$. Finally, if $\{L_c : c \in C\}$ is a chain of logics linearly ordered by inclusion such that $L_c^{\boxdot^{-1}} = L_0$ for each $c \in C$, the logic $L = \bigcup_{c \in C} L_c$ also satisfies $L^{\boxdot^{-1}} = L_0$. It follows from Zorn's lemma that every logic in which L_0 faithfully embeds is included in a *maximal* such logic.

Thus, if $\{L_m : m \in M\}$ is the set of all maximal logics such that $L_m^{\boxdot^{-1}} = L_0$, the set of all logics in which L_0 faithfully embeds by the boxdot translation consists of the union

$$\bigcup_{m \in M} [L_0^{\boxdot}, L_m] \tag{1}$$

of intervals in the lattice of normal modal logics. Notice that L_0 is itself one of the maximal logics L_m, as $L^{\boxdot^{-1}} = L$ for any $L \supseteq L_0$ (or $L \supseteq \mathbf{T}$ for that matter). For a nontrivial example, $\mathbf{A}^* = \mathbf{GL} \oplus \Box\Box p \to \Box(\boxdot p \to q) \vee \Box(\boxdot q \to p)$ is a maximal logic in which **S4Grz** embeds [2, Exer. 9.26].

The original boxdot conjecture states that **T** is the largest logic L such that $L^{\boxdot^{-1}} = \mathbf{T}$. In accordance with this, we define that a nml $L_0 \supseteq \mathbf{T}$ has the *boxdot property*, if

$$L^{\boxdot^{-1}} = L_0 \implies L \subseteq L_0 \tag{BDP}$$

holds for every nml L. In light of the discussion above, BDP for L_0 is equivalent to the claim that there is only one maximal logic L such that $L^{\boxdot^{-1}} = L_0$, or in other words, that the union in (1) reduces to the single interval $[L_0^{\boxdot}, L_0]$.

What we are going to show is that BDP holds for all logics L_0 satisfying a natural semantic condition (which we will define precisely in Section 4). Since the condition applies to **T**, this also establishes the original boxdot conjecture. Moreover, our proof shows that under the same condition, L_0 has the *strong boxdot property*:

$$L^{\boxdot^{-1}} \subseteq L_0 \implies L \subseteq L_0. \tag{SBDP}$$

We do not know whether BDP and SBDP are equivalent in general, though they of course express the same condition when $L_0 = \mathbf{T}$.

2 Preliminaries

We first recall elementary concepts from relational semantics to agree on the notation.

Definition 2.1 A *Kripke frame* is a pair $\mathcal{W} = \langle W, R\rangle$, where R is a binary relation on a set W. A *model* based on the frame $\mathcal{W}$ is a triple $\mathcal{M} = \langle W, R, \vDash\rangle$, where the *valuation* $\vDash$ is a relation between elements of W and modal formulas, written as $\mathcal{M}, w \vDash \varphi$, which satisfies

$$\begin{aligned} \mathcal{M}, w \vDash \varphi \to \psi \quad &\text{iff} \quad \mathcal{M}, w \nvDash \varphi \text{ or } \mathcal{M}, w \vDash \psi, \\ \mathcal{M}, w \vDash \Box\varphi \quad &\text{iff} \quad \forall v \in W\,(w\ R\ v \Rightarrow \mathcal{M}, v \vDash \varphi), \end{aligned}$$

and similarly for other Boolean connectives. A formula φ *holds* in $\mathcal{M}$ if $\mathcal{M}, w \vDash \varphi$ for every $w \in W$, and it is *valid* in $\mathcal{W}$, written as $\mathcal{W} \vDash \varphi$, if $\mathcal{M} \vDash \varphi$ for every model $\mathcal{M}$ based on $\mathcal{W}$. If L is a nml, $\mathcal{W}$ is a *Kripke L-frame* if $\mathcal{W} \vDash \varphi$ for every $\varphi \in L$. A logic L is *sound* wrt a class C of frames if every $\mathcal{W} \in C$ is an L-frame, and it is *complete* wrt C if $\varphi \notin L$ implies $\mathcal{W} \nvDash \varphi$ for some $\mathcal{W} \in C$.

A *cluster* in a transitive frame $\langle W, R\rangle$ is an equivalence class of the equivalence relation $\sim$ on W defined by

$$w \sim v \quad \text{iff} \quad w = v \vee (w\ R\ v \wedge v\ R\ w).$$

We will also consider the general frame semantics. The reason is mostly esthetic—we do not want to encumber our results with the arbitrary restriction to Kripke-complete logics L_0 which does not seem to have anything to do with the problem under investigation. A reader who is only interested in the original boxdot conjecture ($L_0 = \mathbf{T}$), or more generally in BDP or SBDP for Kripke-complete logics L_0, may safely ignore general frames in what follows.

Definition 2.2 A *general frame* is a triple $\mathcal{W} = \langle W, R, A\rangle$, where $\langle W, R\rangle$ is a Kripke frame, and A is a family of subsets of W which is closed under Boolean operations, and under the operation

$$\Box X = \{w \in W : \forall v \in W\,(w\ R\ v \Rightarrow v \in X)\}.$$

Sets $X \in A$ are called *admissible*. A model $\mathcal{M} = \langle W, R, \vDash\rangle$ is based on $\mathcal{W}$ if the set

$$\{w \in W : \mathcal{M}, w \vDash p\}$$

is admissible for every variable p (which implies the same holds for all formulas). A formula is valid in $\mathcal{W}$ if it holds in all models based on $\mathcal{W}$, and the notions of L-frames, soundness, and completeness are defined accordingly. A Kripke frame $\langle W, R\rangle$ can be identified with the general frame $\langle W, R, \mathcal{P}(W)\rangle$.

We will also need basic validity-preserving operations on frames and models.

Definition 2.3 A Kripke frame $\langle W', R'\rangle$ is a *generated subframe* of $\mathcal{W} = \langle W, R\rangle$ if $W' \subseteq W$, and $R' = R \cap (W' \times W)$. (Note that $W' \subseteq W$ is a carrier of a generated subframe of $\mathcal{W}$ iff it is upward closed under R.) If $\langle W, R\rangle$ and $\langle W', R'\rangle$ are Kripke frames, a mapping $f\colon W \to W'$ is a *p-morphism*, provided

(i) $w\ R\ v$ implies $f(w)\ R'\ f(v)$,

(ii) if $f(w)\ R'\ v'$, there is $u \in W$ such that $w\ R\ u$ and $f(u) = v'$,

for every $w, v \in W$ and $v' \in W'$. Notice that the image of a p-morphism is always a generated subframe of $\mathcal{W}'$.

Similarly, a general frame $\langle W', R', A' \rangle$ is a generated subframe of a frame $\langle W, R, A \rangle$ if the Kripke frame $\langle W', R' \rangle$ is a generated subframe of $\langle W, R \rangle$, and $A' = \{X \cap W' : X \in A\}$. A p-morphism from a general frame $\langle W, R, A \rangle$ to $\langle W', R', A' \rangle$ is a p-morphism from the Kripke frame $\langle W, R \rangle$ to $\langle W', R' \rangle$ which additionally satisfies

(iii) $f^{-1}[X'] \in A$ for every $X' \in A'$.

Fact 2.4 ([1, Thm. 3.14, Prop. 5.72]) *Let $\mathcal{W}$ and $\mathcal{W}'$ be Kripke or general frames, and φ a formula valid in $\mathcal{W}$. If $\mathcal{W}'$ is a generated subframe of $\mathcal{W}$, or if there exists a p-morphism from $\mathcal{W}$ onto $\mathcal{W}'$, then $\mathcal{W}' \vDash \varphi$.* □

In more detail, p-morphisms preserve truth in models in the following way.

Fact 2.5 ([1, Prop. 2.14]) *If $\mathcal{M} = \langle W, R, \vDash \rangle$ and $\mathcal{M}' = \langle W', R', \vDash \rangle$ are models, φ is a formula, and f is a p-morphism from $\langle W, R \rangle$ to $\langle W', R' \rangle$ such that $\mathcal{M}, w \vDash p_i \Leftrightarrow \mathcal{M}', f(w) \vDash p_i$ for every $w \in W$ and every variable p_i occurring in φ, then $\mathcal{M}, w \vDash \varphi \Leftrightarrow \mathcal{M}', f(w) \vDash \varphi$.* □

We have been using $\vdash_L \varphi$ as a synonym for $\varphi \in L$, however we will also extend this notation to allow for nonlogical axioms.

Definition 2.6 For any nml L, $\vdash_L$ denotes the global consequence relation of L: if X is a set of formulas, and φ a formula, then $X \vdash_L \varphi$ iff φ has a finite derivation using elements of X, theorems of L, modus ponens, and necessitation.

The global consequence relation satisfies the following version of the deduction theorem, where $\Box^n \varphi = \underbrace{\Box \cdots \Box}_{n \text{ boxes}} \varphi$, and $\Box^{\le n} \varphi = \bigwedge_{i=0}^{n} \Box^i \varphi$.

Fact 2.7 ([2, Thm. 3.51]) *$X \vdash_L \varphi$ iff there is a finite subset $X_0 \subseteq X$ and a natural number n such that $\Box^{\le n} \bigwedge X_0 \to \varphi \in L$.* □

3 Motivation

Before we introduce the semantic condition that will guarantee the SBDP for a logic $L_0 \supseteq \mathbf{T}$, let us give some intuition. This section is mostly informal, its purpose is to explain that the condition does not come out of blue, but follows naturally from the properties of the boxdot translation. However, the argument uses a somewhat heavier machinery than the rest of the paper, hence it is intended for readers familiar with the structure theory of transitive modal logics (see e.g. [2] for more background). It can be skipped without losing continuity, though some of the counterexamples in Example 3.1 may be worth bearing in mind.

Assume that $L \nsubseteq L_0$, we would like to show that $L^{\boxdot^{-1}} \nsubseteq L_0$. If L_0 has the finite model property, there is a finite rooted L_0-frame $\mathcal{F}$ which is not an L-frame. If L and L_0 are transitive (i.e., extensions of **K4**), we can consider a frame formula $A_{\mathcal{F}}$ as in Fine [3]: $A_{\mathcal{F}}$ is invalid in a transitive frame $\mathcal{W}$ iff $\mathcal{F}$ is a p-morphic image of a generated subframe of $\mathcal{W}$. (Note that we will not actually use frame formulas in the proof of our main result below.) Clearly $\nvdash_{L_0} A_{\mathcal{F}}$.

Do we have $\vdash_L A_{\mathcal{F}}^{\boxdot}$? Well, if not, and if L is Kripke-complete for simplicity, there is an L-frame $\mathcal{W} = \langle W, R\rangle$ such that $\mathcal{W} \nvDash A_{\mathcal{F}}^{\boxdot}$. This means that $A_{\mathcal{F}}$ is invalid in the reflexivization $\mathcal{W}^\circ = \langle W, R \cup \mathrm{id}\rangle$, hence $\mathcal{F}$ is a p-morphic image of a generated subframe of $\mathcal{W}^\circ$. Since the class of L-frames is closed under generated subframes, we may assume that there is a p-morphism f from $\mathcal{W}^\circ$ itself onto $\mathcal{F}$.

If f were a p-morphism from $\mathcal{W}$ to $\mathcal{F}$, then $\mathcal{F}$ would be an L-frame contrary to the assumptions. This contradiction would show that $\vdash_L A_{\mathcal{F}}^{\boxdot}$, providing an example for $L^{\boxdot^{-1}} \nsubseteq L_0$. In general, f does not have to be a p-morphism from $\mathcal{W}$ to $\mathcal{F}$, however the only way this can fail is that for some $w \in W$, there is no $u \in W$ such that $w\ R\ u$ and $f(w) = f(u)$. The key observation is that this cannot happen if all clusters of $\mathcal{F}$ have more than one element: then we can fix $v' \neq f(w)$ in the same cluster as $f(w)$; since f is a p-morphism from $\mathcal{W}^\circ$ to $\mathcal{F}$, there must be $w\ R\ v\ R\ u$ such that $f(v) = v'$ and $f(u) = f(w)$ (where necessarily $w \neq v \neq u$), and we have $w\ R\ u$ by transitivity.

This suggests that a (transitive) logic L_0 will satisfy SBDP if the class of L_0-frames is closed under the operation of blowing up each cluster by adding new elements. On the other hand, there are examples implying that some condition of that sort is necessary, which shows that we are on the right track:

Example 3.1 The logic **S4Grz** corresponds to noetherian partially ordered frames. In particular, such frames only have one-element clusters. **S4Grz** does not have the BDP: it is well known that $\mathbf{GL}^{\boxdot^{-1}} = \mathbf{S4Grz}$, but $\mathbf{GL} \nsubseteq \mathbf{S4Grz}$ (in fact, **GL** and **S4Grz** have no consistent common extension).

Similarly, top-most clusters in **S4.1**-frames can only have one element, and BDP duly fails for **S4.1**: for example, $L^{\boxdot^{-1}} = \mathbf{S4.1}$, where $L = \mathbf{K4} \oplus \Diamond\Box\bot$ (this logic is complete wrt finite transitive frames whose top clusters are irreflexive).

The motivational argument above may give the false impression that it is enough for (S)BDP if we can blow up one-element clusters in L_0-frames to have at least two elements, and leave the rest unchanged, but this is only an artifact of the stipulation that L is transitive. For a simple counterexample, let L_0 be the logic of the two-element cluster $\mathcal{C}_2$, and L the logic of the (nontransitive) frame $\mathcal{I}_2 = \langle \{0,1\}, \{\langle 0,1\rangle, \langle 1,0\rangle\}\rangle$. Then $L^{\boxdot^{-1}} = L_0$ as $\mathcal{I}_2^\circ = \mathcal{C}_2$, but $L \nsubseteq L_0$: e.g., $\vdash_L p \wedge \Box(\Box p \to p) \to \Box p$, which is not valid in L_0.

4 The boxdot property

The concept of clusters does not make much sense for nontransitive frames, nevertheless the operation of doubling the size of each cluster can be easily generalized to arbitrary frames while retaining its most salient properties.

Definition 4.1 For any Kripke frame $\mathcal{W} = \langle W, R\rangle$, we define a new frame $2\mathcal{W} = \langle 2W, 2R\rangle$ by putting $2W = W \times \{0, 1\}$, and

$$\langle w, a\rangle \; 2R \; \langle v, b\rangle \quad \text{iff} \quad w \; R \; v$$

for any $w, v \in W$ and $a, b \in \{0, 1\}$. If $\mathcal{W} = \langle W, R, A\rangle$ is a general frame, we put $2\mathcal{W} = \langle 2W, 2R, 2A\rangle$, where

$$2A = \big\{(X \times \{0\}) \cup (Y \times \{1\}) : X, Y \in A\big\}.$$

We remark that a similar construction of modal frames from intuitionistic frames is employed in [2] under the name τ_2 for investigation of the Gödel translation.

The following is immediate from the definition.

Observation 4.2 *If $\mathcal{W}$ is a Kripke or general frame, the natural projection $\pi\colon 2W \to W$ is a p-morphism.* $\square$

Motivated by the discussion in Section 3, we will consider logics with the following property.

Assumption 4.3 *L_0 is a normal modal logic complete with respect to a class C of Kripke or general frames such that $2\mathcal{W}$ is an L_0-frame for every $\mathcal{W} \in C$.*

Notice that even though it is not explicitly demanded, the assumption implies that L_0 is sound wrt C because of Fact 2.4 and Observation 4.2.

Example 4.4 It is readily seen that if $\mathcal{W}$ is reflexive, symmetric, or transitive, then so is $2\mathcal{W}$. Thus, Assumption 4.3 holds for **T**, **KTB**, **S4**, and **S5**. For reflexive transitive frames, $2\mathcal{W}$ has the simple geometric interpretation of expanding each cluster of $\mathcal{W}$ to twice its original size, which implies that Assumption 4.3 also holds for **S4.2** and **S4.3**.

We come to our main result.

Theorem 4.5 *Every logic $L_0 \supseteq \mathbf{T}$ satisfying Assumption 4.3 has the strong boxdot property.*

Proof: Let L be a nml such that $L \nsubseteq L_0$. We fix a formula $\varphi \in L$ such that $\varphi \notin L_0$, and we will construct a formula χ such that $\chi^{\boxdot} \in L$ and $\chi \notin L_0$, witnessing that $L^{\boxdot^{-1}} \nsubseteq L_0$.

Let $\mathrm{Sub}(\varphi)$ denote the set of all subformulas of φ. We choose a propositional variable $q \notin \mathrm{Sub}(\varphi)$, and consider the finite set of formulas

$$X = \big\{\Box(q^e \to \psi) \to \psi : e \in \{0, 1\}, \Box\psi \in \mathrm{Sub}(\varphi)\big\},$$

where $q^1 = q$, $q^0 = \neg q$.

Claim 1

(*i*) $X^{\boxdot} \vdash_{\mathbf{K}} \Box\psi^{\boxdot} \to \psi^{\boxdot}$ *for every* $\Box\psi \in \mathrm{Sub}(\varphi)$ (*even without the use of necessitation*).

(*ii*) $X^{\boxdot} \vdash_{\mathbf{K}} \psi \leftrightarrow \psi^{\boxdot}$ *for every* $\psi \in \mathrm{Sub}(\varphi)$.

Proof: (i): We have

$$\vdash_{\mathbf{K}} q^e \wedge \Box\psi^{\boxdot} \to \boxdot(q^{1-e} \to \psi^{\boxdot})$$

for $e = 0, 1$, and $X^{\boxdot}$ includes the formula $\boxdot(q^{1-e} \to \psi^{\boxdot}) \to \psi^{\boxdot}$, which yields

$$X^{\boxdot} \vdash_{\mathbf{K}} q^e \to (\Box\psi^{\boxdot} \to \psi^{\boxdot}).$$

The result follows using $\vdash_{\mathbf{K}} q \vee \neg q$.

(ii): By induction on the complexity of ψ. The steps for variables and Boolean connectives are straightforward. If $\psi = \Box\psi_0$, we have $X^{\boxdot} \vdash_{\mathbf{K}} \psi_0 \leftrightarrow \psi_0^{\boxdot}$ by the induction hypothesis, hence

$$X^{\boxdot} \vdash_{\mathbf{K}} \Box\psi_0 \leftrightarrow \Box\psi_0^{\boxdot}$$

using necessitation and $\mathbf{K}$. However,

$$X^{\boxdot} \vdash_{\mathbf{K}} \Box\psi_0^{\boxdot} \leftrightarrow \boxdot\psi_0^{\boxdot}$$

by (i), whence $X^{\boxdot} \vdash_{\mathbf{K}} \Box\psi_0 \leftrightarrow \boxdot\psi_0^{\boxdot} = (\Box\psi_0)^{\boxdot}$. □ (Claim 1)

By Claim 1 and the choice of φ, we have $X^{\boxdot} \vdash_L \varphi^{\boxdot}$, hence

$$\vdash_L \Box^{\leq n} \bigwedge X^{\boxdot} \to \varphi^{\boxdot}$$

for some n by Fact 2.7. (In fact, one can take for n the modal degree of φ, but we will not need this.) In other words, $\vdash_L \chi^{\boxdot}$, where

$$\chi = \Box^n \bigwedge X \to \varphi.$$

It remains to show that $\nvdash_{L_0} \chi$. The argument below actually gives the slightly stronger conclusion $X \nvdash_{L_0} \varphi$.

Since $\nvdash_{L_0} \varphi$, there is a frame $\mathcal{W} \in C$ such that $\mathcal{W} \nvDash \varphi$ by Assumption 4.3. We can fix a model $\mathcal{M} = \langle W, R, \vDash \rangle$ based on $\mathcal{W}$, and $w_0 \in W$ such that $\mathcal{M}, w_0 \nvDash \varphi$. We define a model $2\mathcal{M} = \langle 2W, 2R, \vDash \rangle$ based on $2\mathcal{W}$ by putting

$$2\mathcal{M}, \langle w, a \rangle \vDash q \quad \text{iff} \quad a = 1,$$
$$2\mathcal{M}, \langle w, a \rangle \vDash p_i \quad \text{iff} \quad \mathcal{M}, w \vDash p_i$$

for every $w \in W$, $a \in \{0, 1\}$, and every variable p_i distinct from q. We have

(2) $$2\mathcal{M}, \langle w, a \rangle \vDash \psi \quad \text{iff} \quad \mathcal{M}, w \vDash \psi$$

for every $w \in W$, $a \in \{0, 1\}$, and $\psi \in \mathrm{Sub}(\varphi)$ by Fact 2.5 and Observation 4.2, in particular $2\mathcal{M}, \langle w_0, a \rangle \nvDash \varphi$. On the other hand, consider any formula

$$\Box(q^e \to \psi) \to \psi$$

from X. If $2\mathcal{M}, \langle w, a \rangle \nvDash \psi$, we have $\mathcal{M}, w \nvDash \psi$ by (2), hence $\mathcal{M}, v \nvDash \psi$ for some $w \; R \; v$ as $\mathcal{W} \vDash L_0 \supseteq \mathbf{T}$. (If $\mathcal{W}$ is a Kripke frame, it has to be reflexive, in which case we can simply take $v = w$.) Then $2\mathcal{M}, \langle v, e \rangle \vDash q^e \wedge \neg\psi$ by (2), and $\langle w, a \rangle \; 2R \; \langle v, e \rangle$, thus $2\mathcal{M}, \langle w, a \rangle \nvDash \Box(q^e \to \psi)$.

We have verified that $2\mathcal{M} \vDash X$ and $2\mathcal{M} \nvDash \varphi$. By assumption, $2\mathcal{W}$ is an L_0-frame, hence $X \nvdash_{L_0} \varphi$. □

Corollary 4.6 *The logics* **T**, **KTB**, **S4**, **S5**, **S4.2** *and* **S4.3** *have the SBDP. In particular, the boxdot conjecture is true.* □

It is natural to ask how far is Assumption 4.3 from being a *necessary* and sufficient criterion for (S)BDP. On the positive side, it is possible to show using similar techniques as in Section 3 that our condition gives full characterization of (S)BDP for well-behaved transitive logics:

Proposition 4.7 *If* $L_0 \supseteq \mathbf{S4}$ *has the finite model property, the following are equivalent.*

(*i*) L_0 *has the BDP.*

(*ii*) L_0 *has the SBDP.*

(*iii*) L_0 *satisfies Assumption* 4.3 *for some* C.

(*iv*) L_0 *satisfies Assumption* 4.3 *for every class* C *of general frames with respect to which it is sound and complete.*

(*v*) L_0 *is the smallest modal companion of some superintuitionistic logic* (*see [2, §9.6]*).

Proof sketch: (ii) → (i) and (iv) → (iii) are trivial, and (iii) → (ii) is Theorem 4.5. (v) → (iv) follows from the fact that $\mathcal{W}$ and $2\mathcal{W}$ induce the same intuitionistic frame.

(i) → (iii): Let C be the class of all finite rooted L_0-frames. For each $\mathcal{F} \in C$, let $\mathcal{F}^\bullet$ be the (not necessarily transitive) frame obtained from $\mathcal{F}$ by making all points irreflexive, and let L be the logic determined by $\{\mathcal{F}^\bullet : \mathcal{F} \in C\}$. Since $(\mathcal{F}^\bullet)^\circ = \mathcal{F}$, we have $L^{\boxdot^{-1}} = L_0$, hence $L \subseteq L_0$ by (i). Since $L \supseteq \mathbf{wK4} = \mathbf{K} \oplus \boxdot p \to \Box\Box p$, we can use Fine frame formulas as in the transitive case. In particular, $L \subseteq L_0$ implies that for every $\mathcal{F} \in C$, there is a p-morphism $f\colon \mathcal{G}^\bullet \twoheadrightarrow \mathcal{F}$ for some $\mathcal{G} \in C$. We use it to construct a mapping $g\colon G \to 2F$ as follows. If $x \in F$, and c is a maximal cluster of $\mathcal{G}$ intersecting $f^{-1}[x]$, then $|c \cap f^{-1}[x]| \geq 2$ as $\mathcal{G}^\bullet$ is irreflexive while $\mathcal{F}$ is reflexive. Thus, we can split $c \cap f^{-1}[x]$ in two nonempty parts that are respectively mapped to $\langle x, 0\rangle$ and $\langle x, 1\rangle$ by g. Other elements of $f^{-1}[x]$ are mapped, say, to $\langle x, 0\rangle$. It is easy to check that g is a p-morphism of $\mathcal{G}$ onto $2\mathcal{F}$, hence $2\mathcal{F}$ is an L_0-frame. Consequently, L_0 satisfies Assumption 4.3 wrt C.

(iii) → (v): This can be shown using the machinery of Zakharyaschev's canonical formulas (see [2, §9] for an explanation, which is outside the scope of this paper). Let $\alpha(\mathcal{F}, D, \bot)$ be a canonical formula based on a reflexive $\mathcal{F}$, $\mathcal{F}'$ be a frame obtained from $\mathcal{F}$ by shrinking each cluster to (at least) half its original size, and D' be the corresponding set of closed domains. Then any cofinal subreduction of $\mathcal{W} \vDash \mathbf{S4}$ to $\mathcal{F}'$ with the closed domain condition for D' can be lifted to a cofinal subreduction of $2\mathcal{W}$ to $\mathcal{F}$ with CDC for D. Using Assumption 4.3, this shows that $\vdash_{L_0} \alpha(\mathcal{F}, D, \bot)$ implies $\vdash_{L_0} \alpha(\mathcal{F}', D', \bot)$. Repeating this process yields an axiomatization of L_0 over **S4** by formulas $\alpha(\mathcal{F}, D, \bot)$ where $\mathcal{F}$ has only simple clusters, which means that L_0 is of the form (v). □

However, the situation appears to be more complicated in the case of nontransitive L_0.

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