Rules with parameters in modal logic I

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Abstract

We study admissibility of inference rules and unification with parameters in transitive modal logics (extensions of **K4**), in particular we generalize various results on parameter-free admissibility and unification to the setting with parameters.

Specifically, we give a characterization of projective formulas generalizing Ghilardi's characterization in the parameter-free case, leading to new proofs of Rybakov's results that admissibility with parameters is decidable and unification is finitary for logics satisfying suitable frame extension properties (called cluster-extensible logics in this paper). We construct explicit bases of admissible rules with parameters for cluster-extensible logics, and give their semantic description. We show that in the case of finitely many parameters, these logics have independent bases of admissible rules, and determine which logics have finite bases.

As a sideline, we show that cluster-extensible logics have various nice properties: in particular, they are finitely axiomatizable, and have an exponential-size model property. We also give a rather general characterization of logics with directed (filtering) unification.

In the sequel, we will use the same machinery to investigate the computational complexity of admissibility and unification with parameters in cluster-extensible logics, and we will adapt the results to logics with unique top cluster (e.g., **S4.2**) and superintuitionistic logics.

1 Introduction

Admissibility of inference rules is among the fundamental properties of nonclassical propositional logic: a rule is admissible if the set of tautologies of the logic is closed under the rule, or equivalently, if adjunction of the rule to the logic does not lead to derivation of new tautologies. Admissible rules of basic transitive modal logics (**K4**, **S4**, **GL**, **Grz**, **S4.3**, ...) are fairly well understood. Rybakov proved that admissibility in a large class of modal logics is decidable and provided semantic description of their admissible rules, see [22] for a detailed treatment. Ghilardi [8] gave a characterization of projective formulas in terms of extension

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properties of their models, and proved the existence of finite projective approximations. This led to an alternative proof of some of Rybakov's results, and it was utilized by Jeřábek [14, 17] to construct explicit bases of admissible rules, and to determine the computational complexity of admissibility [16]. A sequent calculus for admissible rules was developed by Iemhoff and Metcalfe [12]. Methods used for transitive modal logics were paralleled by a similar treatment of intuitionistic and intermediate logics, see e.g. [22, 7, 10, 11].

Admissibility is closely related to unification [2, 1]: for equational theories corresponding to algebraizable propositional logics, E-unification can be stated purely in terms of logic, namely a unifier of a formula is a substitution which makes it a tautology. Thus, a rule is admissible iff every unifier of the premises of the rule also unifies its conclusion, and conversely the unifiability of a formula can be expressed as nonadmissibility of a rule with inconsistent conclusion. In fact, the primary purpose of Ghilardi [8] was to prove that unification in the modal logics in question is finitary.

In unification theory, it is customary to work in a more general setting that allows for extension of the basic equational theory by free constants. In logical terms, formulas may include atoms (variously called parameters, constants, coefficients, or metavariables) that behave like ordinary propositional variables for most purposes, but are required to be left fixed by substitutions. Some of the above-mentioned results on admissibility in transitive modal logics also apply to admissibility and unification with parameters, in particular Rybakov [22] proved the decidability of admissibility with parameters in basic transitive logics, and he has recently extended his method to show that unification with parameters is finitary in these logics [23, 24]. Nevertheless, a significant part of the theory only deals with parameter-free rules and unifiers.

The purpose of this paper is to (at least partially) remedy this situation by extending some of the results on admissibility in transitive modal logics to the setup with parameters. Our basic methodology is similar to the parameter-free case, however the presence of parameters brings in new phenomena leading to nontrivial technical difficulties that we have to deal with.

For a more detailed overview of the content of the paper, after reviewing basic concepts and notation (Section 2) and establishing some elementary background on multiple-conclusion consequence relations with parameters (Section 2.1), we start in Section 3 with a parametric version of Ghilardi's characterization of projective formulas in transitive modal logics with the finite model property in terms of a suitable model extension property on finite models. In Section 4, we introduce the class of cluster-extensible (clx) logics (and more generally, Par-extensible logics for the case when the set Par of allowed parameters is finite), and we use the results from Section 3 to show that in clx (or Par-extensible) logics, all formulas have projective approximations. As a corollary, this reproves results of Rybakov [22, 23] that such logics L have finitary unification type, and if L is decidable, then admissibility in L is also decidable, and one can compute a finite complete set of unifiers of a given formula. In order to determine which of these logics have unitary unification, we include in Section 4.2 a simple syntactic criterion for directed (filtering) unification, vastly generalizing the result of Ghilardi and Sacchetti [9]. In Section 4.3, we look more closely at semantic and structural properties of clx logics: we show that every clx logic is finitely axiomatizable, decidable, $\forall\exists$-definable on finite frames, and has an exponential-size model property. Moreover, the class of clx logics is closed under joins in the lattice of normal extensions of **K4**. (These results mostly do not have good analogues in the parameter-free case, they exploit the fact that the extension conditions designed to make the other results on admissibility and unification work need to be more restrictive when parameters are considered.)

In Section 5, we introduce (multiple-conclusion) rules corresponding to the existence of a parametric version of tight predecessors, generalizing the parameter-free rules considered in [14, 17]. We investigate their semantic properties, and as the main result of this section, we show that these extension rules form bases of admissible rules for clx or Par-extensible logics. We present single-conclusion variants of these bases in Section 5.1. Finally, in Section 5.2, we modify the extension rules further to provide independent bases of admissible rules with finitely many parameters for Par-extensible logics, and we show that finite bases exist if and only if the logic has bounded branching.

As the name suggests, this paper is to be continued by a sequel, where we will address the computational complexity of admissibility and unification with parameters in clx logics, and modifications of our results to related classes of logics: modal logics whose finite rooted frames have a single top cluster (such as **K4.2** and **S4.2**), and intuitionistic and intermediate logics.

2 Preliminaries and notation

The purpose of this section is to review basic definitions and standard facts we are going to use in order to fix our terminology and notation. For more detailed information, we refer the reader to [3] (modal logic), [22] (admissible rules), [2, 1] (unification), [13] (propositional consequence relations), [25] (multiple-conclusion consequence relations).

We will work with *propositional languages* Form consisting of formulas freely built from a (usually countably infinite) set of *atoms* using a fixed set of finitary *connectives*. (We distinguish two types of atoms: variables and parameters. We will elaborate on this in Section 2.1.) We will usually denote formulas by lowercase Greek letters $\varphi, \psi, \chi, \dots$. We write $\psi \subseteq \varphi$ if ψ is a subformula of φ. $\mathrm{Sub}(\varphi)$ denotes the set of all subformulas of a formula φ, and $|\varphi|$ the length (i.e., the number of symbols) of φ. Finite sets of formulas will be usually denoted by uppercase Greek letters $\Gamma, \Delta, \dots$.

Let us fix a propositional language Form. An *atomic substitution* is a mapping $\sigma\colon \mathrm{Form} \to \mathrm{Form}$ that commutes with connectives, hence it is uniquely determined by its values on atoms. (We reserve the term "substitution" for parameter-preserving substitutions, see below.) A (*propositional*) *logic* L is an atomic single-conclusion consequence relation: a binary relation between finite sets of formulas and formulas (written in infix notation as $\Gamma \vdash_L \varphi$), satisfying

(i) (identity) $\varphi \vdash_L \varphi$,

(ii) (weakening) $\Gamma \vdash_L \varphi$ implies $\Gamma, \Delta \vdash_L \varphi$,

(iii) (cut) $\Gamma \vdash_L \varphi$ and $\Gamma, \varphi \vdash_L \psi$ implies $\Gamma \vdash_L \psi$,

(iv) (substitution) $\Gamma \vdash_L \varphi$ implies $\sigma(\Gamma) \vdash_L \sigma(\varphi)$ for every atomic substitution σ.

Here we employ common conventions for sets of formulas: Γ, Δ denotes $\Gamma \cup \Delta$, φ can stand for $\{\varphi\}$, and $\sigma(\Gamma) = \{\sigma(\varphi) : \varphi \in \Gamma\}$. We also write $\vdash_L \varphi$ instead of $\varnothing \vdash_L \varphi$; such formulas φ are called *L-tautologies*. A logic is *inconsistent* if all formulas are tautologies, otherwise it is *consistent*. Note that our consequence relations are by definition finitary, or more precisely, they are finitary fragments of consequence relations under a more conventional definition; we consider our choice more convenient for the purpose of investigation of admissible rules and unification, as these only concern the finitary fragment of a given logic.

Being a binary relation, a logic is a set of pairs $\langle \Gamma, \varphi \rangle$, where Γ is a finite set of formulas, and φ a formula. Such pairs are called *single-conclusion rules*, and we write them as $\Gamma \,/\, \varphi$. In this context, a formula φ can be identified with the rule $\varnothing \,/\, \varphi$ (an *axiom*). If L is a logic and X a set of single-conclusion rules (or formulas), $L \oplus X$ denotes the smallest logic including L and X. (We reserve $+$ for parametric consequence relations, see below.)

In this paper, we will mostly work with normal modal logics. The basic modal language is generated by the connectives $\to, \bot, \Box$; other common connectives $(\Diamond, \wedge, \vee, \neg, \leftrightarrow, \top)$ are defined as abbreviations in the usual way. We also put $\boxdot\varphi = \varphi \wedge \Box\varphi$, $\diamonddot\varphi = \varphi \vee \Diamond\varphi$. **K4** is the smallest logic in the basic modal language that includes classical propositional tautologies, and the axioms and rules

$$
\begin{gathered}
\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi), \\
\Box\varphi \to \Box\Box\varphi, \\
\varphi, \varphi \to \psi \,/\, \psi, \\
\varphi \,/\, \Box\varphi.
\end{gathered}
$$

A *transitive modal logic* is an axiomatic extension of **K4**, i.e., a logic of the form $\mathbf{K4} \oplus X$, where X is a set of formulas. (Under our definition, normal modal logics are identified with their global consequence relations, whereas in most modal literature they are identified with their sets of tautologies. Nevertheless, we will abuse the notation and write $L \supseteq \mathbf{K4}$ as a short-hand for "L is a transitive modal logic". Local consequence relations or non-normal modal logics do not appear in this paper, hence our usage of $\oplus$ agrees with its standard meaning.)

The set of all transitive modal logics ordered by inclusion is a complete lattice, denoted $\mathrm{NExt}\,\mathbf{K4}$. The meet of a family of logics is just their intersection, and we will write it as such. We will write joins with $\bigvee$, though in the case of binary joins we also have $L_0 \vee L_1 = L_0 \oplus L_1$.

A (*transitive*) *Kripke frame* is a pair $\langle W, < \rangle$, where $<$ is a transitive binary relation on a (possibly empty) set W. We will generally use the same symbol to denote both the frame and its underlying set. We write $u \le v$ for $u < v \vee u = v$, $u \sim v$ for $u \le v \le u$, and $u \lneqq v$ for $u < v \wedge v \not< u$. A point $u \in W$ is *reflexive* if $u < u$, and *irreflexive* otherwise. If $X \subseteq W$, we define $X{\uparrow} = \{v \in W : \exists u \in X\, u < v\}$, $X{\Uparrow} = \{v \in W : \exists u \in X\, u \le v\}$. The *cluster* of $u \in W$ is $\mathrm{cl}(u) := \{v \in W : u \sim v\}$. If $u \in W$, then W_u is the frame $\langle u{\Uparrow}, < \rangle$. If $W = W_u$ for some $u \in W$, W is called a *rooted frame*, any such u is its *root*, and $\mathrm{rcl}(W) := \mathrm{cl}(u)$ its *root cluster*. If $X \subseteq W$, an $x \in X$ is called a *maximal* (or *$<$-maximal*) point of X, if $v \notin X$

for every $v \gtrsim u$. A cluster is *final* if its points are maximal in W, and *inner* otherwise. An $X \subseteq W$ is an *antichain* if $u \not< v$ for any $u, v \in X$, $u \neq v$.

A (*Kripke*) *model* is a triple $\langle W, <, \vDash \rangle$, where $\langle W, < \rangle$ is a Kripke frame, and the *valuation* $\vDash$ is a relation between elements of W and formulas, satisfying the usual conditions for compound formulas. Again, we often use the same symbol for a model and its underlying set, and we write $W, u \vDash \varphi$ instead of $u \vDash \varphi$ when we need to stress which model the $\vDash$ belongs to. We write $W \vDash \varphi$ if $W, u \vDash \varphi$ for every $u \in W$.

If $L \supseteq \mathbf{K4}$, a *Kripke L-frame* is a Kripke frame $\langle W, < \rangle$ such that $W \vDash \varphi$ for every model $\langle W, <, \vDash \rangle$ and every L-tautology φ. An *L-model* is a model $\langle W, <, \vDash \rangle$ such that $\langle W, < \rangle$ is an L-frame. Mod_L denotes the set of all finite rooted L-models. For a formula φ, we put $\mathrm{Mod}_L(\varphi) := \{W \in \mathrm{Mod}_L : W \vDash \varphi\}$. Notice that $\mathrm{Mod}_L(\varphi) = \mathrm{Mod}_L(\boxdot\varphi)$. L has the *finite model property* (*fmp*) if $\mathrm{Mod}_L(\varphi) = \mathrm{Mod}_L$ implies $\vdash_L \varphi$ for every formula φ.

If W is a model and σ a substitution, we define $\sigma(W)$ to be the model based on the same frame such that $\sigma(W), u \vDash \varphi$ iff $W, u \vDash \sigma(\varphi)$ for every formula φ and $u \in W$. Notice that $(\sigma \circ \tau)(W) = \tau(\sigma(W))$.

Let W be a finite model. The *depth* of W is the maximal length of a chain $x_1 \lnsim x_2 \lnsim \cdots \lnsim x_n$ in W. The *branching* of W is the maximal number of immediate successor clusters of any node $u \in W$. The *width* of W is the maximal size of an antichain in any rooted subframe of W.

For any formula φ, we put $\varphi^1 = \varphi$, $\varphi^0 = \neg\varphi$. If Γ is a set of formulas, $\mathbf{2}^\Gamma$ denotes the set of all assignments $e \colon \Gamma \to \mathbf{2}$, where $\mathbf{2} := \{0, 1\}$. If Γ is finite and $e \in \mathbf{2}^\Gamma$, we put $\Gamma^e := \bigwedge_{\varphi \in \Gamma} \varphi^{e(\varphi)}$. (Here and elsewhere, the empty conjunction is defined as $\top$, and the empty disjunction as $\bot$.) Conversely, if W is a Kripke model and $u \in W$, $\mathrm{Sat}_\Gamma(W, u)$ (shortened to $\mathrm{Sat}_\Gamma(u)$ if W is understood from the context) denotes the assignment $e \in \mathbf{2}^\Gamma$ such that $e(\varphi) = 1$ iff $W, u \vDash \varphi$. If W is a model and φ a formula, $W \upharpoonright \varphi$ denotes $\{u \in W : W, u \vDash \varphi\}$.

A *general frame* is $\langle W, <, A \rangle$, where $\langle W, < \rangle$ is a Kripke frame, and $A \subseteq \mathcal{P}(W)$ is a Boolean algebra of sets closed under the operation $\Box X := \{u \in W : \forall v\, (u < v \Rightarrow v \in X)\}$, or equivalently, under $\Diamond X := \{u \in W : \exists v \in X\, u < v\}$. Sets from A are called *admissible* (or *definable*), and their arbitrary intersections are *closed sets*. A Kripke frame $\langle W, < \rangle$ can be identified with the general frame $\langle W, <, \mathcal{P}(W) \rangle$. We will sometimes write just *frame* instead of general frame, however finite frames are always assumed to be Kripke frames. An *admissible valuation* in $\langle W, <, A \rangle$ is a valuation $\vDash$ in $\langle W, < \rangle$ satisfying $W \upharpoonright \varphi \in A$ for every φ.

If κ is a cardinal number, a general frame $\langle W, <, A \rangle$ is *κ-generated* if A is generated as a modal algebra by a subset of size at most κ. (Note that this notion is unrelated to the similarly named generated subframes.)

A general frame $\langle W, <, A \rangle$ is *refined* if for every $u, v \in W$,

$$\forall X \in A\, (u \in \Box X \Rightarrow v \in X) \Rightarrow u < v,$$
$$\forall X \in A\, (u \in X \Rightarrow v \in X) \Rightarrow u = v.$$

(In other words, all sets of the form $\{u\}$ or $u{\uparrow}$ are closed.) A family of sets has the *finite intersection property* (*fip*) if any its finite subfamily has a nonempty intersection. A frame is *compact* if every family of admissible (or closed) sets with fip has a nonempty intersection.

Compact refined frames are called *descriptive*.

If L is a transitive modal logic, a (*general*) *L-frame* is a frame $\langle W, <, A\rangle$ such that $W \vDash \varphi$ for every admissible valuation $\vDash$ and L-tautology φ. Every L is complete with respect to descriptive L-frames.

We will use the following well-known property:

Lemma 2.1 *If $\langle W, <, A\rangle$ is a descriptive frame, $X \subseteq W$ is closed, and $u \in X$, then there exists a $<$-maximal $v \in X$ such that $u \leq v$.*

Proof: If $C \subseteq X$ is a nonempty chain, the set $S = \{X\} \cup \{v{\uparrow\!\!\!\!\!\_} : v \in C\}$ has fip. Since each $v{\uparrow\!\!\!\!\!\_}$ is closed, S has a nonempty intersection, and any $w \in \bigcap S$ is an element of X majorizing C. Thus, $\langle X, \leq\rangle$ satisfies the assumptions of Zorn's lemma, and the result follows. □

Let $\langle W, <, A\rangle$ and $\langle V, \prec, B\rangle$ be general frames. V is a *generated subframe* of W if $V \subseteq W$, $\prec\, = \,<\cap\,(V \times W)$ (which implies $V{\uparrow} \subseteq V$), and $B = \{X \cap V : X \in A\}$. A *p-morphism* from W to V is a mapping $f\colon W \to V$ such that

- $f^{-1}(X) \in A$,
- $f(u) \prec v$ iff there is $u' > u$ such that $f(u') = v$,

for every $u \in W$, $v \in V$, and $X \in B$. The *disjoint union* of frames $\langle W_i, <_i, A_i\rangle$, $i \in I$, is the frame $\langle W, <, A\rangle$ whose underlying set W is the disjoint union $\dot{\bigcup}_{i\in I} W_i$, $< \,= \bigcup_{i\in I} <_i$, and $A = \{X \subseteq W : \forall i \in I\,(X \cap W_i \in A_i)\}$. Generated submodels, and p-morphisms and disjoint unions of models, are defined similarly.

We will usually index sequences of formulas, frames, points, and other objects by nonnegative integers, whose set is denoted ω. In particular, if $n \in \omega$, then $i < n$ (without further qualification such as $1 \leq i < n$) means $i = 0, \ldots, n-1$.

2.1 Parametric consequence relations

As already mentioned, we consider atoms of two kinds: variables and parameters (in unification literature, the latter are usually called constants). The set of all variables is denoted Var, and we assume it is countably infinite. We can enumerate $\mathrm{Var} = \{x_n : n \in \omega\}$, but for ease of reading we will also use letters $x, y, z, \ldots$ for variables. The set of all parameters is denoted Par, and we will use letters such as $p, q, r, \ldots$ for parameters. We assume that Par is at most countable, but we allow it to be infinite or finite (or even empty, so that our results subsume the parameter-free case). If $P \subseteq \mathrm{Par}$ and $V \subseteq \mathrm{Var}$, $\mathrm{Form}(P, V)$ denotes the set of modal formulas in parameters P and variables V.

A *substitution* is an atomic substitution σ such that $\sigma(p) = p$ for every parameter p. A *single-conclusion consequence relation* is a relation between finite sets of formulas and formulas (or equivalently, a set of single-conclusion rules) which satisfies conditions (i)–(iii) from the definition of a logic, as well as

(iv') $\Gamma \vdash \varphi$ implies $\sigma(\Gamma) \vdash \sigma(\varphi)$ for every substitution σ.

More generally, a *multiple-conclusion consequence relation* (or just *consequence relation* for short) is a binary relation between finite sets of formulas, satisfying

(i) (identity) $\varphi \vdash \varphi$,

(ii) (weakening) $\Gamma \vdash \Delta$ implies $\Gamma, \Gamma' \vdash \Delta, \Delta'$,

(iii) (cut) $\Gamma \vdash \varphi, \Delta$ and $\Gamma, \varphi \vdash \Delta$ implies $\Gamma \vdash \Delta$,

(iv) (substitution) $\Gamma \vdash \Delta$ implies $\sigma(\Gamma) \vdash \sigma(\Delta)$ for every substitution σ.

Every consequence relation satisfies the following more general form of the cut property for any finite set of formulas Θ:

$$\text{If } \Gamma, \Pi \vdash \Lambda, \Delta \text{ for every partition } \Pi \mathbin{\dot\cup} \Lambda = \Theta, \text{ then } \Gamma \vdash \Delta. \tag{1}$$

(For non-finitary consequence relations, (1) for arbitrary sets Θ needs to be taken as part of the definition, see [25].)

A consequence relation is thus a set of pairs of finite sets of formulas. We will call such pairs *multiple-conclusion rules*, or just *rules*, and we will write them as $\Gamma \mathbin{/} \Delta$.

For every consequence relation $\vdash$, the set of single-conclusion rules $\Gamma \mathbin{/} \varphi$ such that $\Gamma \vdash \varphi$ is a single-conclusion consequence relation, the *single-conclusion fragment of* $\vdash$. Conversely, for every single-conclusion consequence relation $\vdash_1$, there is a smallest consequence relation $\vdash_m$ whose single-conclusion fragment is $\vdash_1$, namely $\Gamma \vdash_m \Delta$ iff $\Gamma \vdash_1 \varphi$ for some $\varphi \in \Delta$. In particular, if L is a logic, a rule is *L-derivable* if it belongs to the smallest consequence relation extending L (which we identify with L itself).

If L is a consequence relation and X a set of rules, $L+X$ denotes the smallest consequence relation containing L and X.

Let L be a logic. An *L-unifier* of a set of formulas Γ is a substitution σ such that $\vdash_L \sigma(\varphi)$ for every $\varphi \in \Gamma$. A rule $\Gamma \mathbin{/} \Delta$ is *L-admissible* if every unifier of Γ also unifies some $\varphi \in \Delta$. The set of all L-admissible rules forms a consequence relation which we denote $\mathrel{|\!\!\sim}_L$. A *basis* of L-admissible rules is a set of rules B such that $\mathrel{|\!\!\sim}_L = L + B$.

A logic L is (*finitely*) *equivalential* if there is a finite set of formulas $E(x, y)$ such that

$$\begin{aligned} \vdash_L \varepsilon(x, x) &\quad \text{for all } \varepsilon \in E, \\ E(x, y), \varphi(x) \vdash_L \varphi(y) & \end{aligned}$$

for every formula φ, possibly involving other variables not shown. Modal logics are equivalential with $E(x, y) = \{x \leftrightarrow y\}$. Substitutions σ, τ are *equivalent*, written $\sigma =_L \tau$, if $\vdash_L \varepsilon(\sigma(x), \tau(x))$ for every variable x. τ is *more general* than σ, written $\sigma \leq_L \tau$, if $\sigma =_L \upsilon \circ \tau$ for some substitution υ. A *complete set of unifiers* of a set of formulas Γ is a set C of unifiers of Γ such that every unifier of Γ is less general than some $\sigma \in C$. A complete set of unifiers is *minimal* if no $C' \subsetneq C$ is complete, or equivalently, if C consists of pairwise incomparable $\leq_L$-maximal unifiers. A *most general unifier* (*mgu*) of Γ is a unifier of Γ more general than any other unifier of Γ. If Γ has a minimal complete set of unifiers C, its cardinality is an invariant of Γ. We say that Γ is of

- type 1 (unary), if $|C| = 1$ (i.e., Γ has a mgu),
- type ω (finitary), if $1 < |C| < \aleph_0$,
- type ∞ (infinitary), if C is infinite,
- type 0 (nullary), if Γ has no minimal complete set of unifiers.

The *unification type* of L is the maximal type of a unifiable set of formulas Γ, where the types are ordered as $1 < \omega < \infty < 0$. L has *at most finitary* unification, if its unification type is unary or finitary.

A *parametric Kripke frame* is $\langle W, <, \vDash_p \rangle$, where $\langle W, < \rangle$ is a Kripke frame, and $\vDash_p \subseteq W \times \mathrm{Par}$. Similarly, a *parametric (general) frame* is $\langle W, <, A, \vDash_p \rangle$, where $\langle W, <, A \rangle$ is a general frame, and $\vDash_p \subseteq W \times \mathrm{Par}$ satisfies $\{u \in W : u \vDash_p p\} \in A$ for every $p \in P$. An *admissible valuation* in $\langle W, <, A, \vDash_p \rangle$ is an admissible valuation $\vDash$ in $\langle W, <, A \rangle$ such that $\vDash \supseteq \vDash_p$.

A rule $\varrho = \Gamma \mathrel{/} \Delta$ is *satisfied* in a model $\langle W, <, \vDash \rangle$ if $W \nvDash \varphi$ for some $\varphi \in \Gamma$, or $W \vDash \varphi$ for some $\varphi \in \Delta$. A rule ϱ is *valid* in a parametric frame W, written $W \vDash \varrho$, if ϱ is satisfied in any model based on an admissible valuation in W.

Generated subframes and *disjoint unions* of parametric frames are defined in the obvious way. *P-morphisms* of parametric frames are required to preserve the valuation of parameters in both directions. Validity of rules in parametric frames is preserved by p-morphic images. Single-conclusion rules are also preserved under disjoint unions, and premise-free rules under generated subframes.

Let $L \supseteq \mathbf{K4}$, $P \subseteq \mathrm{Par}$, $V \subseteq \mathrm{Var}$. The *canonical frame* $C_L(P, V)$ is the descriptive parametric frame $\langle C, <, A, \vDash_p \rangle$, where C is the collection of maximal L-consistent subsets of $\mathrm{Form}(P, V)$, $X < Y$ iff $\{\varphi : \Box\varphi \in X\} \subseteq Y$ iff $\{\Diamond\varphi : \varphi \in X\} \subseteq Y$, A consists of sets of the form $\{X \in C : \varphi \in X\}$, where $\varphi \in \mathrm{Form}(P, V)$, and for $p \in \mathrm{Par}$, $X \vDash_p p$ iff $p \in X$.

We have $C_L(P, V) \vDash L$. On the other hand, if $\nvdash_L \varphi$, where $\varphi \in \mathrm{Form}(P, V')$ and $|V'| \le |V|$, then $C_L(P, V) \nvDash \varphi$. The following standard lemma follows easily.

Lemma 2.2 *Let $\Gamma \mathrel{/} \Delta$ be a rule whose parameters are included in P.*

(*i*) *If $\Gamma \vdash_L \Delta$, then $C_L(P, V) \vDash \Gamma \mathrel{/} \Delta$ for every $V \subseteq \mathrm{Var}$.*

(*ii*) *If $\Gamma \nvdash_L \Delta$, there is $n \in \omega$ such that $C_L(P, V) \nvDash \Gamma \mathrel{/} \Delta$ whenever $|V| \ge n$.*

If $\mathcal{W}$ is a class of parametric frames, the set of all rules valid in $\mathcal{W}$ is easily seen to be a consequence relation extending $\vdash_{\mathbf{K4}}$. On the other hand, every consequence relation $\vdash \supseteq \vdash_{\mathbf{K4}}$ is complete wrt a class of (finitely generated) descriptive frames. In particular, if $\Gamma \nvdash \Delta$, where $\Gamma \cup \Delta \subseteq \mathrm{Form}(P, V)$, then (1) and Zorn's lemma imply that there exists a partition $\Gamma' \dot\cup \Delta' = \mathrm{Form}(P, V)$ such that $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, and $\Gamma'' \nvdash \Delta''$ for any finite $\Gamma'' \subseteq \Gamma'$, $\Delta'' \subseteq \Delta'$. Then $W := \{X \in C_{\mathbf{K4}}(P, V) : \Gamma' \subseteq X\}$ is a closed (hence descriptive) generated subframe of $C_{\mathbf{K4}}(P, V)$, and one checks easily that $W \models \vdash$ and $W \nvDash \Gamma \mathrel{/} \Delta$ (see e.g. [18, Thm. 2.2] for the parameter-free case).

On a related note, descriptive parametric frames can be embedded in canonical frames. The lemma below holds for arbitrary cardinals κ if we allow uncountable sets of variables and parameters, but we will only need it for finite (hence finitely generated) frames.

Lemma 2.3 *Let $L \supseteq \mathbf{K4}$, $P \subseteq \mathrm{Par}$, and W be a parametric κ-generated descriptive L-frame. If V is a set of variables such that $|V| \geq \kappa$, there is a general frame isomorphism from W onto a closed generated subframe of $C_L(P,V)$, preserving the valuation of parameters from P.*

Proof: Let F be the free L-algebra generated by $P \cup V$, and h a homomorphism from F to the algebra A of admissible sets of W, mapping V onto a set of generators of A, and each $p \in P$ to the element of A given by the valuation of p in W. Since h is onto, the dual p-morphism from W to $C_L(P,V)$ is injective, and it has the required properties. $\Box$

A *nonstructural consequence relation* is a binary relation between finite sets of formulas satisfying conditions (i), (ii), (iii) from the definition of multiple-conclusion consequence relations. We will not refer to nonstructural consequence relations directly, but we will extend the $\vdash$ notation to rules as follows. Let $\vdash$ be a (structural) consequence relation, and $R \cup \{\varrho\}$ a set of rules. We write $R \vdash \varrho$ if ϱ is in the least nonstructural consequence relation containing $\vdash$ and R.

Note that if the rules in R and ϱ are just axioms, $R \vdash \varrho$ iff the same holds for the corresponding formulas under the original consequence relation $\vdash$, thus this overloading of the symbol $\vdash$ does not lead to conflicts. Also, if $\varrho = \Gamma \,/\, \Delta$, then $\varnothing \vdash \varrho$ iff $\Gamma \vdash \Delta$.

($\vdash$ defines a sort of a single-conclusion consequence relation operating with multiple-conclusion rules instead of formulas, but we will not use this terminology in order to avoid unnecessary confusion.)

If $\vdash$ extends $\vdash_{\mathbf{K4}}$, and W is a frame validating $\vdash$, then $R \vdash \varrho$ implies that every admissible valuation $\vDash$ in W that satisfies all rules from R also satisfies ϱ. One can in fact show easily that $\vdash$ is complete with respect to this semantics, but we will not need this. Rather, we will use the following lemma which follows from [14, L. 2.3, 2.4], but we include a direct proof for completeness.

Lemma 2.4 *Let $L \supseteq \mathbf{K4}$ have fmp, and $R \cup \{\varrho\}$ be a finite set of rules. If $R \nvdash_L \varrho$, there is a finite L-model W such that $W \vDash R$, and $W \nvDash \varrho$.*

Proof: Write $\varrho = \Gamma \,/\, \Delta$, and let Σ be the set of all formulas occurring in $R \cup \{\varrho\}$. By (1), there is a partition $\Sigma = X \mathbin{\dot\cup} Y$ such that $\Gamma \subseteq X$, $\Delta \subseteq Y$, and $R \nvdash_L X \,/\, Y$. In particular, if $\Gamma' \,/\, \Delta' \in R$, then $\Gamma' \cap Y \neq \varnothing$ or $\Delta' \cap X \neq \varnothing$, and for every $\psi \in Y$, $\nvdash_L \boxdot \bigwedge X \to \psi$. The latter implies that there are models $W_\psi \in \mathrm{Mod}_L$ with roots u_ψ such that $W_\psi \vDash X$, and $W_\psi, u_\psi \nvDash \psi$. Let W be the disjoint union of all W_ψ, $\psi \in Y$. Then $W \vDash \varphi$ for every $\varphi \in X$, and $W \nvDash \psi$ for every $\psi \in Y$. In particular, $W \vDash \varrho'$ for every $\varrho' \in R$, and $W \nvDash \varrho$. $\Box$

3 Projective formulas

Let us fix a logic $L \supseteq \mathbf{K4}$ with the finite model property. Recall that a formula φ is *projective* if it has a *projective unifier*, which is an L-unifier σ of φ such that

$$\varphi \vdash_L \sigma(x) \leftrightarrow x \tag{2}$$

for every variable x (which implies $\varphi \vdash_L \sigma(\psi) \leftrightarrow \psi$ for every formula ψ). A projective unifier of a formula is also its most general unifier.

Ghilardi [8] described projective formulas in the parameter-free case: they are exactly the formulas whose finite L-models have a certain extension property, and moreover, one can explicitly define for any formula a substitution satisfying (2) which turns out to be a projective unifier whenever the formula is projective. The goal of this section is to generalize this result to projectivity with parameters. Let us start by defining the relevant extension properties and substitutions.

Definition 3.1 Models $F, F' \in \mathrm{Mod}_L$ are *variants* of each other if they are based on the same frame, have the same valuation of parameters, and their valuation of variables can only differ in $\mathrm{rcl}(F)$. A set of models $M \subseteq \mathrm{Mod}_L$ has the *model extension property,* if every $F \in \mathrm{Mod}_L$ whose all rooted generated proper submodels belong to M has a variant in M. A formula φ has the model extension property if this holds for $M = \mathrm{Mod}_L(\varphi)$.

Let $\varphi \in \mathrm{Form}(P, V)$, where P and V are *finite* sets of parameters and variables, respectively. Let $D = \langle d_x : x \in V\rangle$, where each $d_x \colon \mathbf{2}^P \to \mathbf{2}$ is a Boolean function of the parameters. We define the *Löwenheim substitution* $\theta_{\varphi,D}$ by

$$\theta_{\varphi,D}(x) = (\boxdot\varphi \wedge x) \vee (\neg\boxdot\varphi \wedge d_x)$$

for every $x \in V$, where d_x is identified with any Boolean formula representing it. Notice that sequences D as above can be equivalently represented as assignments $D \colon \mathbf{2}^P \times V \to \mathbf{2}$ or $D \colon \mathbf{2}^P \to \mathbf{2}^V$. Let θ_φ be the composition of all $2^{|V|2^{|P|}}$ substitutions of the form $\theta_{\varphi,D}$, in an arbitrary order. We will also write $\theta_D = \theta_{\varphi,D}$ and $\theta = \theta_\varphi$ when φ is clear from the context.

Notice that in the case $P = \varnothing$, D can be identified with a variable assignment $D \colon V \to \mathbf{2}$, and $\theta_{\varphi,D}$ is equivalent to the substitution

$$\theta_{\varphi,D}(x) = \begin{cases} \boxdot\varphi \to x, & \text{if } D(x) = 1, \\ \boxdot\varphi \wedge x, & \text{if } D(x) = 0 \end{cases}$$

considered by Ghilardi.

It is easy to see that substitutions satisfying (2) are closed under composition, hence θ_φ^N satisfies (2) for any $N \in \omega$.

The rest of this section is devoted to the proof of the following characterization.

Theorem 3.2 *Let $L \supseteq$ **K4** have the finite model property, and φ be a formula in finitely many parameters P and variables V. Then the following are equivalent.*

(i) φ is projective.

(ii) φ has the model extension property.

(iii) θ_φ^N is a unifier of φ, where $N = (|B| + 1)\big(2^{|P|} + 1\big)$, $B = \{\psi : \Box\psi \subseteq \varphi\}$.

Remark 3.3 In the parameter-free case, we obtain $N = 2(|B| + 1) \leq 2|\varphi|$. This is a considerable improvement over Ghilardi's original proof, which gives N nonelementary (a tower of exponentials whose height is the modal degree of φ).

For the next few lemmas, let us fix finite sets of parameters P and variables V, a formula $\varphi \in \mathrm{Form}(P, V)$ with the model extension property, and $B = \{\psi : \Box\psi \subseteq \varphi\}$. We aim to show that $\vdash_L \theta^N(\varphi)$, which in view of the fmp of L amounts to $\theta^N(\varphi)$ being true in every finite rooted L-model F.

The basic idea behind the θ_D substitutions is that their successive application leaves unchanged the part of F where φ already holds, while we are making progress on the rest of the model: specifically, a maximal cluster where φ fails can be made to satisfy φ by applying θ_D for a suitably chosen D.

Lemma 3.4 *Let $F \in \mathrm{Mod}_L$ and $D \colon \mathbf{2}^P \to \mathbf{2}^V$.*

(i) If $F \vDash \varphi$, then $\theta_D(F) = F$.

(ii) If $F, u \nvDash \boxdot\varphi$, then $\mathrm{Sat}_V(\theta_D(F), u) = D(\mathrm{Sat}_P(F, u))$.

(iii) If $F' \vDash \varphi$, where F' is the variant of F such that $\mathrm{Sat}_V(F', u) = D(\mathrm{Sat}_P(F, u))$ for every $u \in \mathrm{rcl}(F)$, then $\theta_D(F) \vDash \varphi$.

In particular, since φ has the extension property, we have $\theta(F) \vDash \varphi$ whenever $F \smallsetminus \mathrm{rcl}(F) \vDash \varphi$.

Proof: (i) and (ii) are clear from the definition of θ_D. (iii): If $F \vDash \varphi$, the result follows from (i). Otherwise $F \smallsetminus \mathrm{rcl}(F) \vDash \varphi$, and $\theta_D(F) = F'$ by (i) and (ii). $\Box$

Lemma 3.4 implies that $\theta^k(F) \vDash \varphi$ for any $F \in \mathrm{Mod}_L$ of depth at most k. However, in order to show that some power of θ is a unifier of φ, we need a uniform bound on k independent of F. As in Ghilardi's proof, we will achieve this by defining a rank function on models whose number of possible values depends only on φ, and showing that sufficiently many applications of θ will strictly decrease the rank or make the model satisfy φ.

Ghilardi's rank is based on Fine's n-equivalence [5]. It seems that matters becomes more delicate when we need to deal with valuation of parameters, hence we need a notion of a rank better adapted to our particular situation in order to make the arguments go through. We will use a rank function based on the satisfaction of some formulas related to φ. As a side effect, this leads to much smaller bounds, as already mentioned in Remark 3.3. We will also find it helpful to consider ranks to be the actual sets of formulas rather than just their cardinality.

Definition 3.5 If $F \in \mathrm{Mod}_L$, we put

$$R_0(F) = \{\psi \in B : F \vDash \Box(\boxdot\varphi \to \psi)\},$$
$$R_1(F) = \Big\{e \in \mathbf{2}^P : F \vDash P^e \to \bigvee_{\psi \in B \smallsetminus R_0(F)} \Box(\boxdot\varphi \to \psi)\Big\}.$$

Notice that $R_1(F)$ is a proper subset of $\mathbf{2}^P$, specifically $\mathrm{Sat}_P(F, u) \notin R_1(F)$ for any $u \in \mathrm{rcl}(F)$. The *crude rank* of F is $R_0(F)$, and its *rank* is $R(F) := \langle R_0(F), R_1(F)\rangle$. The rank of a point $u \in F$ is defined as $R(F_u)$. Ranks are ordered lexicographically: if $X, X' \subseteq B$ and $Y, Y' \subsetneq \mathbf{2}^P$, we put

$$\begin{aligned}
\langle X, Y\rangle \subseteq_{\mathrm{Lex}} \langle X', Y'\rangle \quad &\text{iff} \quad X \subsetneq X' \vee (X = X' \wedge Y \subseteq Y'),\\
\langle X, Y\rangle \subsetneq_{\mathrm{Lex}} \langle X', Y'\rangle \quad &\text{iff} \quad X \subsetneq X' \vee (X = X' \wedge Y \subsetneq Y').
\end{aligned}$$

The *numerical rank* of F is $\|R(F)\|$, where $\|\langle X, Y\rangle\| := 2^{|P|}|X| + |Y|$. Notice that $\langle X, Y\rangle \subsetneq_{\mathrm{Lex}} \langle X', Y'\rangle$ implies $\|\langle X, Y\rangle\| < \|\langle X', Y'\rangle\|$.

Lemma 3.6

(*i*) *If $u, v \in F$, $u < v$, then $R(F_u) \subseteq_{\mathrm{Lex}} R(F_v)$.*

(*ii*) *$R(\theta_D(F)) \subseteq_{\mathrm{Lex}} R(F)$.*

Proof: (i) is obvious from the definition. (ii) follows from Lemma 3.4: when passing from F to $\theta_D(F)$, the set of points satisfying $\boxdot\varphi$ can only increase, and the valuation of all formulas $\psi \in B$ in $F \restriction \boxdot\varphi$ is preserved. $\square$

The argument for decreasing rank will be different depending on whether maximal clusters where φ fails are reflexive or irreflexive. We treat the reflexive case first.

Lemma 3.7 *Let $F \in \mathrm{Mod}_L$ be such that all points of $F \restriction \neg\boxdot\varphi$ have the same rank, $R(F) = R(\theta(F))$, and $F \restriction \neg\boxdot\varphi$ has a reflexive $<$-maximal cluster. Then $\theta(F) \vDash \varphi$.*

Proof: Put $R = \langle R_0, R_1\rangle := R(F)$, $G := F \restriction \boxdot\varphi$. If $u \in F \smallsetminus G$ and $\theta = \sigma \circ \tau$, where σ, τ are compositions of θ_D's, we have

$$R = R(\theta(F)) \subseteq_{\mathrm{Lex}} R(\theta(F)_u) = R(\tau(\sigma(F))_u) \subseteq_{\mathrm{Lex}} R(\sigma(F)_u) \subseteq_{\mathrm{Lex}} R(F_u) = R \tag{3}$$

by Lemma 3.6, hence $R(\sigma(F)_u) = R$.

Fix w in a reflexive maximal cluster of $F \smallsetminus G$. We define $D \colon \mathbf{2}^P \to \mathbf{2}^V$ as follows. Let $e \in \mathbf{2}^P$. If $e \in R_1$, we pick an arbitrary $D(e) \in \mathbf{2}^V$. Otherwise $e \notin R_1(\theta(F)_w)$, hence there exists $w_e \geq w$ such that

$$\theta(F), w_e \vDash P^e \wedge \bigwedge_{\psi \in B \smallsetminus R_0} \Diamond(\boxdot\varphi \wedge \neg\psi). \tag{4}$$

We define $D(e) = \mathrm{Sat}_V(\theta(F), w_e)$. Notice that

$$\theta(F), w_e \vDash \boxdot\varphi \wedge P^e \wedge V^{D(e)} \wedge \bigwedge_{\psi \in R_0} \boxdot\psi \wedge \bigwedge_{\psi \in B \smallsetminus R_0} \neg\boxdot\psi : \tag{5}$$

we have $\theta(F), w_e \vDash \boxdot\varphi$ by Lemma 3.4; if $\psi \in R_0$, then $\theta(F)_w \vDash \boxdot(\boxdot\varphi \to \boxdot\psi)$ as $R_0(\theta(F)_w) = R_0$ and w is reflexive; and if $\psi \in B \smallsetminus R_0$, then $\theta(F), w_e \nvDash \boxdot\psi$ by (4).

We can write $\theta = \sigma \circ \theta_D \circ \tau$. We claim $\theta_D(\sigma(F)) \vDash \varphi$, which implies $\theta(F) = \tau(\theta_D(\sigma(F))) \vDash \varphi$. Assume for contradiction $\theta_D(\sigma(F)), u \nvDash \varphi$; we may also assume without loss of generality that $\theta_D(\sigma(F)), v \vDash \varphi$ for every $v \gnsim u$. Since $\sigma(F), u \nvDash \boxdot\varphi$ and $R_0(\theta_D(\sigma(F))_u) = R_0$, we have

$$\theta_D(\sigma(F)), u' \vDash P^{\mathrm{Sat}_P(u')} \wedge V^{D(\mathrm{Sat}_P(u'))} \wedge \bigwedge_{\psi \in R_0} \Box(\boxdot\varphi \to \psi) \wedge \bigwedge_{\psi \in B \smallsetminus R_0} \neg\boxdot\psi \tag{6}$$

for every $u' \sim u$ using Lemma 3.4. Notice that $\mathrm{Sat}_P(u') \notin R_1$. We will show

$$\theta_D(\sigma(F)), u' \vDash \chi \quad \text{iff} \quad \theta(F), w_{\mathrm{Sat}_P(u')} \vDash \chi \tag{7}$$

for every $u' \sim u$ and $\chi \subseteq \varphi$ by induction on the complexity of χ. If $\chi \in P \cup V$, or $\chi = \Box\psi$ with $\psi \notin R_0$, then (7) follows immediately from (5) and (6). The steps for Boolean connectives are trivial. If $\chi = \Box\psi$, $\psi \in R_0$, we have $\theta(F), w_{\mathrm{Sat}_P(u')} \vDash \Box\psi$ by (5). On the other hand, $\theta_D(\sigma(F)), v \vDash \psi$ for every $v \gtrsim u$, $v \not\sim u$ by (6), and for every $v \sim u$ by the induction hypothesis, since $\theta(F), w_{\mathrm{Sat}_P(v)} \vDash \psi$. Thus, $\theta_D(\sigma(F)), u' \vDash \Box\psi$, irrespective of the reflexivity or irreflexivity of $\mathrm{cl}(u)$.

However, (7) and (5) imply $\theta_D(\sigma(F)), u \vDash \varphi$, a contradiction. $\Box$

Definition 3.8 If $F \in \mathrm{Mod}_L$, $F \nvDash \varphi$, we define

$$\begin{aligned} r(F) &:= \max\bigl\{\|R(F_u)\| : F, u \nvDash \varphi\bigr\}, \\ r_0(F) &:= \max\bigl\{|R_0(F_u)| : F, u \nvDash \varphi\bigr\}, \\ r_1(F) &:= \max\bigl\{|R_1(F_u)| : F, u \nvDash \varphi\bigr\}. \end{aligned}$$

If $F \vDash \varphi$, we can put $r(F) = r_0(F) = r_1(F) = -\infty$.

Corollary 3.9 *If $F \in \mathrm{Mod}_L$ is such that $F \nvDash \varphi$, and all $<$-maximal clusters of $F \restriction \neg\boxdot\varphi$ are reflexive, then $r(\theta(F)) < r(F)$.*

Proof: We have $r(\theta(F)) \le r(F)$ by Lemma 3.6. If $r(\theta(F)) = r(F)$, choose $u \in F$ such that $\theta(F), u \nvDash \varphi$ and $\|R\| = r(F)$, where $R = R(\theta(F)_u)$. As in (3), we have $R(F_v) = R$ for every $v \in F_u \restriction \neg\boxdot\varphi$. Thus $\theta(F)_u \vDash \varphi$ by Lemma 3.7, a contradiction. $\Box$

Lemma 3.10 *Let $F \in \mathrm{Mod}_L$ be such that all points of $F \restriction \neg\boxdot\varphi$ have the same crude rank, $F \restriction \neg\boxdot\varphi$ has an irreflexive $<$-maximal point, and $R_0(F) = R_0\bigl(\theta^m(F)\bigr)$, where $m = r_1(F) + 2$. Then $\theta^m(F) \vDash \varphi$.*

Proof: Put $R_0 = R_0(F)$, and fix an irreflexive maximal point $w \in F \restriction \neg\boxdot\varphi$. For any $e \in \mathbf{2}^P$, we can change the valuation of parameters in the root of F_w to match e, and apply the model extension property to obtain its variant G^e such that $G^e \vDash \varphi$. Let $D(e)$ be the valuation of variables in the root of G^e. Since $R_0(F_w) = R_0$, $F, w \vDash \Box\varphi$, and valuation of boxed formulas in w is unaffected by a change of variables or parameters in w, we have

$$G^e, w \vDash \boxdot\varphi \wedge P^e \wedge V^{D(e)} \wedge \bigwedge_{\psi \in R_0} \Box\psi \wedge \bigwedge_{\psi \in B \smallsetminus R_0} \neg\Box\psi. \tag{8}$$

We can write $\theta = \sigma \circ \theta_D \circ \tau$, where σ and τ are compositions of some of the $\theta_{D'}$. Put $\eta = \tau \circ \sigma \circ \theta_D$ and $\eta_k = \sigma \circ \theta_D \circ \eta^k$ for $k = 0, \dots, m-1$, so that $\theta^m = \eta_{m-1} \circ \tau$. Notice that

$$R_0(\eta_k(F)_u) = R_0 \tag{9}$$

for any $u \in F \restriction \neg\boxdot\varphi$ and $k < m$, by the same argument as in (3).

Claim 1 *For any $k < m$, all $<$-maximal clusters of $\eta_k(F) \restriction \neg\boxdot\varphi$ are reflexive.*

Proof: Assume for contradiction that u is an irreflexive maximal point of $\eta_k(F) \restriction \neg\boxdot\varphi$. Let $e = \mathrm{Sat}_P(\eta_k(F), u)$. Since $\eta_k(F)$ is of the form $\theta_D(\cdots)$, we have $\eta_k(F), u \vDash V^{D(e)}$ by Lemma 3.4. Also $R_0(\eta_k(F)_u) = R_0$ by (9), and $\eta_k(F), u \vDash \Box\varphi$, hence

$$\eta_k(F), u \vDash P^e \wedge V^{D(e)} \wedge \bigwedge_{\psi \in R_0} \Box\psi \wedge \bigwedge_{\psi \in B \smallsetminus R_0} \neg\Box\psi.$$

By (8), $\eta_k(F), u$ and G^e, w satisfy the same Boolean combinations of atoms and boxed subformulas of φ. In particular, they agree on the satisfaction of φ itself. However, this contradicts $G^e, w \vDash \varphi$ and $\eta_k(F), u \nvDash \varphi$. $\Box$ (Claim 1)

Assume for contradiction $\theta^m(F) \nvDash \varphi$. Since η is a composition of all $\theta_{D'}$ in some order, we can apply Corollary 3.9 with η in place of θ, obtaining $r(\eta_{k+1}(F)) < r(\eta_k(F))$ for every $k < m$. In view of (9), this implies $r_1(\eta_{k+1}(F)) < r_1(\eta_k(F))$. However, this is impossible, as $r_1(\eta_0(F)) \le r_1(F) = m - 2$ and $r_1(\eta_{m-1}(F)) \ge 0$. $\Box$

Lemma 3.11 *If $F \in \mathrm{Mod}_L$ and $F \nvDash \varphi$, then $r_0\big(\theta^{2^{|P|}+1}(F)\big) < r_0(F)$.*

Proof: Put $m = 2^{|P|} + 1$. We always have $r_0(\theta^m(F)) \le r_0(F)$. Assume for contradiction $r_0(\theta^m(F)) = r_0(F)$, and choose $u \in F$ such that $\theta^m(F), u \nvDash \varphi$ and $|R_0| = r_0(F)$, where $R_0 = R_0(\theta^m(F)_u)$. As in (3), we have $R_0(\theta^k(F)_v) = R_0$ for every $v \in F_u \restriction \neg\boxdot\varphi$ and $k \le m$.

If all maximal clusters of $\theta^k(F)_u \restriction \neg\boxdot\varphi$ are reflexive for every $k < m$, Corollary 3.9 implies that $r(\theta^k(F)_u)$ is strictly decreasing, hence so is $r_1(\theta^k(F)_u)$. However, this contradicts $r_1(F) < 2^{|P|} < m$ and $r_1(\theta^m(F)) \ge 0$.

If some $\theta^k(F)_u \restriction \neg\boxdot\varphi$ has an irreflexive maximal cluster, let k be the smallest such. As above, we have $r_1(\theta^{i+1}(F)_u) < r_1(\theta^i(F)_u)$ for every $i < k$, hence $r_1(\theta^k(F)_u) \le 2^{|P|} - 1 - k = m - k - 2$. But then Lemma 3.10 gives $\theta^m(F)_u = \theta^{m-k}(\theta^k(F))_u \vDash \varphi$, a contradiction. $\Box$

Proof of Theorem 3.2:

(i) $\to$ (ii): Let σ be a projective unifier of φ, and $F \in \mathrm{Mod}_L$ be such that $F \smallsetminus \mathrm{rcl}(F) \vDash \varphi$. Since $\vdash_L \sigma(\varphi)$, we have $\sigma(F) \vDash \varphi$, and (2) implies $\sigma(F \smallsetminus \mathrm{rcl}(F)) = F \smallsetminus \mathrm{rcl}(F)$, hence $\sigma(F)$ is a variant of F.

(ii) $\to$ (iii): Assume $\nvdash_L \theta^N(\varphi)$, hence there exists $F \in \mathrm{Mod}_L$ such that $\theta^N(F) \nvDash \varphi$. Put $s(n) = r_0\big(\theta^{(2^{|P|}+1)n}(F)\big)$. We have $s(0) \le |B|$ and $s(|B| + 1) = r_0(\theta^N(F)) \ge 0$. However, s is strictly decreasing by Lemma 3.11, a contradiction.

(iii) $\to$ (i) follows from the fact that θ^N satisfies (2). $\Box$

4 Cluster-extensible logics

Most of our results on admissibility and unification with parameters will be stated for logics satisfying a suitable extensibility condition on finite frames that we introduce in this section. We will call logics satisfying the full condition *cluster-extensible*; the condition can be somewhat relaxed if the set Par of all parameters is finite, leading to the definition of Par-*extensible logics*. The primary reason our methods work for these logics is that they have finite projective approximations, and we will prove this in Section 4.1. We slightly digress in Section 4.2 to give a rather general characterization of logics with directed unification, which enables us to distinguish Par-extensible logics with unary and finitary unification type. In Section 4.3, we investigate the behaviour of cluster-extensible logics in terms of various properties commonly studied in modal logic; while these properties do not directly involve admissibility or unification, we will use them as tools later.

The Par-extensibility condition is similar to Assumption 1.2 of Ghilardi [8]. The differences are that on the one hand, we need to work with proper clusters as roots in order to accommodate parameters, on the other hand we make the condition more fine-grained by taking into consideration the number of successors of the root; this makes our results applicable to logics with bounded branching at almost no additional cost.

Definition 4.1 A *cluster type* is an isomorphism type of a finite cluster. We will denote the irreflexive cluster type by $\bullet$, and the k-point reflexive cluster type by $\textcircled{k}$. If C is a cluster type and $n \in \omega$, a finite rooted frame F is of *type* $\langle C, n\rangle$ if $\mathrm{rcl}(F)$ has type C and n immediate successor clusters.

A logic $L \supseteq \mathbf{K4}$ is $\langle C, n\rangle$-*extensible* if whenever F is a type-$\langle C, n\rangle$ frame such that $F \smallsetminus \mathrm{rcl}(F)$ is an L-frame, then F is an L-frame itself, unless $\mathrm{Par} = \varnothing$, $n = 1$, and $F \smallsetminus \mathrm{rcl}(F)$ has a reflexive root cluster. L is a Par-*extensible logic* if it has fmp, and it is $\langle C, n\rangle$-extensible for every n and C such that there exists at least one L-frame of type $\langle C, n\rangle$, and $|C| \leq 2^{|\mathrm{Par}|}$. Note that the condition $|C| \leq 2^{|\mathrm{Par}|}$ is satisfied automatically if Par is infinite. Logics Par-extensible with respect to infinite Par will also be called *cluster-extensible* (*clx*) *logics.*

The purpose of the seemingly odd exception in the case $\mathrm{Par} = \varnothing$ and $n = 1$ is to make Theorem 5.14 hold. The underlying reason is that if F' is a frame with a reflexive root, and F the frame obtained from F' by adding a new root cluster C, then the mapping that contracts C to a fixed element of $\mathrm{rcl}(F')$ is a p-morphism from F to F'. (This is no longer true for parametric frames when $\mathrm{Par} \neq \varnothing$, as C and $\mathrm{rcl}(F')$ may have incompatible valuation of parameters.)

Example 4.2 Table 1 lists some important extensions of **K4**, along with conditions characterizing their finite frames. All these logics as well as **K4** itself are cluster-extensible, as can be readily seen from their frame conditions. We will prove later (Corollary 4.39) that joins of clx logics are themselves clx, hence arbitrary combinations of logics from the table are also cluster-extensible. We will denote joins of logics by stacking axiom labels on top of a name of a base logic, so that $\mathbf{D4.3} = \mathbf{D4} \oplus \mathbf{K4.3}$, $\mathbf{S4GrzBB}_k = \mathbf{S4} \oplus \mathbf{K4Grz} \oplus \mathbf{K4BB}_k$, etc. (The logic **S4.1.4**, whose name we take from Zeman [27], is an exception: this is not a systematic name, but a meaningless numerical label.) The axiomatization of the bounded branching logics $\mathbf{K4BB}_k$ comes from [19], and it is only valid for $k > 0$. We can put $\mathbf{K4BB}_0 = \mathbf{K4B}$ to maintain the frame condition. Notice that also $\mathbf{K4.3} = \mathbf{K4BB}_1$ and $\mathbf{K4Grz} = \mathbf{K4BC}_1$, but we still prefer to call **K4B**, **K4.3**, and **K4Grz** by their more common names.

The trivial logics $\mathbf{Triv} = \mathbf{K4} \oplus x \leftrightarrow \Box x = \mathbf{S5Grz}$, $\mathbf{Verum} = \mathbf{K4} \oplus \Box\bot = \mathbf{GLBB}_0$, and $\mathbf{Form} = \mathbf{K4} \oplus \bot = \mathbf{S4} \oplus \mathbf{GL}$ are also cluster-extensible.

logic	axiomatization over **K4**	frame condition
S4	$\Box x \to x$	reflexive
D4	$\Diamond\top$	final clusters reflexive
GL	$\Box(\Box x \to x) \to \Box x$	irreflexive
K4Grz	$\Box\big(\Box(x \to \Box x) \to x\big) \to \Box x$	no proper clusters
K4.1	$\boxdot\Diamond x \to \Diamond\Box x$	no proper final clusters
K4.3	$\Box(\boxdot x \to y) \vee \Box(\Box y \to x)$	width 1
K4B	$x \to \Box\Diamond x$	depth 1
S5	**S4** $\oplus$ **K4B**	reflexive, depth 1
K4BB$_k$	$\Box\Big(\bigvee_{i\le k}\Box\Big(\boxdot x_i \to \bigvee_{j\ne i} x_j\Big) \to \bigvee_{i\le k}\boxdot x_i\Big) \to \bigvee_{i\le k}\Box\bigvee_{j\ne i} x_j$	branching at most k
K4BC$_k$	$\bigwedge_{i=1}^{k}\Box\Big(\Box\Big(\bigwedge_{j<i} x_j \to x_i \vee \Box x_0\Big) \to x_0\Big) \to \Box x_0$	cluster size at most k
S4.1.4	$\Box\big(\Box(x \to \Box x) \to x\big) \to (\Box\Diamond\Box x \to x)$	reflexive, no inner proper clusters

Table 1: Some cluster-extensible logics

We note that some of the axioms have robust definitions only over **S4**, whereas their definitions over **K4** vary in the literature. In particular, the Grzegorczyk axiom is often stated ending with $\cdots \to x$ (which defines over **K4**, or even over **K**, the same logic as our **S4Grz**), and **K4.1** is often defined as **K4** $\oplus\, \Box\Diamond x \to \Diamond\Box x$ (which is our **D4.1**). We chose the definitions given because they seem to be most natural in a potentially irreflexive context, and as just mentioned, the alternative definitions are covered under other names.

A notable example of a well-known logic that is not clx is **S4.2** $=$ **S4** $\oplus\, \Diamond\Box x \to \Box\Diamond x$; this and other logics whose rooted finite frames have a single top cluster will be dealt with separately in the sequel. For other examples, the logics of bounded depth (**K4BD**$_k$) or width (**K4BW**$_k$) are not clx for $k > 1$.

Let $m \ge 1$, and L be the logic of finite **S4**-frames such that clusters of depth 3 or more have size at most m. Then L is not cluster-extensible, but it is Par-extensible if $|\mathrm{Par}| \le \log_2 m$.

4.1 Projective approximations

Definition 4.3 Let $L \supseteq$ **K4** and φ be a formula. A *projective approximation of* φ is a finite set of L-projective formulas Π_φ such that

(i) $\varphi \vdash_L \Pi_\varphi$,

(ii) $\psi \vdash_L \varphi$ for every $\psi \in \Pi_\varphi$.

Notice that if σ_ψ is a projective unifier of ψ for every $\psi \in \Pi_\varphi$, then $\{\sigma_\psi : \psi \in \Pi_\varphi\}$ is a complete set of unifiers of φ.

The definition of Par-extensible logics is motivated by the following result, generalizing the parameter-free case proved by Ghilardi [8].

Theorem 4.4 *If L is a* Par*-extensible logic, then any formula φ has a projective approximation Π_φ such that every $\psi \in \Pi_\varphi$ is a Boolean combination of subformulas of φ.*

First we need a simple lemma on preservation of formulas when attaching new root clusters to models, which we will also use later on. (The lemma is easier to prove than to formulate.)

Definition 4.5 Let Σ be a set of formulas closed under subformulas. If W_0, W_1 are models and $u_i \in W_i$, $i = 0, 1$, we will write $W_0, u_0 \equiv_\Sigma W_1, u_1$ if $W_0, u_0 \vDash \psi \Leftrightarrow W_1, u_1 \vDash \psi$ for every $\psi \in \Sigma$ (i.e., $\mathrm{Sat}_\Sigma(W_0, u_0) = \mathrm{Sat}_\Sigma(W_1, u_1)$).

Lemma 4.6 *For every $i = 0, 1$, let W_i be a Kripke model, $C_i = \{u_{i,j} : j \in J\} \subseteq W_i$, and $X_i = \{w_{i,k} : k \in K\} \subseteq W_i$, where either*

$$u_{i,j}\!\uparrow = X_i\!\updownarrow \tag{10}$$

for every $i = 0, 1$, $j \in J$, or

$$u_{i,j}\!\uparrow = \{u_{i,j'} : j' \in J\} \cup X_i\!\updownarrow \tag{11}$$

for every $i = 0, 1$, $j \in J$. Let Σ be a set of formulas closed under subformulas, and assume that $W_0, w_{0,k} \equiv_\Sigma W_1, w_{1,k}$ for every $k \in K$, and $W_0, u_{0,j}$ satisfies the same atoms from Σ as $W_1, u_{1,j}$ for every $j \in J$.

Then $W_0, u_{0,j} \equiv_\Sigma W_1, u_{1,j}$.

Proof: We will show $u_{0,j} \vDash \psi \Leftrightarrow u_{1,j} \vDash \psi$ for every $j \in J$ by induction on the complexity of $\psi \in \Sigma$. The statement holds for atoms by assumption, and steps for Boolean connectives are obvious. Assume $u_{0,j} \nvDash \Box\psi$. If (11) holds, and $u_{0,j'} \nvDash \psi$ for some $j' \in J$, then $u_{1,j'} \nvDash \psi$ by the induction hypothesis, hence $u_{1,j} \nvDash \Box\psi$. Otherwise there is $k \in K$ such that $w_{0,k} \nvDash \psi$ or $w_{0,k} \nvDash \Box\psi$. Since $w_{0,k} \equiv_\Sigma w_{1,k}$, this implies $w_{1,k} \nvDash \psi$ or $w_{1,k} \nvDash \Box\psi$, hence $u_{1,j} \nvDash \Box\psi$. The reverse direction is symmetric. $\Box$

Proof of Theorem 4.4*:* Let $\Sigma = \mathrm{Sub}(\varphi)$, $B(\Sigma)$ be its Boolean closure, and Π_φ the set of all $\psi \in B(\Sigma)$ such that ψ is projective and $\psi \vdash_L \varphi$. It suffices to show that every unifier σ of φ also unifies some $\psi \in \Pi_\varphi$. Define

$$\psi = \bigvee \bigl\{ \Sigma^{\mathrm{Sat}_\Sigma(\sigma(G), v)} : G \in \mathrm{Mod}_L, v \in G \bigr\}.$$

Clearly, $\psi \in B(\Sigma)$, and as $\vdash_L \sigma(\varphi)$, we have $\vdash_L \psi \to \varphi$. The fmp of L also implies $\vdash_L \sigma(\psi)$, thus the only thing left to prove is that ψ is projective. Using Theorem 3.2, it suffices to show that ψ has the model extension property. Notice that

$$\mathrm{Mod}_L(\psi) = \{F \in \mathrm{Mod}_L : \forall u \in F\, \exists G \in \mathrm{Mod}_L\, \exists v \in G\, (F, u \equiv_\Sigma \sigma(G), v)\}.$$

Let $F_0 \in \mathrm{Mod}_L$ be such that $F_0 \smallsetminus \mathrm{rcl}(F_0) \vDash \psi$. If $\mathrm{Par} = \varnothing$ and $F_0 \smallsetminus \mathrm{rcl}(F_0)$ has a reflexive root cluster, the mapping contracting $\mathrm{rcl}(F_0)$ to a fixed point $r \in \mathrm{rcl}(F_0 \smallsetminus \mathrm{rcl}(F_0))$ is a p-morphism, hence we can define a variant of F_0 satisfying ψ by copying the valuation of variables from r to $\mathrm{rcl}(F_0)$.

Otherwise, let F be the model obtained from F_0 by identifying points of $\mathrm{rcl}(F_0)$ with the same valuation of parameters (valuation of variables in $\mathrm{rcl}(F)$ is immaterial), and let $\langle C, n\rangle$ be its type. Notice that $|C| \le 2^{|\mathrm{Par}|}$. Pick elements $\{u_i : i < n\} \subseteq F$, one in every immediate successor cluster of $\mathrm{rcl}(F)$. Since $F_{u_i} \vDash \psi$, there exists $G_i \in \mathrm{Mod}_L$ and $v_i \in G_i$ such that $\sigma(G_i), v_i \equiv_\Sigma F, u_i$. We may assume $v_i \in \mathrm{rcl}(G_i)$. We define a model G as follows: we take the disjoint union of G_i, $i < n$, and attach a new root cluster of type C. We may identify elements of $\mathrm{rcl}(G)$ with elements of $\mathrm{rcl}(F)$; we define the valuation of parameters in $\mathrm{rcl}(G)$ identically to F, the valuation of variables is arbitrary.

Since L is $\langle C, n\rangle$-extensible, G is based on an L-frame. Let F' be the variant of F such that

$$F', w \vDash x \quad \text{iff} \quad \sigma(G), w \vDash x$$

for every variable x and $w \in \mathrm{rcl}(F)$. Then $F', w \equiv_\Sigma \sigma(G), w$ for every $w \in \mathrm{rcl}(F)$ by Lemma 4.6, which means $F' \vDash \psi$. Finally, we can define a variant $F'_0 \vDash \psi$ of the original F_0 by lifting the valuation from F' by the parameter-preserving p-morphism of F_0 onto F. $\Box$

Corollary 4.7 *If L is a* Par*-extensible logic, every formula φ of length n has a projective approximation consisting of at most 2^{2^n} formulas of length $O(n2^n)$.* $\Box$

Corollary 4.8 *Let L be a* Par*-extensible logic, and Γ / Δ a rule.*

(*i*) $\Gamma \mathrel{|\!\!\sim}_L \Delta$ *iff for every projective formula ψ that is a Boolean combination of subformulas of Γ, if $\psi \vdash_L \varphi$ for every $\varphi \in \Gamma$, then $\psi \vdash_L \varphi$ for some $\varphi \in \Delta$.*

(*ii*) *If L is decidable, $\mathrel{|\!\!\sim}_L$ is decidable.*

Proof: (i): This follows from the facts that the set of projective unifiers of formulas from $\Pi_{\bigwedge\Gamma}$ is a complete set of unifiers of Γ, and if σ is a projective unifier of ψ, then $\vdash_L \sigma(\varphi)$ iff $\psi \vdash_L \varphi$.

(ii): Projectivity, hence the criterion from (i), is decidable by condition (iii) of Theorem 3.2. $\Box$

Corollary 4.9 *Every* Par*-extensible logic L has at most finitary unification type. If L is decidable, we can compute a complete set of unifiers for any given formula.* $\Box$

Corollaries 4.8 (ii) and 4.9 were proved for a class of transitive modal logics by Rybakov [22, 23] using a different approach.

In Corollary 4.40, we will see that the assumption of decidability of L in Corollaries 4.8 and 4.9 is redundant if Par is infinite (i.e., for clx logics). We will prove more precise estimates on the computational complexity of $\mathrel{|\!\!\sim}_L$ in the sequel.

Example 4.10 The bounds in Corollary 4.7 cannot be substantially improved, even in the parameter-free case.

If L is a $\langle \bullet, 2\rangle$-extensible logic (e.g., **K4** or **GL**), consider the formulas

$$\varphi_m = \bigwedge_{i<m} (\Box x_i \vee \Box\neg x_i) \to \Box y \vee \Box\neg y$$

of length $n = O(m)$. We claim that φ_m has a projective approximation Π_{φ_m} consisting of the formulas

$$\psi_f = \bigwedge_{i<m} (\boxdot x_i \vee \boxdot\neg x_i) \to (y \leftrightarrow f(\vec{x})),$$

where $f\colon \mathbf{2}^X \to \mathbf{2}$ is any Boolean function in the m variables $X = \{x_i : i < m\}$. The formulas ψ_f have the model extension property (we can modify valuation of y to match $f(\vec{x})$), and $\psi_f \vdash_L \varphi_m$ follows from $\vdash_{\mathbf{K4}} \bigwedge_{i<m}(\Box x_i \vee \Box\neg x_i) \to \Box f(\vec{x}) \vee \Box\neg f(\vec{x})$. In order to show $\varphi_m \vdash_L \Pi_{\varphi_m}$, let σ be any unifier of φ_m. For every $e \in \mathbf{2}^X$, there is $f(e) \in \mathbf{2}$ such that

$$\vdash_L \sigma\bigl(\boxdot X^e \to y^{f(e)}\bigr) : \tag{12}$$

If not, we could find models $F_0, F_1 \in \mathrm{Mod}_L$ with roots u_0, u_1 (resp.) such that $\sigma(F_i) \vDash X^e$, $\sigma(F_0), u_0 \vDash \neg y$, and $\sigma(F_1), u_1 \vDash y$. Let $F \in \mathrm{Mod}_L$ be the disjoint union of F_0 and F_1, endowed with a new irreflexive root u. Then there is no way to define valuation in u so that $\sigma(F), u \vDash \varphi_m$, contradicting $\vdash_L \sigma(\varphi_m)$.

This defines a function $f\colon \mathbf{2}^X \to \mathbf{2}$, and (12) implies $\vdash_L \sigma(\psi_f)$. Thus, Π_{φ_m} is indeed a projective approximation of φ_m. Since $\psi_f \nvdash_L \psi_g$ for $f \neq g$, *every* projective approximation Π of φ_m must contain at least $2^{2^m} = 2^{2^{\Omega(n)}}$ formulas, and by a counting argument, most of these formulas must have length $\Omega(\log|\Pi|) = 2^{\Omega(n)}$.

If L is a $\langle ①, 2\rangle$-extensible logic (such as **S4** or **S4Grz**), we can use in a similar way the slightly more complicated formulas

$$\varphi'_m = \bigwedge_{i<m} (\Box\Diamond x_i \vee \Box\Diamond\neg x_i) \to \Box y \vee \Box\neg y,$$

whose projective approximation consists of the 2^{2^m} formulas

$$\psi'_f = \bigwedge_{e\in\mathbf{2}^X} \Bigl(\bigwedge_{i<m} \boxdot\Diamond x_i^{e(x_i)} \to y^{f(e)}\Bigr),$$

where again $f\colon \mathbf{2}^X \to \mathbf{2}$.

4.2 Directed unification

Corollary 4.9 tells us that the parametric unification type of any Par-extensible logic is at most finitary, but it does not specify whether it is of type 1 or ω. We will resolve this with the help of the following concept.

Definition 4.11 A logic L has *directed* (or *filtering*) *unification* if for any formula φ, the preorder of L-unifiers of φ is directed, i.e., for every unifiers σ_0 and σ_1 of φ, there exists a unifier of φ more general than either of σ_0, σ_1.

Clearly, if a formula has a mgu, then its preorder of unifiers is directed, whereas if it has at least two incomparable maximal unifiers, it is not directed. Thus, if L has non-nullary unification type, then it has unitary unification if and only if it has directed unification.

Ghilardi and Sacchetti [9] discovered a criterion for directedness of parameter-free unification in transitive modal logics: namely, $L \supseteq \mathbf{K4}$ has directed unification if and only if it includes the logic

$$\mathbf{K4.2} := \mathbf{K4} \oplus \Diamond\boxdot x \to \Box\diamonddot x.$$

We will give a simple syntactic proof of this result that also applies to unification with parameters, as well as a much more general class of logics.

We are temporarily leaving the realm of modal logics, the theorem below works for logics given by arbitrary structural consequence relations satisfying the stated conditions. In this context, we will write $\Gamma \vdash \bot$ as a short-hand for "Γ is inconsistent", i.e., $\Gamma \vdash \varphi$ for every formula φ. Note that we are working with single-conclusion systems here, we will use $\Gamma \vdash \Delta$ as an abbreviation for $\Gamma \vdash \varphi$ for every $\varphi \in \Delta$.

Theorem 4.12 *Let L be a logic such that:*

(a) L is equivalential with respect to a set of formulas $E(x, y)$.

(b) There is a finite set of formulas $D(x, y)$ such that

$$\Gamma, D(\varphi, \psi) \vdash_L \chi \quad \textit{iff} \quad \Gamma, \varphi \vdash_L \chi \textit{ and } \Gamma, \psi \vdash_L \chi$$

for every finite set of formulas Γ, and formulas φ, ψ, and χ.

(c) There are formulas $S(x, y_0, y_1)$, $C_0(x)$, and $C_1(x)$ such that for $i = 0, 1$,

$$C_i(x) \vdash_L E(S(x, y_0, y_1), y_i).$$

(d) There is a formula $B(x)$ such that for every Γ and φ,

$$x \vdash_L C_1(B(x)),$$
$$\Gamma, \varphi \vdash_L \bot \;\Rightarrow\; \Gamma \vdash_L C_0(B(\varphi)).$$

Then the following are equivalent:

(i) L has directed (parametric) unification.

(ii) There is a formula α such that $\vdash_L D(C_0(\alpha), C_1(\alpha))$, and $C_0(\alpha)$ and $C_1(\alpha)$ are L-unifiable.

(iii) $\vdash_L D(C_0(B(C_0(x))), C_0(B(C_1(x))))$.

Moreover, (i) is equivalent to (ii) for any logic L that satisfies (a), (b), (c), and

(d') $C_0(x)$ and $C_1(x)$ are L-unifiable,

where we can allow C_0 and C_1 to be finite sets instead of single formulas.

Remark 4.13 The mnemonics for the letters are Equivalence, Disjunction, Switch, truth Constant, and Box.

In general, we allow parameters (not indicated by the notation) to appear in all formulas and unifiers mentioned in Theorem 4.12. However, if the assumptions are satisfied with parameter-free E, D, C_i, B (which is the common case), then we can assume without loss of generality that α and the unifiers in (ii), (d') are also parameter-free. Consequently, L has directed parametric unification iff it has directed parameter-free unification.

We could allow the variable x in C_i and S to be a list $x_1, \dots, x_k$ of variables instead, with obvious modifications (we would have $\alpha_1, \dots, \alpha_k$ and $B_1, \dots, B_k$ to go with these).

Corollary 4.14 *A logic $L \supseteq$ **K4** has directed unification if and only if $L \supseteq$ **K4.2**.*

More generally, let L be an n-transitive multimodal logic (i.e., L has finitely many boxes $\Box_1, \dots, \Box_k$, and the combined modality $\Box x := \Box_1 x \land \dots \land \Box_k x$ satisfies $\vdash_L \Box^{\le n} x \to \Box^{n+1} x$). Then L has directed unification iff it proves $\Diamond^{\le n} \Box^{\le n} x \to \Box^{\le n} \Diamond^{\le n} x$.

Proof: Apply Theorem 4.12 with $E(x, y) = \{x \leftrightarrow y\}$, $D(x, y) = \{\Box^{\le n} x \lor \Box^{\le n} y\}$, $C_1(x) = x$, $C_0(x) = \neg x$, $S(x, y_0, y_1) = (x \land y_1) \lor (\neg x \land y_0)$, $B(x) = \Box^{\le n} x$. $\Box$

Corollary 4.15 *Let $L \supseteq$ **K4** be a* Par*-extensible logic. Then L has unification of type* 1 *if L is linear (see §5.1), and type ω otherwise.* $\Box$

Proof of Theorem 4.12*:* Using (c) and (a), we have $C_0(x), C_1(x) \vdash_L E(y_0, y_1)$, hence

$$C_0(x), C_1(x) \vdash_L \bot. \tag{13}$$

Notice also that (d) implies (d'): by (d), we have $\vdash_L C_1(B(\top))$ for any tautology $\top$, which implies $\vdash_L C_0(B(C_0(B(\top))))$ by (d) and (13).

(i) $\to$ (ii): Let σ_i be a unifier of C_i, $i = 0, 1$. Both σ_i are unifiers of $D(C_0(x), C_1(x))$, hence by (i), this formula has a unifier τ such that $\sigma_0, \sigma_1 \le_L \tau$. Then $\alpha := \tau(x)$ is as desired.

(ii) $\to$ (i): Let τ_0, τ_1 be unifiers of Γ, and define

$$\tau(x_j) = S(\alpha, \tau_0(x_j), \tau_1(x_j))$$

for every variable x_j occurring in φ. We may assume that α (hence $C_i(\alpha)$) shares no variables with $\tau_i(x_j)$. We have

$$C_i(\alpha) \vdash_L E(\tau(x_j), \tau_i(x_j)) \tag{14}$$

by (c), hence

$$C_i(\alpha) \vdash_L \tau(\Gamma)$$

by (a), which means

$$\vdash_L D(C_0(\alpha), C_1(\alpha)) \vdash_L \tau(\Gamma)$$

using (b). Moreover, if σ_i is a unifier of $C_i(\alpha)$ identical on variables not occurring in α, then (14) gives

$$\vdash_L E(\sigma_i(\tau(x_j)), \tau_i(x_j)),$$

i.e., $\tau_i \leq_L \tau$ via σ_i.

(ii) $\to$ (iii): Let σ_i be a unifier of $C_i(\alpha)$, and define

$$\tau(x_j) = S(x, \sigma_0(x_j), \sigma_1(x_j)).$$

We have

$$\begin{aligned}
C_i(x) &\vdash_L E(\tau(x_j), \sigma_i(x_j)) && \text{by (c),} \\
C_i(x) &\vdash_L \tau(C_i(\alpha)) && \text{by (a),} \\
C_i(x), \tau(C_{1-i}(\alpha)) &\vdash_L \bot && \text{by (13),} \\
\tau(C_{1-i}(\alpha)) &\vdash_L C_0(B(C_i(x))) && \text{by (d),} \\
\tau(D(C_0(\alpha), C_1(\alpha))) &\vdash_L D(C_0(B(C_0(x))), C_0(B(C_1(x)))) && \text{by (b),} \\
&\vdash_L D(C_0(B(C_0(x))), C_0(B(C_1(x)))) && \text{by (ii).}
\end{aligned}$$

(iii) $\to$ (ii): Put $\alpha = B(C_0(B(C_1(x))))$. We have

$$\begin{aligned}
C_1(x) &\vdash_L C_1(B(C_1(B(C_1(x))))) && \text{by (d),} \\
C_1(x), C_0(B(C_1(B(C_1(x))))) &\vdash_L \bot && \text{by (13),} \\
C_0(B(C_1(B(C_1(x))))) &\vdash_L C_0(B(C_1(x))) && \text{by (d),} \\
C_0(B(C_1(B(C_1(x))))) &\vdash_L C_1(\alpha) && \text{by (d),} \\
C_0(B(C_0(B(C_1(x))))) &\vdash_L C_0(\alpha) && \text{by definition,} \\
&\vdash_L D(C_0(\alpha), C_1(\alpha)) && \text{by (b) and (iii).}
\end{aligned}$$

Let σ_i be a unifier of $C_i(x)$, $i = 0, 1$. We have $\sigma_0(C_1(x)) \vdash_L \bot$ by (13), hence

$$\vdash_L \sigma_0(C_0(B(C_1(x)))) \vdash_L \sigma_0(C_1(B(C_0(B(C_1(x)))))) = \sigma_0(C_1(\alpha))$$

by (d). Similarly,

$$\begin{aligned}
&\vdash_L \sigma_1(C_1(B(C_1(x)))) && \text{by (d),} \\
\sigma_1(C_0(B(C_1(x)))) &\vdash_L \bot && \text{by (13),} \\
&\vdash_L \sigma_1(C_0(B(C_0(B(C_1(x)))))) = \sigma_1(C_0(\alpha)) && \text{by (d).}
\end{aligned}$$

□

Remark 4.16 For readers familiar with substructural logics (see [6]): Theorem 4.12 can be applied to a large class of logics as follows.

Let L be an extension (not necessarily simple, i.e., L may have a richer language) of the $\{\to, \wedge, \vee, 0, 1\}$-fragment of $\mathbf{FL_o}$, where $\to$ is either of the two residua. Assume that L is equivalential with respect to the formula $E(x, y) = (x \to y) \wedge (y \to x)$ (note that this holds automatically for simple axiomatic extensions of fragments of $\mathbf{FL_o}$), and it has the deduction-detachment theorem in the form

$$\Gamma, \varphi \vdash_L \psi \quad \text{iff} \quad \Gamma \vdash_L \Delta\varphi \to \psi \tag{15}$$

for some formula $\Delta(x)$. (In systems that have it, one can usually take Baaz delta for Δ.) Put $\neg\varphi := \varphi \to 0$. Then L satisfies the assumptions of Theorem 4.12 with $D(x, y) = \Delta x \vee \Delta y$, $S(x, y_0, y_1) = (1 \wedge x \to y_1) \wedge (1 \wedge \neg x \to y_0)$, $C_1(x) = x$, $C_0(x) = \neg x$, $B(x) = \Delta x$. Thus, the following are equivalent:

(i) L has directed unification.

(ii) There is a formula α such that $\vdash_L \Delta\alpha \vee \Delta\neg\alpha$, and α and $\neg\alpha$ are unifiable.

(iii) $\vdash_L \Delta\neg\Delta x \vee \Delta\neg\Delta\neg x$.

For example, this subsumes Corollary 4.14 by taking $\Delta x = \Box^{\leq n} x$. For another example, let L be an n-contractive simple axiomatic extension of $\mathbf{FL_{ew}}$. (Note that the case $n = 1$ covers superintuitionistic logics.) Then we can take $\Delta x = x^n$, hence L has directed unification iff it proves $\bigl(\neg x^n\bigr)^n \vee \bigl(\neg(\neg x)^n\bigr)^n$. In fact, unification in some of these logics has been proved unitary by Dzik [4].

4.3 Structure of cluster-extensible logics

While there are continuum many extensible logics in the parameter-free case (see e.g. [15]), extensibility is a much tighter constraint if there are infinitely many parameters. This is to be expected: we defined cluster-extensible logics for the purpose that their admissible rules have bases consisting of subsets of certain explicitly defined rules (Theorem 5.17). Unlike the parameter-free case, it is impossible for a logic to inherit the admissible rules of its proper sublogic if we have infinitely many parameters: for any consistent logic L and a parameter-free formula $\varphi(\vec{x})$, we have

$$\begin{aligned} \vdash_L \varphi(\vec{x}) \quad &\text{iff} \quad \mathrel{|\!\sim}_L \varphi(\vec{p}), \\ \nvdash_L \varphi(\vec{x}) \quad &\text{iff} \quad \varphi(\vec{p}) \mathrel{|\!\sim}_L \bot, \end{aligned} \tag{16}$$

where $\vec{p}$ are pairwise distinct parameters. Thus, each clx logic is uniquely determined by a set of extension rules, and the relatively simple structure of these rules carries over to the corresponding class of logics.

In this section, we will show that all clx logics have various nice properties that will be helpful for description of their admissible rules and their complexity: in particular, clx logics are finitely axiomatizable, have the exponential-size model property, and are first-order ($\forall\exists$) definable on finite frames. Moreover, clx logics are closed under joins (hence they form a complete lattice). On the other hand, the class of clx logics includes most of the best known particular transitive monomodal logics (a notable exception being logics with a single top cluster such as $\mathbf{S4.2}$, which require special treatment).

Unless stated otherwise, we assume Par is infinite for the rest of this section.

Definition 4.17 An *extension condition* is a pair $\langle C, n\rangle$, where $n \in \omega \cup \{\infty\}$, and C is a cluster type or $\textcircled{\scriptsize$\infty$}$. An extension condition $\langle C, n\rangle$ is *finite* if $C \neq \textcircled{\scriptsize$\infty$}$ and $n \neq \infty$. The set of all extension conditions is denoted by EC^∞, and the set of finite extension conditions by EC.

We generalize the notion of a $\langle C, n\rangle$-extensible logic to arbitrary extension conditions so that L is $\langle \textcircled{\scriptsize$\infty$}, n\rangle$-extensible iff it is $\langle \textcircled{\scriptsize$k$}, n\rangle$-extensible for every $0 < k \in \omega$, and L is $\langle \textcircled{\scriptsize$k$}, \infty\rangle$-extensible iff it is $\langle \textcircled{\scriptsize$k$}, n\rangle$-extensible for every $0 < n \in \omega$. If T is a set of extension conditions, L is T-extensible if it is t-extensible for every $t \in T$.

Let $\leq_0$ be the partial order on $\omega \cup \{\infty\}$ such that $n \leq_0 m \leq_0 \infty$ for every $0 < n \leq m \in \omega$, and 0 is incomparable to any other element. If C and D are cluster types, we define $D \preceq C$ iff both C, D are irreflexive, or both are reflexive and $|D| \leq |C|$. We also put $\textcircled{\scriptsize$k$} \preceq \textcircled{\scriptsize$\infty$}$ for every $0 < k \in \omega$. (Notice that if we identify $\bullet$ with 0, and $\textcircled{\scriptsize$k$}$ with k for $0 < k \leq \infty$, then $\preceq$ is the same order as $\leq_0$.) If $\langle C, n\rangle$ and $\langle D, m\rangle$ are extension conditions, we put $\langle C, n\rangle \preceq \langle D, m\rangle$ iff $C \preceq D$ and $n \leq_0 m$.

The *closure* of a set T of extension conditions is the smallest set $\overline{T} \supseteq T$ downward closed under $\preceq$, and closed under the rules

- if $\langle C, n\rangle \in \overline{T}$ for every $0 < n \in \omega$, then $\langle C, \infty\rangle \in \overline{T}$,
- if $\langle \textcircled{\scriptsize$k$}, n\rangle \in \overline{T}$ for every $0 < k \in \omega$, then $\langle \textcircled{\scriptsize$\infty$}, n\rangle \in \overline{T}$.

Two sets of extension conditions are *equivalent* if they have the same closure.

Recall that a *well partial order* (*wpo*) is a partial order $\leq$ on a set X satisfying any of the following equivalent conditions:

- Every subset $Y \subseteq X$ has a finite *basis*: a finite set $B \subseteq Y$ such that $B{\Uparrow} \supseteq Y$.
- $<$ is well founded, and there are no infinite antichains.
- For every sequence $\{x_i : i \in \omega\} \subseteq X$, there are $i < j$ such that $x_i \leq x_j$.

It is easily seen that the class of wpo contains all well-ordered sets, and it is closed under subsets, finite unions and Cartesian products, and homomorphic images [21].

Lemma 4.18

(*i*) *If T and T' are equivalent sets of extension conditions, then a logic is T-extensible iff it is T'-extensible.*

(*ii*) *$\preceq$ is a well partial order on EC^∞.*

(*iii*) *Every set of extension conditions is equivalent to a unique finite set of extension conditions that is an antichain wrt $\preceq$.*

Proof:

(i): The cases involving infinite conditions are clear from the definition. If L is $\langle \textcircled{\scriptsize$k$}, n\rangle$-extensible and $0 < l \leq k$, then L is $\langle \textcircled{\scriptsize$l$}, n\rangle$-extensible as every type-$\langle \textcircled{\scriptsize$l$}, n\rangle$ frame is a p-morphic image of a type-$\langle \textcircled{\scriptsize$k$}, n\rangle$ frame with the same $F \smallsetminus \mathrm{rcl}(F)$. Finally, assume that L is $\langle C, n\rangle$-extensible, and F is a type-$\langle C, m\rangle$ frame such that $F \smallsetminus \mathrm{rcl}(F)$ is an L-frame, where $m \leq_0 n$.

logic L	bas(L)	xcb(L)
K4	$\langle \bullet/\otimes, 0/\infty\rangle$	
S4	$\langle \otimes, 0/\infty\rangle$	$\langle \bullet, 0/1\rangle$
K4Grz	$\langle \bullet/\textcircled{1}, 0/\infty\rangle$	$\langle \textcircled{2}, 0/1\rangle$
S4Grz	$\langle \textcircled{1}, 0/\infty\rangle$	$\langle \bullet/\textcircled{2}, 0/1\rangle$
K4.3	$\langle \bullet/\otimes, 0/1\rangle$	$\langle \bullet/\textcircled{1}, 2\rangle$
K4BB$_k$	$\langle \bullet/\otimes, 0/k\rangle$	$\langle \bullet/\textcircled{1}, k+1\rangle$
K4BC$_k$	$\langle \bullet/\textcircled{k}, 0/\infty\rangle$	$\langle \textcircled{k+1}, 0/1\rangle$
S4.1.4	$\langle \otimes, 0\rangle, \langle \textcircled{1}, \infty\rangle$	$\langle \bullet, 0/1\rangle, \langle \textcircled{2}, 1\rangle$

logic L	bas(L)	xcb(L)
K4B	$\langle \bullet/\otimes, 0\rangle$	$\langle \bullet/\textcircled{1}, 1\rangle$
S5	$\langle \otimes, 0\rangle$	$\langle \bullet, 0/1\rangle, \langle \textcircled{1}, 1\rangle$
GL	$\langle \bullet, 0/\infty\rangle$	$\langle \textcircled{1}, 0/1\rangle$
GL.3	$\langle \bullet, 0/1\rangle$	$\langle \textcircled{1}, 0/1\rangle, \langle \bullet, 2\rangle$
S4.3	$\langle \otimes, 0/1\rangle$	$\langle \bullet, 0/1\rangle, \langle \textcircled{1}, 2\rangle$
Triv	$\langle \textcircled{1}, 0\rangle$	$\langle \bullet/\textcircled{2}, 0\rangle, \langle \bullet/\textcircled{1}, 1\rangle$
Verum	$\langle \bullet, 0\rangle$	$\langle \textcircled{1}, 0\rangle, \langle \bullet/\textcircled{1}, 1\rangle$
Form		$\langle \bullet/\textcircled{1}, 0/1\rangle$

logic L	bas(L)	xcb(L)
D4	$\langle \otimes, 0\rangle, \langle \bullet/\otimes, \infty\rangle$	$\langle \bullet, 0\rangle$
K4.1	$\langle \bullet/\textcircled{1}, 0\rangle, \langle \bullet/\otimes, \infty\rangle$	$\langle \textcircled{2}, 0\rangle$
S4.1	$\langle \textcircled{1}, 0\rangle, \langle \otimes, \infty\rangle$	$\langle \bullet, 0/1\rangle, \langle \textcircled{2}, 0\rangle$

Table 2: Extension characteristics of some clx logics

Choose $\{w_i : i < m\}$ such that $F \smallsetminus \mathrm{rcl}(F) = \bigcup_{i<m} w_i{\uparrow}$, let f be a surjection of $\{0, \dots, n-1\}$ onto $\{0, \dots, m-1\}$, and let G be the frame consisting of $\dot\bigcup_{i<n} F_{w_{f(i)}}$ together with a copy of $\mathrm{rcl}(F)$ as its root cluster. Then G is an L-frame as it has type $\langle C, n\rangle$, and F is a p-morphic image of G.

(ii): $\preceq$ is the product of two partial orders, each of which is a disjoint union of a singleton and a well order of type $\omega + 1$, and as such it is a wpo.

(iii): Every set is equivalent to its closure, hence we may assume that T is a closed set of extension conditions. Since any chain in T has a supremum in T, the set M of maximal elements of T is cofinal in T by Zorn's lemma, and therefore equivalent to T. Clearly, M is an antichain, hence it is finite by (ii). The closure of any finite set of conditions is its downward closure, hence distinct antichains have distinct closures. □

Observation 4.19 *There is a bijective correspondence between closed subsets $T \subseteq EC^\infty$, and upward closed subsets $U \subseteq EC$, given by $U = EC \smallsetminus T$ and $T = \overline{EC \smallsetminus U}$.* □

Definition 4.20 If L is a clx logic, its *type* $\mathrm{tp}(L)$ is the set of all extension conditions $\langle C, n\rangle$ such that L is $\langle C, n\rangle$-extensible.

Its *basis* $\mathrm{bas}(L)$ consists of maximal elements of $\mathrm{tp}(L)$. Notice that $\mathrm{tp}(L) = \overline{\mathrm{tp}(L)}$ by Lemma 4.18 (i), hence $\mathrm{bas}(L)$ is the unique finite antichain equivalent to $\mathrm{tp}(L)$ by the proof of (iii), and $\langle D, m\rangle \in \mathrm{tp}(L)$ iff $\langle D, m\rangle \preceq \langle C, n\rangle$ for some $\langle C, n\rangle \in \mathrm{bas}(L)$.

The *exclusion type of* L is $\mathrm{xcl}(L) = EC \smallsetminus \mathrm{tp}(L)$, and its *exclusion basis* $\mathrm{xcb}(L)$ is the set of all minimal elements of $\mathrm{xcl}(L)$. By Lemma 4.18 (ii), $\mathrm{xcb}(L)$ is finite, and $\mathrm{xcl}(L)$ is its upward closure, hence $\langle D, m\rangle \notin \mathrm{tp}(L)$ iff $\langle C, n\rangle \preceq \langle D, m\rangle$ for some $\langle C, n\rangle \in \mathrm{xcb}(L)$.

If $U \subseteq EC$ is upward closed, then Fr_U is the class of all finite frames F such that there is no $u \in F$ for which the type of F_u belongs to U, and $\mathbf{Clx}_U$ is the logic of Fr_U.

Example 4.21 Bases and exclusion bases of some concrete clx logics are listed in Table 2. The table employs abbreviations to save space: for example, the line for **K4.3** means that $\mathrm{bas}(\mathbf{K4.3}) = \{\langle\bullet,0\rangle,\langle\bullet,1\rangle,\langle\otimes,0\rangle,\langle\otimes,1\rangle\}$ and $\mathrm{xcb}(\mathbf{K4.3}) = \{\langle\bullet,2\rangle,\langle①,2\rangle\}$.

Theorem 4.22 *If $U \subseteq EC$ is upward closed, then $\mathbf{Clx}_U$ is the unique clx logic of exclusion type U. In particular, every clx logic is uniquely determined by either of* $\mathrm{tp}(L)$*,* $\mathrm{bas}(L)$*,* $\mathrm{xcl}(L)$*, or* $\mathrm{xcb}(L)$*.*

Proof: Since Fr_U is closed under generated subframes, finite disjoint unions, and (due to the upward closure of U) under p-morphic images, it is the class of all finite $\mathbf{Clx}_U$-frames ([3, Exercise 9.34]). By the definition, $\mathbf{Clx}_U$ has fmp, and no finite rooted $\mathbf{Clx}_U$-frame has type $t \in U$. On the other hand, $\mathbf{Clx}_U$ is t-extensible for every $t \in EC \smallsetminus U$ by the definition of Fr_U. Thus, $\mathbf{Clx}_U$ is cluster-extensible, and $\mathrm{xcl}(\mathbf{Clx}_U) = U$.

If L is any clx logic of exclusion type U, then every finite L-frame belongs to Fr_U by the definition of type. On the other hand, if $F \in \mathrm{Fr}_U$, we can show that F is an L-frame by induction on $|F|$, using the fact that L is $(EC \smallsetminus U)$-extensible. Thus, Fr_U is the class of all finite L-frames, and as L has fmp, $L = \mathbf{Clx}_U$. □

Corollary 4.23 *There are countably many clx logics.*

Proof: There are countably many choices for $\mathrm{bas}(L)$ or $\mathrm{xcb}(L)$. □

Corollary 4.24 *The set* CLX *of all clx logics is a complete lattice under inclusion, and the mappings $L \mapsto \mathrm{xcl}(L)$ and $U \mapsto \mathbf{Clx}_U$ are mutually inverse isomorphisms of* CLX *to the lattice of all upward closed sets of finite extension conditions. Alternatively, $L \mapsto \mathrm{tp}(L)$ and $T \mapsto \mathbf{Clx}_{EC \smallsetminus T}$ are mutually inverse dual isomorphisms of* CLX *to the lattice of closed sets of extension conditions.*

Proof: Upward closed sets of finite extension conditions are closed under arbitrary intersections and unions, hence they form a complete lattice. The rest is clear from Theorem 4.22 and Observation 4.19 and the definitions. □

Corollary 4.25 *There is no strictly increasing infinite sequence of clx logics.*

Proof: Assume that $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$ are clx logics, and let L be the join $\bigvee_n L_n$ in CLX. We have $\mathrm{xcl}(L) = \bigcup_n \mathrm{xcl}(L_n)$. As $\mathrm{xcb}(L)$ is finite, we must have $\mathrm{xcb}(L) \subseteq \mathrm{xcl}(L_n)$ for some $n \in \omega$, hence $\mathrm{xcl}(L) = \mathrm{xcl}(L_n)$, and $L = L_n$. □

Remark 4.26 Let S be a set of clx logics, and L its join in NExt **K4**. Since a finite frame is an L-frame iff it is an L'-frame for every $L' \in S$, the logic determined by finite L-frames is the clx logic of exclusion type $\bigcup\{\mathrm{xcl}(L') : L' \in S\}$, i.e., the join of S in CLX. However, there is no a priori reason why L itself should have the finite model property.

Nevertheless, we will establish later that this is indeed the case, hence CLX is a complete join-subsemilattice of NExt **K4** (Corollary 4.39).

Example 4.27 Cluster-extensible logics are not closed under intersections, hence CLX is not a sublattice of NExt $\mathbf{K4}$. For example, consider the logics $\mathbf{S4Grz}, \mathbf{S5} \in$ CLX. Finite rooted frames of $L = \mathbf{S4Grz} \cap \mathbf{S5}$ are exactly those that are either $\mathbf{S4Grz}$-frames or $\mathbf{S5}$-frames. In particular, L has a type-$\langle ①, 2\rangle$ frame, but it is not $\langle ①, 2\rangle$-extensible, as there is no type-$\langle ①, 2\rangle$ L-frame including a proper cluster.

The meet $L' = \mathbf{S4Grz} \wedge \mathbf{S5}$ in CLX is in fact the clx logic satisfying $\mathrm{tp}(L') = \mathrm{tp}(\mathbf{S5}) \cup \mathrm{tp}(\mathbf{S4Grz})$ and $\mathrm{xcl}(L') = \mathrm{xcl}(\mathbf{S5}) \cap \mathrm{xcl}(\mathbf{S4Grz})$, namely $L' = \mathbf{S4.1.4}$.

In contrast, we have:

Proposition 4.28 *If S is a chain (or more generally, a downward directed set) of clx logics, then $\bigcap S$ is a clx logic, and* $\mathrm{xcl}\big(\bigcap S\big) = \bigcap\{\mathrm{xcl}(L) : L \in S\}$.

Proof: The logic $L_0 = \bigcap S$ has fmp, and a finite rooted frame is an L_0-frame iff it is an L-frame for some $L \in S$. Let F be a finite frame of type $\langle C, n\rangle$ such that $F \smallsetminus \mathrm{rcl}(F)$ is an L_0-frame. For every $u \in F \smallsetminus \mathrm{rcl}(F)$, there is $L_u \in S$ such that F_u is an L_u-frame. If L_0 has a type-$\langle C, n\rangle$ frame, then so does some $L' \in S$. Let $L \in S$ be such that $L \subseteq L'$ and $L \subseteq L_u$ for every u. Then $F \smallsetminus \mathrm{rcl}(F)$ is an L-frame and $\langle C, n\rangle \in \mathrm{tp}(L)$, hence F is an L-frame, and a fortiori an L_0-frame. □

Theorem 4.29 *Every clx logic L is $\forall\exists$-definable on finite frames.*

Proof: Let $U = \mathrm{xcb}(L)$. We know that U is a finite set of finite extension conditions, and a finite frame is an L-frame iff it has no rooted generated subframe of type $\langle D, m\rangle \succeq \langle C, n\rangle$, where $\langle C, n\rangle \in U$, hence it suffices to express the latter property by a $\forall\exists$ formula $\xi_{C,n}$. (Notice that transitivity is defined by a universal formula.) This is easy, we can take, e.g.,

$$
\begin{aligned}
\xi_{\bullet,0} &= \forall u\, \exists v\, (u < v),\\
\xi_{\bullet,n} &= \forall u, w_0, \dots, w_{n-1}\, \exists v\, \big(\neg(u < u) \wedge \alpha_n(u, \vec{w}) \to \beta_n(u, v, \vec{w})\big),\\
\xi_{ⓚ,0} &= \forall u_0, \dots, u_{k-1}\, \exists v\, \big(\gamma_k(\vec{u}) \to u_0 < v \wedge \neg(v < u_0)\big),\\
\xi_{ⓚ,n} &= \forall u_0, \dots, u_{k-1}, w_0, \dots, w_{n-1}\, \exists v\, \big(\gamma_k(\vec{u}) \wedge \alpha_n(u_0, \vec{w}) \to \beta_n(u_0, v, \vec{w})\big)
\end{aligned}
$$

for any $k, n \in \omega \smallsetminus \{0\}$, where

$$
\begin{aligned}
\alpha_n(u, w_0, \dots, w_{n-1}) &= \bigwedge_{i<n} \big(u < w_i \wedge \neg(w_i < u)\big) \wedge \bigwedge_{i<j<n} \neg(w_i < w_j \vee w_i = w_j \vee w_j < w_i),\\
\beta_n(u, v, w_0, \dots, w_{n-1}) &= \bigvee_{i<n} \big(u < v \wedge \neg(v < u) \wedge v < w_i \wedge \neg(w_i < v)\big),\\
\gamma_k(u_0, \dots, u_{k-1}) &= \bigwedge_{i,j<k} u_i < u_j \wedge \bigwedge_{i<j<k} \neg(u_i = u_j).
\end{aligned}
$$

(Note that the last conjunction in the definition of α_1 and γ_1 is empty and represents $\top$.) □

Having established all we could about clx logics using more-or-less trivial methods, we now turn to the problem of their finite axiomatizability. We will use an indirect approach: we will define a kind of frame formulas semantically corresponding to extension conditions, and we will show that every logic axiomatized by these formulas has the finite model property. It will follow easily that any clx logic is axiomatizable by a set of these formulas, which can be taken finite due to Lemma 4.18. As a byproduct of our proof of the fmp we obtain an exponential bound on the size of countermodels, and the form of canonical axiom sets we provide for clx logics also shows that the class of clx logics is closed under joins, as alluded to in Remark 4.26. Last but not least, being a finitely axiomatizable logic with the fmp, every clx logic is decidable.

Definition 4.30 Let $\langle C, n\rangle \in EC$. The frame $F^{\bullet}_{C,n} = \langle F_{C,n}, <_{\bullet}\rangle$ consists of a root cluster $\{c_e : e < k\}$ of type C (where $k = |C|$), and its n immediate irreflexive successors $\{s_i : i < n\}$. The frame $F^{\circ}_{C,n} = \langle F_{C,n}, <_{\circ}\rangle$ is defined similarly, but the s_i's are reflexive.

If $\langle W, <, A\rangle$ is a general frame and $n > 0$, a *weak morphism from W to $F_{C,n}$* is a partial mapping f from W onto $F_{C,n}$ such that for every $u \in \mathrm{dom}(f)$ and $v \in F_{C,n}$,

(i) $f^{-1}(v) \in A$,

(ii) $u' < u$ implies $u' \in \mathrm{dom}(f)$ and $f(u') <_{\circ} f(u)$,

(iii) if $f(u) <_{\bullet} v$, there is $u' \in \mathrm{dom}(f)$ such that $u < u'$ and $f(u') = v$.

(That is, f is essentially a p-morphism of a downward closed definable subframe of W onto $F_{C,n}$, except that the s_i's are hermaphroditic.) For $n = 0$, a weak morphism from W to $F_{C,0}$ is defined to be a p-morphism from W to $F_{C,0}$ (i.e., to C). We define

$$\begin{aligned}
\alpha_{\bullet,0} &= \Diamond\top,\\
\alpha_{\bullet,1} &= \Box y \to y \vee \Box\bot,\\
\alpha_{\bullet,n} &= \bigwedge_{i<j<n} \Box(\boxdot x_i \vee \boxdot x_j) \to \bigvee_{i<n} \Box x_i \qquad (n > 1),\\
\alpha_{\textcircled{k},0} &= \bigvee_{e<k} \dot{\Diamond}\Box\Big(\bigwedge_{d<e} y_d \to y_e\Big),\\
\alpha_{\textcircled{k},n} &= \boxdot\beta_{\textcircled{k},n} \to y_0 \qquad (n > 0),
\end{aligned}$$

where

$$\begin{aligned}
\beta_{\textcircled{k},n} = \bigwedge_{d<e<k} (y_d \vee y_e) \wedge \bigwedge_{d,e<k} (\Box y_d \to y_e) \wedge \bigwedge_{i\neq j<n} (x_i \vee \boxdot x_j)\\
\wedge \bigwedge_{\substack{e<k\\ i<n}} \big((\Box x_i \to y_e) \wedge (x_i \vee \boxdot y_e)\big) \wedge \bigwedge_{i<n} \Big(x_i \wedge \bigwedge_{e<k} y_e \to \Box x_i\Big).
\end{aligned}$$

Lemma 4.31 *For any $t \in EC$ and a general frame W, the following are equivalent.*

(i) α_t is not valid in W.

(ii) There is $w \in W$ and a weak morphism from the generated subframe W_w to F_t.

Proof: We consider $t = \langle ⓚ, n\rangle$ with $n > 0$, the other cases are left to the reader.

(i) $\to$ (ii): Assume that $w \in W$, and $\vDash$ is an admissible valuation that makes $W_w \vDash \beta_{ⓚ,n}$ and $w \nvDash y_0$. Define a partial function f from W_w to $F_{ⓚ,n}$ by

$$
\begin{aligned}
f(u) = c_e \quad &\text{iff} \quad u \nvDash y_e, \\
f(u) = s_i \quad &\text{iff} \quad u \nvDash x_i.
\end{aligned}
\tag{17}
$$

The truth of $\beta_{ⓚ,n}$ (namely, the clauses $y_d \vee y_e$, $x_i \vee x_j$, and $x_i \vee y_e$) ensures that f is well-defined, and clearly $f^{-1}(v)$ is admissible in W_w for every $v \in F_{ⓚ,n}$, so condition (i) from Definition 4.30 is satisfied. Every u mapped to c_e sees some points mapped to each element of $F_{ⓚ,n}$ (using the conjuncts $\Box y_d \to y_e$ and $\Box x_i \to y_e$), which ensures condition (iii), and in view of $f(w) = c_0$, also the surjectivity of f. Points mapped to s_i can only see points mapped to s_i (due to $x_i \vee \Box x_j$ and $x_i \vee \Box y_e$), hence condition (ii) holds whenever $u' \in \mathrm{dom}(f)$. The last conjunct of $\beta_{ⓚ,n}$ implies that $u' \in \mathrm{dom}(f)$ whenever $u' < u$ is such that $f(u) = s_i$. Finally, if $u' < u$ and $f(u) = c_e$, then there is $u < u''$ such that $f(u'') = s_0$, hence $u' \in \mathrm{dom}(f)$ as well.

(ii) $\to$ (i): Let f be a weak morphism from W_w to $F_{ⓚ,n}$. Since f is onto, we may assume that $f(w) = c_0$. Define an admissible valuation in W_w by (17). Then by inspection $W_w \vDash \beta_{ⓚ,n}$, and $w \nvDash y_0$, hence $\alpha_{ⓚ,n}$ is not valid in W_w. Since W_w is a generated subframe of W, it is not valid in W either. $\Box$

Corollary 4.32 *If $t \preceq t'$, then $\mathbf{K4} \oplus \alpha_t$ proves $\alpha_{t'}$.* $\Box$

Remark 4.33 The axioms $\alpha_{C,n}$ are variants of Zakharyaschev's canonical formulas [3]. Using the refutation criterion from Lemma 4.31, one can show that $\alpha_{C,0}$ generates over $\mathbf{K4}$ the same logic as Zakharyaschev's $\alpha(C, \bot)$, and for $n > 0$, $\mathbf{K4} \oplus \alpha_{C,n}$ can be axiomatized by the $n + 1$ canonical formulas $\alpha(F_i, D)$ ($i \le n$), where F_i is the version of $F_{C,n}$ with i irreflexive and $n - i$ reflexive leaves, and D consists of all sets of leaves of size at least 2.

Lemma 4.34 *If $t \in EC$, a finite frame F validates α_t iff it has no rooted generated subframe of type $t' \succeq t$.*

Proof: If F_u has type $\langle D, m\rangle \succeq \langle C, n\rangle = t$, we can define a weak morphism from F_u to $F_{C,n}$ by fixing a surjection of $\mathrm{cl}(u)$ to $\mathrm{rcl}(F_{C,n})$, and picking n distinct immediate successor clusters of u, each of which is mapped to one leaf of $F_{C,n}$.

Conversely, let f be a weak morphism from F_u to $F_{C,n}$. Without loss of generality, we may assume that $\mathrm{cl}(u)$ is a $<$-maximal cluster intersecting $f^{-1}(c_0)$. Let $\langle D, m\rangle$ be the type of F_u. Since f is a p-morphism of $\mathrm{cl}(u)$ to C, we have $C \preceq D$. If $n = 0$, then $F_u = \mathrm{dom}(f) = \mathrm{cl}(u)$, hence $m = 0$. Otherwise, choose a $<$-minimal point $w_i > u$ such that $f(w_i) = s_i$ for each $i < n$. Since $\mathrm{dom}(f)$ is downward closed, w_i must be an immediate successor of u, and we have $w_i \not\le w_j$ for $i \ne j$, hence $m \ge n$. $\Box$

The following crucial lemma as well as the combinatorial principle in Lemma 4.36 are inspired by the proof of the exponential model property for cofinal-subframe logics by Zakharyaschev [26, Thm. 4.3].

Lemma 4.35 *If $U \subseteq EC$, then $\mathbf{K4} \oplus \{\alpha_t : t \in U\}$ has the finite model property.*

Proof: Put $L = \mathbf{K4} \oplus \{\alpha_t : t \in U\}$, and assume $\Gamma \nvdash_L \varphi$, hence there exists a descriptive L-frame W and an admissible valuation $\vDash$ in W such that $W \vDash \Gamma$ and $W \nvDash \varphi$. Let Σ be the set of all subformulas of $\Gamma \cup \{\varphi\}$, and $B = \{\varphi\} \cup \{\psi : \Box\psi \in \Sigma\}$. For any $u \in W$, we put

$$\mathrm{bd}(u) = \{\psi \in B : u \vDash \boxdot\psi\}.$$

Notice that $u \leq v$ implies $\mathrm{bd}(u) \subseteq \mathrm{bd}(v)$, and in particular, $u \sim v$ implies $\mathrm{bd}(u) = \mathrm{bd}(v)$. The set of *critical formulas of u* is

$$\mathrm{crit}(u) = \bigcap_{v \gnsim u} \mathrm{bd}(v) \smallsetminus \mathrm{bd}(u).$$

Clearly, $\mathrm{crit}(u) = \mathrm{crit}(v)$ if $u \sim v$ (hence we can also write $\mathrm{crit}(C)$ when C is a cluster), and $\mathrm{crit}(u) \cap \mathrm{crit}(v) = \varnothing$ if $u \lnsim v$. By Lemma 2.1,

$$B \smallsetminus \mathrm{bd}(u) = \bigcup_{v \geq u} \mathrm{crit}(v). \tag{18}$$

We are going to construct a finite subtree $T \subseteq \omega^{<\omega}$, a labelling of T by finite clusters $\{C_\sigma : \sigma \in T\}$ (including valuation of atoms) that together define a finite rooted model $F = \bigcup_\sigma C_\sigma$, and a mapping $f \colon F \to W$ such that

(i) a and $f(a)$ satisfy the same atoms,

(ii) $a < b$ implies $f(a) < f(b)$ (hence $a \sim b$ implies $f(a) \sim f(b)$),

(iii) $a \lnsim b$ implies $f(a) \lnsim f(b)$,

(iv) $f(a) \lnsim u$ implies $\mathrm{bd}(f(a)) \subsetneq \mathrm{bd}(u)$,

(v) either $C_\sigma = \{a_\sigma\}$ and $\mathrm{crit}(f(a_\sigma)) = \varnothing$, or $C_\sigma = \{a_{\sigma,\psi} : \psi \in \mathrm{crit}(f(C_\sigma))\}$ (hence $\mathrm{crit}(f(C_\sigma)) \neq \varnothing$) and $f(a_{\sigma,\psi}) \nvDash \psi$ for every $\psi \in \mathrm{crit}(f(C_\sigma))$

for every $a, b \in F$ and $u \in W$. Note that the elements $a_{\sigma,\psi}$ are not necessarily distinct. We denote by D_σ the cluster of W including $f(C_\sigma)$.

We build T and F from bottom up. First, we put the root ϵ in T, and we find a cluster $D_\epsilon \subseteq W$ such that $\varphi \in \mathrm{crit}(D_\epsilon)$ using (18).

Assume that $\sigma \in T$ and a cluster $D_\sigma \subseteq W$ has been defined in such a way that

$$\mathrm{bd}(D_\sigma) \subsetneq \mathrm{bd}(u) \text{ for every } u \gnsim D_\sigma. \tag{19}$$

If $\mathrm{crit}(D_\sigma) = \varnothing$, we pick any point $u_\sigma \in D_\sigma$, and we let $C_\sigma = \{a_\sigma\}$ be a copy of u_σ, putting $f(a_\sigma) = u_\sigma$. Otherwise, we find an $\subseteq$-minimal subset $\{u_{\sigma,i} : i < k_\sigma\} \subseteq D_\sigma$ such that every $\psi \in \mathrm{crit}(D_\sigma)$ is refuted in $u_{\sigma,i(\psi)}$ for some $i(\psi) < k_\sigma$. We make $C_\sigma = \{a_{\sigma,i} : i < k_\sigma\}$ a copy of $\{u_{\sigma,i} : i < k_\sigma\}$, and we put $f(a_{\sigma,i}) = u_{\sigma,i}$ and $a_{\sigma,\psi} = a_{\sigma,i(\psi)}$. Notice that the minimality of $\{u_{\sigma,i} : i < k_\sigma\}$ implies $C_\sigma = \{a_{\sigma,\psi} : \psi \in \mathrm{crit}(D_\sigma)\}$.

If D_σ is a maximal cluster of W, σ will be a leaf of T, and we put $n_\sigma = 0$. Otherwise, let $S_\sigma \neq \varnothing$ be the collection of all $\subseteq$-minimal sets in $\{\mathrm{bd}(u) : u \gnsim D_\sigma\}$. If $|S_\sigma| \geq 2$ and some $d \in S_\sigma$ includes $\bigcap\{d' \in S_\sigma : d' \neq d\}$, we remove d from S_σ. Continuing in the same way, we eventually obtain a subset $R_\sigma \subseteq S_\sigma$ such that either $|R_\sigma| = 1$, or

$$d \not\supseteq \bigcap_{\substack{d' \in R_\sigma \\ d' \neq d}} d' \tag{20}$$

for every $d \in R_\sigma$. We fix an enumeration $R_\sigma = \{d_{\sigma,i} : i < n_\sigma\}$. For every $i < n_\sigma$, we add $\sigma^\frown i$ into T, and we choose a $<$-maximal cluster $D_{\sigma^\frown i} \subseteq W$ such that $D_\sigma \lnsim D_{\sigma^\frown i}$ and $\mathrm{bd}(D_{\sigma^\frown i}) = d_{\sigma,i}$ using Lemma 2.1. Notice that (19) holds for $D_{\sigma^\frown i}$, hence we can carry on with the construction.

Assume that the construction has been completed. Since $\mathrm{bd}(D_\sigma) \subsetneq \mathrm{bd}(D_{\sigma^\frown i})$ for every $\sigma^\frown i \in T$, each $\sigma \in T$ has $|\sigma| \leq |B|$, and in particular, T and F are finite. Properties (i)–(v) are clearly satisfied. We claim $a \equiv_\Sigma f(a)$ for every $a \in F$. We will prove

$$F, a \vDash \psi \quad \text{iff} \quad W, f(a) \vDash \psi \qquad (a \in C_\sigma) \tag{21}$$

by outer top-down induction on $\sigma \in T$, and inner induction on the complexity of $\psi \in \Sigma$. The steps for atoms and Boolean connectives are trivial. If $f(a) \vDash \Box\psi$, then $f(b) \vDash \psi$ for every $b > a$ by (ii), hence $b \vDash \psi$ by the induction hypothesis. Thus, $a \vDash \Box\psi$. Conversely, assume that $f(a) \nvDash \Box\psi$. If there is $u \gnsim D_\sigma$ such that $u \nvDash \boxdot\psi$, i.e., $\psi \notin \mathrm{bd}(u)$, then $\psi \notin d$ for some $d \in S_\sigma$, hence also $\psi \notin d$ for some $d \in R_\sigma$. Putting $d_{\sigma,i} = d$, we have $D_{\sigma^\frown i} \nvDash \boxdot\psi$. If we pick any $b \in C_{\sigma^\frown i}$, then $f(b) \nvDash \psi$ or $f(b) \nvDash \Box\psi$, hence $b \nvDash \psi$ or $\Box\psi$ by the induction hypothesis for $\sigma^\frown i$, hence $a \nvDash \Box\psi$, as $a < b$. On the other hand, if $\psi \in \mathrm{bd}(u)$ for every $u \gnsim D_\sigma$, then D_σ (and C_σ) must be reflexive, and $\psi \in \mathrm{crit}(D_\sigma)$. By (v), we have $f(a_{\sigma,\psi}) \nvDash \psi$, hence $a_{\sigma,\psi} \nvDash \psi$ by the induction hypothesis for ψ, which implies $a \nvDash \Box\psi$ as $a \sim a_{\sigma,\psi}$.

In particular, (21) implies that $F \vDash \Gamma$ and $F, a_{\epsilon,\varphi} \nvDash \varphi$. It remains to show that F is an L-frame. Assume that $a \in C_\sigma$ has type $\langle C, n\rangle$, and let $W' = W_{f(a)}$. Notice that $|C| = k_\sigma$, and $n = n_\sigma$. By (19), D_σ is definable in W'. If $k_\sigma = 1$, we put $E_0 = D_\sigma$. Otherwise $D_\sigma \supseteq \{u_{\sigma,i} : i < k_\sigma\}$, and each $u_{\sigma,i}$ refutes a formula $\psi \in \mathrm{crit}(D_\sigma)$ that holds in all $u_{\sigma,j}$, $j \neq i$. Thus, we can partition D_σ into k_σ nonempty subsets E_i, $i < k_\sigma$, definable in W'.

If $n_\sigma = 0$, then $W' = D_\sigma$. Otherwise, let $A_i = \{u \in W' : \mathrm{bd}(u) = d_{\sigma,i}\}$ for every $i < n_\sigma$. Clearly, A_i is definable in W', disjoint from D_σ, and as $d_{\sigma,i}$ and $d_{\sigma,j}$ are incomparable for $i \neq j$, we have $u \not\leq v$ for any $u \in A_i$, $v \in A_j$. Moreover, $A_i \cup D_\sigma$ is downward closed: if $u \in W'$, $u < v \in A_i$, then $\mathrm{bd}(D_\sigma) \subseteq \mathrm{bd}(u) \subseteq d_{\sigma,i}$. Since $d_{\sigma,i} \in S_\sigma$, this means that either $\mathrm{bd}(u) = d_{\sigma,i}$, i.e., $u \in A_i$, or $\mathrm{bd}(u) = \mathrm{bd}(D_\sigma)$, i.e., $u \in D_\sigma$ by (19). It follows that the partial mapping

$$\begin{aligned} g(u) = c_i \quad &\text{iff} \quad u \in E_i, \\ g(u) = s_i \quad &\text{iff} \quad u \in A_i \end{aligned}$$

is a weak morphism of W' to $F_{C,n}$. Since W, and therefore W', is an L-frame, we must have $\langle C, n\rangle \not\leq t$ for every $t \in U$. Thus, F is an L-frame. $\Box$

Lemma 4.36 *Let T be a tree with root ϱ, labelled by finite sets $\{X_\sigma : \sigma \in T\}$. If $\sigma \in T$, and $\{\sigma_i : i < n\}$ is the set of all immediate successors of σ, we assume that*

(i) $X_{\sigma_i} \subsetneq X_\sigma$,

(ii) if $n > 1$, then $X_{\sigma_i} \nsubseteq \bigcup_{j \neq i} X_{\sigma_j}$

for every $i < n$. Then $|T| < 3 \cdot 2^{|X_\varrho|-1}$.

Proof: For every $\sigma \in T$, let T_σ be the subtree of T rooted at σ, and $m_\sigma = |X_\sigma|$. We will prove

$$|T_\sigma| < 3 \cdot 2^{m_\sigma - 1} \tag{22}$$

by induction on m_σ. Let $\{\sigma_i : i < n\}$ be the set of immediate successors of σ, and assume that (22) holds for every τ such that $m_\tau < m_\sigma$, and in particular, for every σ_i in view of (i).

If $n = 0$, then $|T_\sigma| = 1 < 3 \cdot 2^{m_\sigma - 1}$. If $n = 1$, then

$$|T_\sigma| = 1 + |T_{\sigma_0}| < 1 + 3 \cdot 2^{m_{\sigma_0} - 1} < 3 \cdot 2^{m_{\sigma_0}} \leq 3 \cdot 2^{m_\sigma - 1}$$

by the induction hypothesis, as $1 < 3/2$.

If $n \geq 2$, then for every $i < n$, there exists $x_i \in X_{\sigma_i} \subseteq X_\sigma$ such that $x_i \notin X_{\sigma_j}$ for every $j \neq i$ by (ii). This means $X_{\sigma_i} \subseteq X_\sigma \smallsetminus \{x_j : j \neq i\}$, hence $m_{\sigma_i} \leq m_\sigma - n + 1$. Also $m_{\sigma_i} > 0$, hence by the induction hypothesis, we obtain

$$|T_\sigma| = 1 + \sum_{i<n} |T_{\sigma_i}| \leq 1 + n\bigl(3 \cdot 2^{m_\sigma - n} - 1\bigr) \leq 1 + 2\bigl(3 \cdot 2^{m_\sigma - 2} - 1\bigr) = 3 \cdot 2^{m_\sigma - 1} - 1,$$

using the fact that $n(3 \cdot 2^{m_\sigma - n} - 1)$ is nonincreasing as a function of n. □

Example 4.37 Let $m \geq 1$, and T' be the full binary tree of height m. Label the root by an m-element set, and for every inner node, label its two immediate successors by two distinct subsets of its label of size smaller by one. Notice that nodes of depth d have labels of size d, in particular, the labels of the leaves are singletons. Let T be obtained from T' by adding a new leaf with empty label over every leaf of T'. Then T satisfies the assumptions of Lemma 4.36 with $|X_\varrho| = m$, and $|T| = 2^m - 1 + 2^{m-1} = 3 \cdot 2^{m-1} - 1$.

Theorem 4.38 *Let L be a clx logic.*

(i) L is finitely axiomatizable. Specifically,

$$L = \mathbf{K4} \oplus \{\alpha_t : t \in \mathrm{xcb}(L)\}.$$

(ii) L has an exponential-size model property: if $\Gamma \nvdash_L \varphi$, where $\Gamma \cup \{\Box\varphi\}$ has b boxed subformulas, then $\Gamma \mathbin{/} \varphi$ can be refuted in a rooted L-frame that is a tree of clusters of total size $< 3 \cdot 2^{b-1}$, depth $\leq b + 1$, cluster size $\leq b$, and branching $\leq \max\{b - 1, 1\}$.

Proof:

(i): The logic $L' = \mathbf{K4} \oplus \{\alpha_t : t \in \mathrm{xcb}(L)\}$ has the same finite frames as L by Lemma 4.34, and enjoys fmp by Lemma 4.35, hence $L = L'$.

(ii): If $\Gamma \nvdash_L \varphi$, then $\Gamma \nvdash_{L'} \varphi$. Consider the finite L'-model $F \vDash \Gamma$, $F \nvDash \varphi$, constructed in the proof of Lemma 4.35. As already observed there, F has depth at most $|B| + 1 = b + 1$. By property (v) of f, F has cluster size at most b. If C_σ has $n_\sigma > 1$ immediate successor clusters, then (20) implies that for every $i < n_\sigma$, we can choose $\psi_i \in B$ such that $\psi_i \notin d_{\sigma,i}$, and $\psi_i \in d_{\sigma,j}$ for every $j \neq i$. Moreover, since $\varphi \in \mathrm{crit}(D_\epsilon)$, we have $\psi_i \in B \smallsetminus \{\varphi\}$ for every i. Thus, $n_\sigma \leq b - 1$.

We will bound $|F|$ using Lemma 4.36. If $C_\sigma = \{a_\sigma\}$, we label σ with $X_\sigma := B \smallsetminus \mathrm{bd}(D_\sigma)$. Otherwise, $k_\sigma = |C_\sigma| \leq |\mathrm{crit}(D_\sigma)|$. We first "linearize C_σ" by replacing σ with a chain $\{\sigma^i : i < k_\sigma\}$, where σ^{i+1} is a successor of σ^i, and we choose labels X_{σ^i} so that $B \smallsetminus \mathrm{bd}(D_\sigma) = X_{\sigma^0} \supsetneq X_{\sigma^1} \supsetneq \cdots \supsetneq X_{\sigma^{k_\sigma - 1}} \supsetneq B \smallsetminus \bigcap_{u \gneqq D_\sigma} \mathrm{bd}(u) = B \smallsetminus (\mathrm{bd}(D_\sigma) \cup \mathrm{crit}(D_\sigma))$. In this way, we obtain a tree T' of the same size as F, and its labelling satisfies the assumptions of Lemma 4.36 due to (20). Thus, $|F| = |T'| < 3 \cdot 2^{|X_\epsilon| - 1} \leq 3 \cdot 2^{b-1}$. □

The fact that the bounds in Theorem 4.38 depend only on the number of boxed subformulas and not on the overall size of $\Gamma \cup \{\varphi\}$ will be important in the proof of Theorem 5.23.

Corollary 4.39 *If S is a set of clx logics, the join of S in* $\mathrm{NExt}\,\mathbf{K4}$ *is also a clx logic.*

Proof: By Theorem 4.38, $L' = \bigvee S$ can be axiomatized by $\{\alpha_t : t \in U\}$, where $U = \bigcup_{L \in S} \mathrm{xcb}(L)$. Thus, $L' = \mathbf{Clx}_{U'}$ using Theorem 4.38 again, where U' is the upward closure of U. □

Corollary 4.40 *If L is a clx logic, then L and $\vdash_L$ are decidable, and given any formula, we can compute its projective approximation and a complete set of unifiers.* □

5 Extension rules

We now proceed to the main part of this paper, namely the construction of bases of admissible rules for Par-extensible logics, and their semantic description. We introduce certain rules related to extension conditions, called extension rules, and we show that their validity in nice (i.e., descriptive or Kripke) parametric frames corresponds to the existence of (a suitable version of) tight predecessors for finite sets of points in the frame (Theorems 5.3 and 5.4). As a consequence, we obtain a characterization of Par-extensible logics as those that admit appropriate sets of extension rules (Theorem 5.14). We prove that consequence relations axiomatized by extension rules are complete with respect to locally finite Kripke frames (Theorem 5.16), and with the help of the description of projective formulas from Section 3, we derive our main result (Theorem 5.17) stating that Par-extensible logics have bases of admissible rules consisting of extension rules, and that the consequence relation $\vdash_L$ is sound and complete with respect to frames having enough tight predecessors. Such frames are nearly always infinite, which may be sometimes inconvenient; for this reason, we also give a description of admissible rules in terms of suitable finite (in fact, exponentially bounded) models (Theorems 5.22 and 5.23).

While the natural form of extension rules has multiple conclusions, we indicate in Section 5.1 how to turn them into single-conclusion rules providing bases of single-conclusion rules for Par-extensible logics. Further properties of bases are investigated in Section 5.2: we show how to modify extension rules so as to obtain independent bases of admissible rules for finite sets of parameters, and we characterize which Par-extensible logics have finite bases.

We start by defining our rules. The irreflexive case is a straightforward modification of the parameter-free rules given in [14, 17], but the reflexive case is more peculiar.

Definition 5.1 Let $n \in \omega$, $P \subseteq \mathrm{Par}$ be finite, and $e \in \mathbf{2}^P$. (If Par itself is finite, it suffices to consider $P = \mathrm{Par}$.) The *irreflexive extension rule* $\mathrm{Ext}_{\bullet,n,\{e\}}$ is

$$P^e \wedge \Box y \to \bigvee_{i<n} \Box x_i \Big/ \{\boxdot y \to x_i : i < n\}.$$

Let $\mathrm{Ext}^{\mathrm{Par}}_{\bullet,n}$ denote the set of all rules of the form $\mathrm{Ext}_{\bullet,n,\{e\}}$.

If n and P are as above, $E \subseteq \mathbf{2}^P$, and $e_0 \in E$, the *reflexive extension rule* $\mathrm{Ext}_{\circ,n,E,e_0}$ is

$$P^{e_0} \wedge \boxdot\Big(y \to \bigvee_{e\in E} \Box(P^e \to y)\Big) \wedge \bigwedge_{e\in E} \boxdot\big(\Box(P^e \to \Box y) \to y\big) \to \bigvee_{i<n} \Box x_i \Big/ \{\boxdot y \to x_i : i < n\}.$$

Let $\mathrm{Ext}_{\circ,n,E} = \{\mathrm{Ext}_{\circ,n,E,e_0} : e_0 \in E\}$. If $k \in \omega \smallsetminus \{0\}$, let $\mathrm{Ext}^{\mathrm{Par}}_{\textcircled{k},n}$ denote the set of all rules of the form $\mathrm{Ext}_{\circ,n,E,e_0}$, where $|E| \leq k$.

We remark that the P^{e_0} conjunct in the reflexive rules can be dropped for $n \neq 1$, as long as the corresponding rules for $n = 1$ (with the conjunct) are present, cf. Remark 5.11 and Lemma 5.36. However, one can check that other elements of the definition are essential, using variants of the construction from the proof of Lemma 5.36.

The semantics of extension rules will be given in terms of the notion of tight predecessor defined below. The main difference from the parameter-free case is that the predecessors are no longer just singletons, we also need to take care of proper clusters whose individual points are distinguishable by a valuation of parameters.

Definition 5.2 Let W be a parametric general frame, $P \subseteq \mathrm{Par}$ finite, $n \in \omega$, and $X = \{w_i : i < n\} \subseteq W$ (where the w_i are not necessarily distinct).

If $e \in \mathbf{2}^P$, a *tight* $\langle\bullet,\{e\}\rangle$*-predecessor* ($\langle\bullet,\{e\}\rangle$*-tp*) of X is $\{u\} \subseteq W$ such that

$$W, u \vDash P^e, \qquad u{\uparrow} = X{\updownarrow}.$$

Tight $\langle\bullet,\{e\}\rangle$-predecessors are also collectively called *irreflexive tight predecessors.* (However, notice that when v is a reflexive smallest element of X satisfying P^e, then $\{v\}$ is a $\langle\bullet,\{e\}\rangle$-tp of X, despite not being irreflexive.) W is $\langle\bullet,n,\{e\}\rangle$*-extensible* if every $\{w_i : i < n\} \subseteq W$ has a $\langle\bullet,\{e\}\rangle$-tp, and it is $\langle\bullet,n\rangle$*-extensible* if it is $\langle\bullet,n,\{e\}\rangle$-extensible for every finite P and $e \in \mathbf{2}^P$.

Similarly, if $E \subseteq \mathbf{2}^P$, $E \neq \varnothing$, then a *tight* $\langle\circ,E\rangle$*-predecessor* ($\langle\circ,E\rangle$*-tp*) of X is $\{u_e : e \in E\} \subseteq W$ such that

$$W, u_e \vDash P^e, \qquad u_e{\uparrow} = X{\updownarrow} \cup \{u_f : f \in E\}.$$

Any tight $\langle \circ, E\rangle$-predecessor is also called a *reflexive tight predecessor.* (Again, a $\langle \circ, E\rangle$-tp may be included in $X{\uparrow}$ when X has a reflexive smallest element whose cluster realizes every $e \in E$.) W is $\langle \circ, n, E\rangle$-*extensible* if every $\{w_i : i < n\} \subseteq W$ has a $\langle \circ, E\rangle$-tp, and it is $\langle \textcircled{k}, n\rangle$-*extensible* if it is $\langle \circ, n, E\rangle$-extensible for every finite P and $E \subseteq \mathbf{2}^P$ such that $0 < |E| \leq k$.

As already mentioned at the beginning of this section, our basic technical tool will be the correspondence of extension rules to extensible parametric frames. While this is straightforward to prove for irreflexive rules, the reflexive case is more intricate, hence we start with the former.

Theorem 5.3 *Let $P \subseteq \mathrm{Par}$ be finite, W a parametric general frame, $n \in \omega$, and $e \in \mathbf{2}^P$.*

(i) If W is $\langle \bullet, n, \{e\}\rangle$-extensible, then $W \vDash \mathrm{Ext}_{\bullet,n,\{e\}}$.

(ii) If W is a descriptive or Kripke frame, and $W \vDash \mathrm{Ext}_{\bullet,n,\{e\}}$, then W is $\langle \bullet, n, \{e\}\rangle$-extensible.

Proof: (i): Let $\vDash$ be an admissible valuation in W that refutes the conclusion of $\mathrm{Ext}_{\bullet,n,\{e\}}$. Pick $w_i \in W$ such that $w_i \vDash \boxdot y \land \neg x_i$, and let $\{u\}$ be a $\langle \bullet, \{e\}\rangle$-tp of $\{w_i : i < n\}$. Then u refutes the premise of $\mathrm{Ext}_{\bullet,n,\{e\}}$.

(ii): Assume first that $W \vDash \mathrm{Ext}_{\bullet,n,\{e\}}$ is a Kripke frame. Given $X = \{w_i : i < n\} \subseteq W$, define a valuation in W by

$$\begin{aligned} u \vDash x_i \quad &\text{iff} \quad u \neq w_i, \\ u \vDash y \quad &\text{iff} \quad u \in X\underline{\uparrow}. \end{aligned}$$

This valuation refutes the conclusion of $\mathrm{Ext}_{\bullet,n,\{e\}}$ as $w_i \nvDash \boxdot y \to x_i$, hence there exists $u \in W$ such that

$$u \vDash P^e \land \Box y \land \bigwedge_{i<n} \Diamond \neg x_i.$$

Thus, $\{u\}$ is a $\langle \bullet, \{e\}\rangle$-tp of X.

Now, let $\langle W, <, A, \vDash_p\rangle \vDash \mathrm{Ext}_{\bullet,n,\{e\}}$ be a descriptive parametric frame, and $X = \{w_i : i < n\} \subseteq W$. We will write P^e for the set of points of W where the formula P^e holds. We claim that the set

$$U = \{P^e\} \cup \{\Box B : B \in A, X \subseteq \boxdot B\} \cup \{\Diamond C : C \in A, C \cap X \neq \varnothing\}$$

has fip. Indeed, assume $X \subseteq \boxdot B_j$, $w_i \in C_{i,j}$, $j < m$. Let $\vDash$ be the valuation that makes x_i true on $W \smallsetminus \bigcap_j C_{i,j}$, and y true on $\bigcap_j B_j$. We have $w_i \nvDash \boxdot y \to x_i$, hence by $\mathrm{Ext}_{\bullet,n,\{e\}}$, there is u such that $u \vDash P^e \land \Box y \land \bigwedge_{i<n} \Diamond \neg x_i$. Then $u \in P^e \cap \bigcap_j \Box B_j \cap \bigcap_{i,j} \Diamond C_{i,j}$.

Since W is compact, there is a point $u \in \bigcap U$. Clearly, $u \vDash P^e$. Since $u \in \Diamond C$ for every $C \in A$ such that $w_i \in C$, and W is refined, we have $u < w_i$, thus $u{\uparrow} \supseteq X\underline{\uparrow}$. On the other hand, if $v \notin X\underline{\uparrow}$, we can find $B_i, B_i' \in A$ such that $w_i \in B_i \cap \Box B_i'$, $v \notin B_i \cup B_i'$ using the refinedness of W. Putting $B = \bigcup_i (B_i \cup B_i')$, we have $X \subseteq \boxdot B$, hence $u \in \Box B$. However, $v \notin B$, hence $u \not< v$. Thus, $\{u\}$ is a $\langle \bullet, \{e\}\rangle$-tp of X. $\Box$

Theorem 5.4 *Let $P \subseteq \mathrm{Par}$ be finite, W a parametric general frame, $n \in \omega$, and $E \subseteq \mathbf{2}^P$, $E \neq \varnothing$.*

(i) If W is $\langle \circ, n, E\rangle$-extensible, then $W \vDash \mathrm{Ext}_{\circ,n,E}$.

(ii) If W is a descriptive or Kripke frame, and $W \vDash \mathrm{Ext}_{\circ,n,E}$, then W is $\langle \circ, n, E\rangle$-extensible.

Proof: (i): Let $e_0 \in E$, and $\vDash$ be an admissible valuation in W that refutes the conclusion of $\mathrm{Ext}_{\circ,n,E,e_0}$. Pick $w_i \in W$ such that $w_i \vDash \boxdot y \wedge \neg x_i$, and let $\{u_e : e \in E\}$ be a $\langle \circ, E\rangle$-tp of $X = \{w_i : i < n\}$. Then u_{e_0} refutes the premise of $\mathrm{Ext}_{\circ,n,E,e_0}$: clearly $u_{e_0} \vDash P^{e_0}$, and $u_{e_0} \vDash \bigwedge_i \neg\Box x_i$ as $u_{e_0} < w_i$. If $u_{e_0} \leq v \vDash y$, then either $v \in X{\uparrow}$, in which case $v \vDash \Box y$, or $v = u_e$ for some $e \in E$, in which case $v \vDash \Box(P^e \to y)$. If $v \geq u$ and $v \nvDash y$, then $v \notin X{\uparrow}$, thus for every $e \in E$, $v < u_e \vDash P^e \wedge \neg\Box y$ (as $u_e < v$), hence $v \nvDash \Box(P^e \to \Box y)$.

(ii): The proof is a bit involved, hence we defer it to Lemmas 5.7 and 5.10 below. $\Box$

Corollary 5.5 *Let W be a parametric frame, $n \in \omega$, and C a cluster type. Then $W \vDash \mathrm{Ext}^{\mathrm{Par}}_{C,n}$ if W is $\langle C, n\rangle$-extensible. If W is a descriptive or Kripke frame, the converse implication also holds.* $\Box$

We begin the proof of Theorem 5.4 (ii) with the Kripke case. We will need the following variant of Katětov's lemma on three sets [20], whose proof we include for completeness.

Lemma 5.6 *Let $f\colon A \to A$ be a function such that $f^k(x) \neq x$ for every $x \in A$ and odd k. Then we can partition A into disjoint sets A_0 and A_1 such that $f(A_i) \subseteq A_{1-i}$, $i = 0, 1$.*

Proof: For $x, y \in A$, we write $x \approx y$ if $f^n(x) = f^m(y)$ for some $n, m \in \omega$, and $x \sim y$ if in addition $n \equiv m \pmod 2$. It is easy to see that $\approx$ and $\sim$ are equivalence relations, and $x \approx y$ iff $x \sim y$ or $f(x) \sim y$. On the other hand, $x \nsim f(x)$, as otherwise $f^n(x) = f^k(f^n(x))$ for some n and an odd k. Let $B \subseteq A$ be a set containing one point in each equivalence class of $\approx$, and put $A_0 = \{x \in A : \exists y \in B\, (x \sim y)\}$, $A_1 = A \smallsetminus A_0 = \{x \in A : \exists y \in B\, (f(x) \sim y)\}$. Then the properties of $\approx$ and $\sim$ ensure $f(A_0) \subseteq A_1$ and $f(A_1) \subseteq A_0$. $\Box$

Lemma 5.7 *Let $P \subseteq \mathrm{Par}$ be finite, $n \in \omega$, and $E \subseteq \mathbf{2}^P$, $E \neq \varnothing$. If W is a Kripke frame such that $W \vDash \mathrm{Ext}_{\circ,n,E}$, then W is $\langle \circ, n, E\rangle$-extensible.*

Proof: Assume that $X = \{w_i : i < n\} \subseteq W$ does not have a $\langle \circ, E\rangle$-tp, we will show that $W \nvDash \mathrm{Ext}_{\circ,n,E,e_0}$ for some $e_0 \in E$. If there is $i_0 < n$ such that $w_{i_0} < w_i$ for every $i < n$ (including itself), then $\mathrm{cl}(w_{i_0})$ contains a $\langle \circ, E\rangle$-tp of X unless some $e \in E$ is not realized in $\mathrm{cl}(w_{i_0})$, and we choose e_0 to be one such e. Otherwise, we can take an arbitrary $e_0 \in E$.

For every $i < n$ and $u \in W$, we define

$$u \vDash x_i \quad \text{iff} \quad u \neq w_i.$$

Put $S = W \smallsetminus X{\uparrow}$. Let us say that an *$E$-cluster* is a reflexive cluster $C \subseteq W$ such that $\mathrm{Sat}_P(u) \in E$ for every $u \in C$, and conversely, for every $e \in E$ there is a unique $u \in C$ such that $\mathrm{Sat}_P(u) = e$. We also consider the condition

$$\forall e \in E\, \forall v \in u{\uparrow} \cap S\, \exists w \in v{\uparrow} \cap S\, w \vDash P^e \tag{23}$$

on $u \in W$. Notice that a point u in a $<$-maximal cluster of S satisfies (23) iff it is reflexive and for every $e \in E$, $\mathrm{cl}(u)$ includes a point realizing P^e (in particular, this holds if it is an E-cluster). We define a valuation of y by cases:

- If $u \in X{\updownarrow}$, we put $u \vDash y$.
- If $u \in S$, and (23) does not hold, then $u \nvDash y$.
- If C is an E-cluster that is a $<$-maximal cluster of S, we put $u \nvDash y$ for every $u \in C$.
- Let C be a $<$-maximal cluster of S that satisfies (23), but is not an E-cluster. Then there is a point $v \in C$ such that $\mathrm{Sat}_P(v) \notin E$, or a point $v \in C$ such that $v \equiv_P v'$ for some $v' \in C$, $v' \neq v$. We pick one such v, and make $v \vDash y$, $u \nvDash y$ for $u \in C \smallsetminus \{v\}$.
- If $u \in S$ satisfies (23), sees a maximal cluster in S, but is not itself in such a maximal cluster, then $u \vDash y$.
- Let T be the set of all $u \in S$ that satisfy (23), but do not see any maximal cluster of S, and for every $e \in E$, let $T^e = \{u \in T : u \vDash P^e\}$. Since condition (23) is preserved upwards in S, for every $u \in T$ there is $v \gtrsim u$, $v \in T$, which in turn sees some $w \in T^e$ for any $e \in E$. Thus, we can choose a function $f_e \colon T^e \to T^e$ such that $u \lnsim f_e(u)$ for every $u \in T^e$. Since f_e is strictly increasing, it is cycle-free, hence by Lemma 5.6, we can write T^e as a disjoint union $T^e_0 \dot\cup T^e_1$ such that $f_e(T^e_i) \subseteq T^e_{1-i}$, $i = 0, 1$. We put $u \vDash y$ for $u \in T^e_0$, and $u \nvDash y$ for $u \in T^e_1$. Satisfaction of y in $T \smallsetminus \bigcup_{e\in E} T^e$ is arbitrary.

Claim 1 *Let $u \in S$ satisfy* (23)*.*

(i) For every $e \in E$, there is $v \in S$ such that $u < v \vDash P^e \wedge \neg y$.

(ii) If $\mathrm{cl}(u)$ is not an E-cluster maximal in S, then there is $v \in S$ such that $u \leq v \vDash y$.

Proof: (i): If u sees a maximal cluster C of S, then C is reflexive and contains $v \vDash P^e$ by (23). We have $v \nvDash y$, unless C contains another point $v' \vDash P^e$, which then does not satisfy y.

If $u \in T$, there is $v \in S$ such that $u < v \vDash P^e$ by (23). We have $v \in T^e$, hence either $v' = v$ or $v' = f_e(v) > v$ belongs to T^e_1, i.e., $v' \vDash P^e \wedge \neg y$.

(ii): If $\mathrm{cl}(u)$ is a maximal cluster of S, then $\mathrm{cl}(u)$ is not an E-cluster, hence there is $v \in \mathrm{cl}(u)$ such that $v \vDash y$.

If $\mathrm{cl}(u)$ sees a maximal cluster of S, but is not a maximal cluster itself, then $u \vDash y$.

Otherwise, $u \in T$. Fix $e \in E$. As above, there is $v \in T^e$, $v > u$, and either $v' = v$ or $v' = f_e(v) > v$ is an element of $T^e \subseteq S$ satisfying y. □ (Claim 1)

Clearly, $w_i \nvDash \boxdot y \to x_i$, we will show

$$u \vDash P^{e_0} \wedge \boxdot\Big(y \to \bigvee_{e\in E} \Box(P^e \to y)\Big) \wedge \bigwedge_{e\in E} \boxdot\big(\Box(P^e \to \Box y) \to y\big) \to \bigvee_{i<n} \Box x_i$$

for every $u \in W$. We distinguish several cases:

- Let $u \in X\updownarrow$. If $u \not< w_i$ for some $i < n$, then $u \vDash \Box x_i$. Otherwise $u \sim w_{i_0}$, and $u \nvDash P^{e_0}$ by the choice of e_0.
- If $u \in S$ does not satisfy (23), let $e \in E$ and $v \geq u$, $v \in S$ be such that $w \nvDash P^e$ for every $w > v$, $w \in S$. Then $v \vDash \Box(P^e \to \Box y)$ as y holds outside S, but $v \nvDash y$, hence $u \nvDash \boxdot\big(\Box(P^e \to \Box y) \to y\big)$.
- If $\mathrm{cl}(u)$ is an E-cluster maximal in S, then $u \not< w_i$ for some $i < n$, as otherwise $\mathrm{cl}(u)$ would be a $\langle \circ, E\rangle$-tp of X. Consequently, $u \vDash \Box x_i$.
- In other cases, Claim 1 gives a $v \in S$ such that $u \leq v \vDash y$. Also, for every $e \in E$, there is $w \in S$ such that $v < w \vDash P^e \wedge \neg y$, hence $v \vDash \bigwedge_{e \in E} \neg\Box(P^e \to y)$, and $u \nvDash \boxdot\big(y \to \bigvee_{e\in E} \Box(P^e \to y)\big)$.

Thus, $\mathrm{Ext}_{\circ,n,E,e_0}$ is refuted in W. □

In order to prove a similar characterization for descriptive frames, we will need a variant of $\mathrm{Ext}_{\circ,n,E}$ with multiple y's (see Corollary 5.9). It is easy to see that such a rule is valid in any $\langle \circ, n, E\rangle$-extensible frame, which suggests that it is indeed equivalent to $\mathrm{Ext}_{\circ,n,E}$, but we cannot infer this directly from Lemma 5.7, as we do not a priori know that $\mathbf{K4} + \mathrm{Ext}_{\circ,n,E}$ is Kripke complete. We could try to take a frame where the variant rule fails (and thus some $\langle \circ, E\rangle$-tp is missing), and mimic the proof of Lemma 5.7 to produce a valuation refuting the original extension rule. However, we made various non-definable choices in the construction of the valuation in Lemma 5.7, and this is the main obstacle we need to overcome. We will handle this by means of the following lemma, whose parameter-free special case appeared in [17].

Lemma 5.8 *Let $m \in \omega$, $P \subseteq \mathrm{Par}$ be finite, and $E \subseteq \mathbf{2}^P$. Then there exists a formula $\alpha(P, y_0, \dots, y_{m-1})$ such that* $\mathbf{K4}$ *proves*

$$\bigwedge_{j<m} \boxdot y_j \to \alpha,$$

and

$$\boxdot\Big(\alpha \to \bigvee_{e\in E} \Box(P^e \to \alpha)\Big) \wedge \bigwedge_{e \in E} \boxdot\big(\Box(P^e \to \Box\alpha) \to \alpha\big)$$
$$\to \boxdot\Big(y_j \to \bigvee_{e\in E} \Box(P^e \to y_j)\Big) \wedge \bigwedge_{e\in E} \boxdot\big(\Box(P^e \to \Box y_j) \to y_j\big)$$

for every $j < m$.

Proof: If $E = \varnothing$, we can take $\alpha = \top$, hence we may assume $E \neq \varnothing$. Put

$$\beta = \bigwedge_{j<m} \boxdot y_j,$$
$$\gamma^e_j = \boxdot(P^e \to y_j) \vee \boxdot(P^e \wedge y_j \to \beta), \qquad j < m, e \in E,$$
$$\alpha = \beta \vee \bigwedge_{e\in E}\Big[\boxdot\big(\Box(P^e \to \beta) \to \beta\big) \wedge \Big(P^e \to \bigvee_{j<m}\Big[\bigwedge_{i<j} \gamma^e_i \wedge \neg\gamma^e_j \wedge \big(\Box(\gamma^e_j \to \beta) \to y_j\big)\Big]\Big)\Big].$$

Clearly, $\vdash \beta \to \alpha$.

Claim 1 **K4** *proves*

(*i*) $\neg\beta \wedge \bigwedge_{e\in E} \boxdot\big(\Box(P^e \to \beta) \to \beta\big) \to \bigwedge_{e\in E} \Diamond(P^e \wedge \neg\alpha)$,

(*ii*) $\boxdot\alpha \to \beta$.

Proof: (i): Assume $F \in \mathrm{Mod}_{\mathbf{K4}}$, $u \in F$, and $u \vDash \neg\beta \wedge \bigwedge_e \boxdot\big(\Box(P^e \to \beta) \to \beta\big)$. Fix $e \in E$, and let $v \geq u$ be maximal such that $v \nvDash \beta$. Since $v \vDash \Box(P^e \to \beta) \to \beta$, we have $v \vDash \Diamond(P^e \wedge \neg\beta)$, hence v is reflexive, and P^e is realized in some $v_e \sim v$. If $\mathrm{cl}(v) \vDash \gamma^e_j$ for every $j < m$, then $v_e \nvDash \alpha$. Otherwise let j be minimal such that $\mathrm{cl}(v) \nvDash \gamma^e_j$. Since $v \nvDash \boxdot(P^e \to y_j)$, we may assume without loss of generality $v_e \nvDash y_j$. Since also $\mathrm{cl}(v) \vDash \Box(\gamma^e_j \to \beta)$, we have $v_e \nvDash \alpha$. Either way, $u \vDash \Diamond(P^e \wedge \neg\alpha)$.

(ii): Let $u \vDash \alpha \wedge \neg\beta$. Then $u \vDash \bigwedge_{e\in E} \boxdot\big(\Box(P^e \to \beta) \to \beta\big)$, hence $u \vDash \Diamond(P^e \wedge \neg\alpha)$ for any $e \in E$ by (i), in particular $u \nvDash \Box\alpha$. $\Box$ (Claim 1)

Now, let $u \in F \in \mathrm{Mod}_L$, and assume

$$u \vDash \boxdot\Big(\alpha \to \bigvee_{e\in E} \Box(P^e \to \alpha)\Big) \wedge \bigwedge_{e\in E} \boxdot\big(\Box(P^e \to \Box\alpha) \to \alpha\big), \tag{24}$$

we have to show

$$u \vDash \boxdot\Big(y_j \to \bigvee_{e\in E} \Box(P^e \to y_j)\Big) \wedge \bigwedge_{e\in E} \boxdot\big(\Box(P^e \to \Box y_j) \to y_j\big) \tag{25}$$

for every $j < m$. If $u \vDash \beta$, then (25) follows, hence we may assume $u \nvDash \beta$. If $v \vDash \Box(P^e \to \beta) \wedge \neg\beta$ for some $v \geq u$ and $e \in E$, then $v \vDash \Box(P^e \to \Box\alpha)$. On the other hand, Claim 1 implies $v \nvDash \boxdot\alpha$, contradicting (24). Thus,

$$u \vDash \boxdot\big(\Box(P^e \to \beta) \to \beta\big) \qquad (e \in E). \tag{26}$$

Consequently, if $v \geq u$ is such that $v \vDash \alpha \wedge \neg\beta$, then $v \vDash \bigwedge_{e\in E} \Diamond(P^e \wedge \neg\alpha)$ by Claim 1, which again contradicts (24). Thus,

$$u \vDash \boxdot(\alpha \leftrightarrow \beta). \tag{27}$$

Also, (26) implies that any $v \geq u$, $v \vDash \bigwedge_{e\in E} \neg P^e$, satisfies α, hence

$$u \vDash \boxdot\Big(\beta \vee \bigvee_{e\in E} P^e\Big). \tag{28}$$

We claim

$$u \vDash \boxdot(P^e \to \gamma^e_j) \qquad (e \in E, j < m). \tag{29}$$

Assume for contradiction that $v \geq u$ satisfies $P^e \wedge \neg\gamma^e_j$ for some j. W.l.o.g., v is maximal with this property, and $v \vDash \bigwedge_{i<j} \gamma^e_i$. In particular, $v \vDash \diamonddot(P^e \wedge \neg y_j)$, hence $v \nvDash \beta$. We obtain $v \nvDash \alpha$ by (27), which is only possible if $v \vDash \Box(\gamma^e_j \to \beta) \wedge \neg y_j$. By the maximality of v and (26), this

implies $w \vDash \beta$ for every $w \gtrsim v$. Since $v \nvDash \gamma_j^e$, there is $w \geq v$, $w \vDash P^e \wedge \neg\beta \wedge y_j$. We must have $w \sim v$. But then $w \vDash \alpha$, hence $w \vDash \beta$, a contradiction.

Now we can complete the proof of (25). Let $v \geq u$ be such that $v \vDash y_j$. If $v \vDash \beta$, then $v \vDash \Box(P^e \to y_j)$ for any $e \in E$, hence we can assume $v \nvDash \beta$. Then $v \vDash P^e$ for some $e \in E$ by (28), hence $v \vDash \gamma_j^e$ by (29). Since $v \nvDash P^e \wedge y_j \to \beta$, we must have $v \vDash \boxdot(P^e \to y_j)$.

Let $e \in E$ and $v \geq u$ be such that $v \nvDash y_j$. By (28), there is $f \in E$ such that $v \vDash P^f$, and we have $v \vDash \gamma_j^f$ by (29), which means $v \vDash \boxdot(P^f \wedge y_j \to \beta)$. By (26), there is $w > v$ such that $w \vDash P^e \wedge \neg\beta$, and $z > w$ such that $z \vDash P^f \wedge \neg\beta$. Then $z \vDash \neg y_j$, hence $w \nvDash \Box y_j$, and $v \nvDash \Box(P^e \to \Box y_j)$. $\Box$

Corollary 5.9 *If $n, m \in \omega$, $P \subseteq \mathrm{Par}$ is finite, and $e_0 \in E \subseteq \mathbf{2}^P$, then $\mathbf{K4}+\mathrm{Ext}_{\circ,n,E,e_0}$ proves the rule $\mathrm{Ext}^m_{\circ,n,E,e_0}$:*

$$\frac{P^{e_0} \wedge \bigwedge_{j<m} \boxdot\Big(y_j \to \bigvee_{e\in E} \Box(P^e \to y_j)\Big) \wedge \bigwedge_{\substack{j<m\\ e\in E}} \boxdot\big(\Box(P^e \to \Box y_j) \to y_j\big) \to \bigvee_{i<n} \boxdot x_i}{\Big\{\bigwedge_{j<m} \boxdot y_j \to x_i : i < n\Big\}}.$$

$\Box$

Lemma 5.10 *Let $P \subseteq \mathrm{Par}$ be finite, $n \in \omega$, and $E \subseteq \mathbf{2}^P$, $E \neq \varnothing$. If W is a descriptive frame such that $W \vDash \mathrm{Ext}_{\circ,n,E}$, then W is $\langle \circ, n, E\rangle$-extensible.*

Proof: Let A be the algebra of admissible sets of W. Let $X = \{w_i : i < n\} \subseteq W$, we have to find a $\langle \circ, E\rangle$-tp of X.

Assume there is $i_0 < n$ such that $w_{i_0} < w_i$ for every i. If $\mathrm{cl}(w_{i_0})$ realizes every $e \in E$, then there is a $\langle \circ, E\rangle$-tp of X included in $\mathrm{cl}(w_{i_0})$. Otherwise, we can fix $e_0 \in E$ not realized in $\mathrm{cl}(w_{i_0})$. If there is no such i_0, we let $e_0 \in E$ be arbitrary.

As in Theorem 5.3, we identify P^{e_0} with the set of points of W where it is satisfied. We also use connectives to denote the corresponding operations on sets from A. By Corollary 5.9, $W \vDash \mathrm{Ext}^m_{\circ,n,E,e_0}$ for every $m \in \omega$, hence the set

$$\begin{aligned} U = {} & \{P^{e_0}\} \cup \{\Diamond B : B \in A, B \cap X \neq \varnothing\} \\ & \cup \Big\{\boxdot\Big(C \to \bigvee_{e\in E} \Box(P^e \to C)\Big) \wedge \bigwedge_{e\in E} \boxdot\big(\Box(P^e \to \Box C) \to C\big) : C \in A, C \supseteq X\!\uparrow\Big\} \end{aligned}$$

has fip, and there is $u_{e_0} \in \bigcap U$ as W is compact. We have $u_{e_0} \vDash P^{e_0}$ and $u_{e_0} < w_i$ for every $i < n$. The choice of e_0 ensures that $u_{e_0} \notin X\!\uparrow$.

Claim 1 *Let $u_{e_0} \leq u \notin X\!\uparrow$.*

- (i) *For every $e \in E$, there is $u < v < u$ such that $v \vDash P^e$. In particular, u is reflexive.*
- (ii) *Putting $e = \mathrm{Sat}_P(u)$, we have $e \in E$, and there is no $u \leq v \notin X\!\uparrow$ such that $v \vDash P^e$, other than u itself.*

Proof: Since W is refined, for every $i < n$ there exists $C_i \in A$ such that $w_i \in \boxdot C_i$, and $u \notin C_i$. If we put $C = \bigcup_{i<n} C_i$, we have $C \supseteq X{\Uparrow}$, and $u \notin C$.

(i): Using the definition of U, $u \in \Box(P^e \to \Box(C \vee \neg D)) \to \neg D$ for every $D \in A$. Thus, the set

$$\{P^e\} \cup \{B \in A : u \in \Box B\} \cup \{\Diamond D : D \in A, u \in D\}$$

has fip, and consequently its intersection contains an element v. Clearly, $v \vDash P^e$, and the refinedness of W implies $u < v < u$.

(ii): Since $u \in P^e \vee \Box C \supseteq X{\Uparrow}$, the definition of U implies that there is $e' \in E$ such that $u \in \Box\big(P^{e'} \to P^e \vee \Box C\big)$. If $e' \neq e$, this would in fact mean $u \in \Box(P^{e'} \to \Box C)$, contradicting (i). Thus, $e' = e$, which implies $e \in E$.

Assume $u < v \notin X{\Uparrow}$, $v \vDash P^e$. By reducing C if necessary, we may assume $v \notin C$. For every $D \in A$ such that $u \in D$, there is $e' \in E$ such that $u \in \Box\big(P^{e'} \to (P^e \wedge D) \vee \Box C\big)$. As above, we must have $e' = e$, hence $u \in \Box(P^e \to D \vee C)$, and $v \in D$. As D was arbitrary, and W is refined, we obtain $v = u$. $\Box$ (Claim 1)

Part (i) of Claim 1 implies that $\mathrm{cl}(u_{e0})$ is reflexive, and for every $e \in E$, there is $u_e \in \mathrm{cl}(u_{e0})$ such that $u_e \vDash P^e$. By (ii), this u_e is a unique point in $u_{e0}{\Uparrow} \smallsetminus X{\Uparrow}$ satisfying P^e, and every point of $u_{e0}{\Uparrow} \smallsetminus X{\Uparrow}$ satisfies some P^e, $e \in E$, hence it equals u_e. Thus, $u_e{\uparrow} = \{u_{e'} : e' \in E\} \cup X{\Uparrow}$, and $\{u_e : e \in E\}$ is a $\langle \circ, E\rangle$-tp of X. $\Box$

This completes the proof of Theorem 5.4.

Remark 5.11 The proof of Lemma 5.10 shows a bit more: if n, P, and E are as in the lemma, $e \in E$, $W \vDash \mathrm{Ext}_{\circ,n,E,e}$ is descriptive, and $X = \{w_i : i < n\} \subseteq W$ either does not have a reflexive root, or its root cluster avoids P^e, then X has a $\langle \circ, E\rangle$-tp in W.

Remark 5.12 Exploiting compactness, one can show that descriptive frames validating certain extension rules are also extensible wrt infinite subsets.

First, if W is a descriptive frame such that $W \vDash \mathrm{Ext}_{*,\infty,E} := \{\mathrm{Ext}_{*,n,E} : 0 < n \in \omega\}$ (where $* \in \{\bullet, \circ\}$, and $|E| = 1$ if $* = \bullet$), and if $\varnothing \neq X \subseteq W$ is *closed,* then X has a $\langle *, E\rangle$-tp in W.

Second, if P is infinite, $n \in \omega$, $e \in \mathbf{2}^P$, and W validates

$$\mathrm{Ext}_{\bullet,n,\{e\}} := \{\mathrm{Ext}_{\bullet,n,\{e\restriction P'\}} : P' \subseteq P \text{ finite}\},$$

then every $X = \{w_i : i < n\} \subseteq W$ has a $\langle \bullet, \{e\}\rangle$-tp. The reflexive case is slightly more complicated: if $E \subseteq \mathbf{2}^P$ is closed (in the product topology on $\mathbf{2}^P$), E_0 is a set of isolated points of E, and W satisfies appropriate instances of the extension rules, then any X as above has a tp cluster consisting of one point realizing e for each $e \in E_0$, and one or more points realizing e for each $e \in E \smallsetminus E_0$. This can be generalized to closed infinite X as above. We leave the details to the interested reader, as we have no further use for these properties.

We are now going to show that the admissibility of $\mathrm{Ext}^{\mathrm{Par}}_{C,n}$ in a logic L is equivalent to $\langle C, n\rangle$-extensibility of L. The basic idea is that $\mathrm{Ext}^{\mathrm{Par}}_{C,n}$ is admissible iff it holds in canonical frames $C_L(P, V)$ by Lemma 2.2, which is equivalent to extensibility of $C_L(P, V)$ by Theorems

5.3 and 5.4. Since every finite frame can be embedded in a canonical frame, the existence of tight predecessors in $C_L(P,V)$ is equivalent to extensibility of the logic. (This is not quite true as not all rooted subframes of $C_L(P,V)$ are finite, but one can make it work anyway.)

Recall Definition 4.17. If $D \preceq C$, then $\mathrm{Ext}^{\mathrm{Par}}_{D,n} \subseteq \mathrm{Ext}^{\mathrm{Par}}_{C,n}$ by definition. Also, since we can identify some of the variables x_i in extension rules by a substitution, we have:

Observation 5.13 *If $\langle D,m\rangle \preceq \langle C,n\rangle$, then $\mathbf{K4} + \mathrm{Ext}^{\mathrm{Par}}_{C,n}$ proves $\mathrm{Ext}^{\mathrm{Par}}_{D,m}$, $\mathbf{K4} + \mathrm{Ext}_{*,n,E}$ proves $\mathrm{Ext}_{*,m,E}$, and $\mathbf{K4} + \mathrm{Ext}_{\circ,n,E,e}$ proves $\mathrm{Ext}_{\circ,m,E,e}$.* □

Theorem 5.14 *Let $L \supseteq \mathbf{K4}$ have fmp, $\langle C,n\rangle \in EC$, and $e \in E \subseteq \mathbf{2}^P$ for some finite $P \subseteq \mathrm{Par}$, where $|E| = |C|$ (note that if Par is finite, this is only possible when $|C| \leq 2^{|\mathrm{Par}|}$). Then the following are equivalent.*

(i) L is $\langle C,n\rangle$-extensible.

(ii) L admits $\mathrm{Ext}_{\bullet,n,E}$ if $C = \bullet$, and $\mathrm{Ext}_{\circ,n,E,e}$ if C is reflexive.

(iii) L admits $\mathrm{Ext}^{\mathrm{Par}}_{D,m}$ for every $\langle D,m\rangle \preceq \langle C,n\rangle$.

Proof: (iii) $\to$ (ii) is trivial.

(ii) $\to$ (i): Let F be a finite rooted frame of type $\langle C,n\rangle$ such that $F \smallsetminus \mathrm{rcl}(F)$ is an L-frame, and if $\mathrm{Par} = \varnothing$ and $n = 1$, then F does not have a reflexive root. We can endow F with a valuation of P such that if $F \smallsetminus \mathrm{rcl}(F)$ has a reflexive root r, then $\mathrm{cl}(r) \vDash \neg P^e$. By Lemma 2.3, there is a finite set V of variables such that $F \smallsetminus \mathrm{rcl}(F)$ can be identified with a generated subframe of the canonical frame $C_L(P,V)$, including the valuation of P. Choose $X = \{w_i : i < n\} \subseteq F$ such that $F \smallsetminus \mathrm{rcl}(F) = X{\upharpoonleft}$. We have $C_L(P,V) \vDash \mathrm{Ext}_{\bullet,n,E}$ ($\mathrm{Ext}_{\circ,n,E,e}$, respectively) by Lemma 2.2, hence X has a $\langle *,E\rangle$-tp U by Theorem 5.3 and Remark 5.11. The choice of the valuation of P in $F \smallsetminus \mathrm{rcl}(F)$ ensures that U is disjoint from $F \smallsetminus \mathrm{rcl}(F)$, hence F is isomorphic to the generated subframe $U \cup (F \smallsetminus \mathrm{rcl}(F))$ of $C_L(P,V)$ (minus its valuation), and as such it is an L-frame.

(i) $\to$ (iii): In view of Observation 5.13, we may assume $m = n$. Let $P' \subseteq \mathrm{Par}$ be finite, $e' \in E' \subseteq \mathbf{2}^{P'}$, $|E'| \leq |C|$. Let σ be a substitution such that $\nvdash_L \sigma(\Box y \to x_i)$ for every $i < n$. Since L has fmp, we can find $F_i \in \mathrm{Mod}_L$ with root w_i such that $\sigma(F_i) \vDash y$ and $\sigma(F_i), w_i \nvDash x_i$. If $\mathrm{Par} = \varnothing$, $n = 1$, and w_0 is reflexive, we put $F = F_0$ and $u_{e'} = w_0$. Otherwise, let F' be the disjoint union of $\{F_i : i < n\}$ extended by a new root cluster of type C, and F a similar model where the root cluster is shrunk to size $|E'|$. F' is an L-frame as L is $\langle C,n\rangle$-extensible, and F is its p-morphic image. We enumerate elements of $\mathrm{rcl}(F)$ as $\{u_f : f \in E'\}$, and we define $F, u_f \vDash P^f$; the valuation of variables in u_f is arbitrary.

Either way, $\{u_f : f \in E'\}$ is a $\langle *,E'\rangle$-tp of $\{w_i : i < n\}$, hence $\sigma(F), u_{e'}$ refutes the premise of $\sigma(\mathrm{Ext}_{\bullet,n,e'})$ ($\sigma(\mathrm{Ext}_{\circ,n,E',e'})$, resp.) by the proof of Theorem 5.3 (5.4). □

Definition 5.15 A frame W is *locally finite* if $u{\upharpoonleft}$ is finite for every $u \in W$.

Theorem 5.16 *Let T be a set of conditions of the form $\langle *,n,E\rangle$, where $* \in \{\bullet,\circ\}$, $n \in \omega$, $\varnothing \neq E \subseteq \mathbf{2}^P$ for a finite $P \subseteq \mathrm{Par}$ (not necessarily the same for each $t \in T$), and if $* = \bullet$,*

$|E| = 1$. *Assume that* $L \supseteq \mathbf{K4}$ *has fmp, and is* $\langle\bullet, n\rangle$*-extensible* ($\langle \textcircled{k}, n\rangle$*-extensible*) *whenever* $\langle\bullet, n, E\rangle \in T$ ($\langle\circ, n, E\rangle \in T$ *with* $k = |E|$, *respectively*). *The following are equivalent for any rule* $\Gamma \,/\, \Delta$.

(*i*) $L + \{\mathrm{Ext}_t : t \in T\}$ *proves* $\Gamma \,/\, \Delta$.

(*ii*) $\Gamma \,/\, \Delta$ *holds in every parametric general* L*-frame that is* t*-extensible for every* $t \in T$.

(*iii*) $\Gamma \,/\, \Delta$ *holds in every parametric countable locally finite Kripke* L*-frame that is* t*-extensible for every* $t \in T$.

Proof: (ii) $\to$ (iii) is trivial, and (i) $\leftrightarrow$ (ii) follows from Theorems 5.3 and 5.4, as any parametric consequence relation is complete with respect to parametric descriptive frames.

(iii) $\to$ (ii): Let W be an L-frame t-extensible for every $t \in T$, and let $\vDash$ be a valuation on W such that $W \nvDash \Gamma \,/\, \Delta$. Let Σ be the set of subformulas of Γ, and put

$$\psi = \bigvee \bigl\{\Sigma^{\mathrm{Sat}_\Sigma(W,w)} : w \in W\bigr\}.$$

We have $\vdash_{\mathbf{K4}} \psi \to \varphi$ for every $\varphi \in \Gamma$, hence it suffices to find a model of the requested form refuting $\psi \,/\, \Delta$.

For every $\varphi \in \Delta$, W witnesses that $\psi \nvdash_L \varphi$. Since L has fmp, we can find a finite L-model whose root refutes $\boxdot\psi \to \varphi$, and by taking a disjoint union of these, we obtain a finite L-model F_0 such that $F_0 \vDash \psi$, and $F_0 \nvDash \varphi$ for every $\varphi \in \Delta$. We will construct a sequence $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ of finite L-models such that $F_k \vDash \psi$, F_k is a generated submodel of F_{k+1}, and $F := \bigcup_k F_k$ is t-extensible for every $t \in T$. Then F is an L-frame, and $F \nvDash \psi \,/\, \Delta$, hence completing the proof.

We may assume that F_0 is included in a countable set Z, and we will choose all the models F_k so that their underlying set is also included in Z. Let $\{\langle t_k, X_k\rangle : k \in \omega\}$ be an enumeration of all pairs of $t_k = \langle *_k, n_k, E_k\rangle \in T$ and $X_k = \{z_{k,0}, \dots, z_{k,n_k-1}\} \subseteq Z$, where each pair occurs infinitely many times in the enumeration.

Starting with F_0, we define the models $F_k \vDash \psi$ by induction on k. Assume that F_k has already been defined. If $X_k \nsubseteq F_k$, or X_k has a $\langle *_k, E_k\rangle$-tp in F_k, we put $F_{k+1} := F_k$. Otherwise, we have $F_k \vDash \psi$ by the induction hypothesis, hence for every $i < n_k$, we can find $w_i \in W$ such that $F_k, z_{k,i} \equiv_\Sigma W, w_i$. Since W is t_k-extensible, we can find a $\langle *_k, E_k\rangle$-tp $\{u_e : e \in E_k\} \subseteq W$ of $\{w_i : i < n_k\}$. Choose distinct elements $\{v_e : e \in E_k\} \subseteq Z \smallsetminus F_k$, and put $F_{k+1} = F_k \cup \{v_e : e \in E_k\}$, where the accessibility relation and valuation of parameters is defined so that $\{v_e : e \in E_k\}$ is a $\langle *_k, E_k\rangle$-tp of X_k, and the valuation of variables in v_e is the same as in u_e. By Lemma 4.6, we have $v_e \equiv_\Sigma u_e$, hence $F_{k+1} \vDash \psi$. Also, F_{k+1} is based on an L-frame: it suffices to show this for the rooted subframe generated by the new elements, which is indeed an L-frame by the extensibility assumptions on L (note that the exceptional case when $\mathrm{Par} = \varnothing$, $n_k = 1$ cannot happen: if $z_{k,0}$ is reflexive, it already has the requisite tight predecessor in F_k, namely itself). $\square$

A more constructive proof of Theorem 5.16 will be given in the course of proving Theorem 5.22.

We can now give the main result of this section. The only part left to prove is that a failure of a rule in an extensible frame implies its nonadmissibility; we do this by "approximating" the frame by a set of finite models with the model extension property, and using the characterization from Section 3 to extract a projective formula whose mgu witnesses nonadmissibility of the rule.

Theorem 5.17 *Let L be a* Par*-extensible logic, and*

$$T = \{\langle C, n\rangle : L \textit{ has a type-}\langle C, n\rangle \textit{ frame, and } |C| \leq 2^{|\mathrm{Par}|}\}.$$

Then the following are equivalent for any rule $\Gamma \,/\, \Delta$.

(i) $\Gamma \mathrel{|\!\!\sim}_L \Delta$.

(ii) $\Gamma \,/\, \Delta$ *holds in every parametric L-frame, $\langle C, n\rangle$-extensible for every $\langle C, n\rangle \in T$.*

(iii) $\Gamma \,/\, \Delta$ *is derivable in* $L + \{\mathrm{Ext}^{\mathrm{Par}}_{C,n} : \langle C, n\rangle \in T\}$.

Moreover, it suffices to consider only countable, locally finite Kripke frames in (ii).

In particular, $\{\mathrm{Ext}^{\mathrm{Par}}_{C,n} : \langle C, n\rangle \in T\}$ *is a basis of L-admissible rules.*

Proof: (ii) → (iii) → (i) follow from Theorems 5.16 and 5.14, respectively.

(i) → (ii): Let W be a parametric L-frame, $\langle C, n\rangle$-extensible for every $\langle C, n\rangle \in T$, and fix an admissible valuation in W that refutes $\Gamma \,/\, \Delta$. Let Σ be the set of all subformulas of formulas occurring in Γ, and define

$$\psi = \bigvee \{\Sigma^{\mathrm{Sat}_\Sigma(W,v)} : v \in W\}.$$

Clearly, $\psi \vdash_L \varphi$ for every $\varphi \in \Gamma$, and $\psi \nvdash_L \varphi$ for every $\varphi \in \Delta$, as $W \vDash \psi$. The same argument as in the proof of Theorem 4.4 shows that

$$\mathrm{Mod}_L(\psi) = \{F \in \mathrm{Mod}_L : \forall u \in F\, \exists v \in W\, (F, u \equiv_\Sigma W, v)\}$$

has the model extension property, hence ψ is projective by Theorem 3.2. Thus, if σ is the projective unifier of ψ, we have $\vdash_L \sigma(\varphi)$ for every $\varphi \in \Gamma$, but $\nvdash_L \sigma(\varphi)$ for every $\varphi \in \Delta$, which implies $\Gamma \mathrel{|\!\!\not\sim}_L \Delta$. □

As a sort of converse to Theorem 5.17, one can show that if Par is infinite, and a logic $L \supseteq \mathbf{K4}$ has a basis of admissible rules consisting of a set of extension rules, then L is a clx logic.

We will also give a characterization of consequences of extension rules using finite models, which will be helpful in the sequel for determination of the computational complexity of admissibility in clx logics. The characterization is very similar to criteria for admissibility in a class of modal logics proved by Rybakov [22, §6.1].

Our assumptions are somewhat different: on the one hand, clx logics have the generalized property of branching below 1 in Rybakov's terminology, on the other hand, we do not need to assume any analogue of the effective m-drop point property. (In fact, one can use the proof of Lemma 4.35 to show that clx logics satisfy this property automatically. We suspect that the property actually holds for *all* logics with the generalized property of branching below m.) We also obtain better bounds: the models we construct in Theorem 5.23 have size exponential in the size of the rule, whereas the bounds in [22] are at least doubly exponential.

In order to keep the notation manageable, we will only state the result for combinations of $\mathrm{Ext}^{\mathrm{Par}}_{C,n}$, rather than individual $\mathrm{Ext}_{*,n,E}$ rules; this is of course enough for the application to admissibility.

Definition 5.18 We generalize the $\mathrm{Ext}^{\mathrm{Par}}_{C,n}$ notation to infinite extension conditions (Definition 4.17) by putting $\mathrm{Ext}^{\mathrm{Par}}_{\otimes,n} := \bigcup_{k=1}^{\infty} \mathrm{Ext}^{\mathrm{Par}}_{ⓚ,n}$, and $\mathrm{Ext}^{\mathrm{Par}}_{C,\infty} := \bigcup_{n=1}^{\infty} \mathrm{Ext}^{\mathrm{Par}}_{C,n}$. We generalize in a similar way the notion of $\langle C, n\rangle$-extensible frames.

Moreover, if $T \subseteq EC^{\infty}$, we put $\mathrm{Ext}^{\mathrm{Par}}_{T} := \bigcup_{t\in T} \mathrm{Ext}^{\mathrm{Par}}_{t}$, and a frame is T-extensible if it is t-extensible for every $t \in T$.

Recall that every set of extension conditions is equivalent to a finite one by Lemma 4.18.

Observation 5.19 *If T, T' are equivalent sets of extension conditions, then $\mathbf{K4} + \mathrm{Ext}^{\mathrm{Par}}_{T} = \mathbf{K4} + \mathrm{Ext}^{\mathrm{Par}}_{T'}$, and a frame is T-extensible iff it is T'-extensible.* □

Definition 5.20 Let Σ be a finite set of formulas closed under subformulas, $P = \Sigma \cap \mathrm{Par}$, and F be a model.

If $X \subseteq F$, and $e \in \mathbf{2}^P$, then a *tight $\langle\bullet, \{e\}\rangle$-pseudopredecessor* (*$\langle\bullet, \{e\}\rangle$-tpp*) *of X wrt Σ* is $\{u\} \subseteq F$ such that $u \vDash P^e$, and for every $\Box\psi \in \Sigma$,

$$u \vDash \Box\psi \quad \text{iff} \quad w \vDash \boxdot\psi \text{ for every } w \in X.$$

If $n \in \omega \cup \{\infty\}$, then F is *$\langle\bullet, n\rangle$-pseudoextensible wrt Σ,* if every finite $X \subseteq F$ such that $|X| \leq_0 n$ has a $\langle\bullet, \{e\}\rangle$-tpp wrt Σ for every $e \in \mathbf{2}^P$.

If $X \subseteq F$, and $\varnothing \neq E \subseteq \mathbf{2}^P$, a *tight $\langle\circ, E\rangle$-pseudopredecessor* (*$\langle\circ, E\rangle$-tpp*) *of X wrt Σ* is $\{u_e : e \in E\} \subseteq F$ such that for every $e \in E$ and $\Box\psi \in \Sigma$, we have $u_e \vDash P^e$, and

$$u_e \vDash \Box\psi \quad \text{iff} \quad w \vDash \boxdot\psi \text{ for every } w \in X \text{ and } u_f \vDash \psi \text{ for every } f \in E. \tag{30}$$

If $n, k \in \omega \cup \{\infty\}$, $k \neq 0$, then F is *$\langle ⓚ, n\rangle$-pseudoextensible wrt Σ,* if every finite $X \subseteq F$ such that $|X| \leq_0 n$ has a $\langle\circ, E\rangle$-tpp wrt Σ for every $E \subseteq \mathbf{2}^P$ such that $|E| \leq_0 k$.

If T is a set of extension conditions, F is *T-pseudoextensible wrt Σ* if it is t-pseudoextensible for every $t \in T$.

Note that every $\langle *, E\rangle$-tp is also a $\langle *, E\rangle$-tpp wrt Σ, and a T-extensible frame is T-pseudoextensible wrt Σ. Essentially, $\langle *, E\rangle$-tpp's wrt Σ are sets of points that behave as if they were $\langle *, E\rangle$-tp's as far as formulas from Σ are concerned.

Let $B = \{\varphi : \Box\varphi \in \Sigma\}$, and let $\mathrm{PExt}^{\Sigma}_{T}$ consist of the following rules:

- If $\langle\bullet, \infty\rangle \in T$, rules of the form

  $$P^e \wedge \bigwedge_{\varphi\in B_+} \Box\varphi \to \bigvee_{\psi\in B_-} \Box\psi \Big/ \Big\{ \bigwedge_{\varphi\in B_+} \boxdot\varphi \to \psi : \psi \in B_- \Big\}, \tag{31}$$

  where $e \in \mathbf{2}^P$, $B = B_+ \mathbin{\dot\cup} B_-$, $B_- \neq \varnothing$ (the case of $B_- = \varnothing$ is actually the rule for $\langle\bullet, 0\rangle$ below).

- If $\langle \bullet, n\rangle \in T$, $n\in\omega$, rules of the form

$$P^e \wedge \bigwedge_{\varphi\in B_+} \Box\varphi \to \bigvee_{\psi\in B_-} \Box\psi \Big/ \Big\{ \bigwedge_{\varphi\in B_+} \boxdot\varphi \to \bigvee_{\psi\in B_i} \boxdot\psi : i<n \Big\}, \tag{32}$$

  where $e\in \mathbf{2}^P$, $B = B_+ \dot\cup B_-$, $B_- = \bigcup_{i<n} B_i$.

- If $\langle ⓚ, \infty\rangle \in T$, rules of the form

$$\frac{\Big\{ P^{S(f)} \wedge \bigwedge_{\varphi\in B_+\smallsetminus D} \boxdot\varphi \to \bigvee_{\substack{\psi\in D\\ f(\psi)=S(f)}} \psi \vee \bigvee_{\psi\in B_-\cup D} \Box\psi : D\subseteq B_+, f\colon D\to E \Big\}}{\Big\{ \bigwedge_{\varphi\in B_+} \boxdot\varphi \to \psi : \psi\in B_- \Big\}}, \tag{33}$$

  where $E\subseteq \mathbf{2}^P$, $|E| \le_0 k$, $B = B_+ \dot\cup B_-$, $B_- \neq \varnothing$, $S\colon \dot\bigcup_{D\subseteq B_+} E^D \to E$. (Here, E^D denotes the set of all functions $f\colon D\to E$.)

- If $\langle ⓚ, n\rangle \in T$, $n\in\omega$, rules of the form

$$\frac{\Big\{ P^{S(f)} \wedge \bigwedge_{\varphi\in B_+\smallsetminus D} \boxdot\varphi \to \bigvee_{\substack{\psi\in D\\ f(\psi)=S(f)}} \psi \vee \bigvee_{\psi\in B_-\cup D} \Box\psi : D\subseteq B_+, f\colon D\to E \Big\}}{\Big\{ \bigwedge_{\varphi\in B_+} \boxdot\varphi \to \bigvee_{\psi\in B_i} \boxdot\psi : i<n \Big\}}, \tag{34}$$

  where $E\subseteq \mathbf{2}^P$, $|E| \le_0 k$, $B = B_+ \dot\cup B_-$, $B_- = \bigcup_{i<n} B_i$, $S\colon \dot\bigcup_{D\subseteq B_+} E^D \to E$.

Notice that PExt_T^Σ is finite, and all formulas occurring in PExt_T^Σ are Boolean combinations of Σ-formulas. The reader should think of PExt_T^Σ as rule *instances* rather than rule schemata, as we will use them in a context where they do not get closed under substitution. In fact, the gist of Theorem 5.22 below is that PExt_T^Σ axiomatizes the consequences of $\mathrm{Ext}_T^{\mathrm{Par}}$ involving only (Boolean combinations of) Σ-formulas. But first we need a semantic characterization of PExt_T^Σ:

Lemma 5.21 *Let Σ be a finite set of formulas closed under subformulas, T a set of extension conditions, and F a model. Then F is T-pseudoextensible wrt Σ iff $F \vDash \mathrm{PExt}_T^\Sigma$.*

In particular, PExt_T^Σ is provable in $\mathbf{K4} + \mathrm{Ext}_T^{\mathrm{Par}}$.

Proof: We will show the lemma for the most complicated case of $t = \langle ⓚ, n\rangle \in T$, $n\in\omega$, the other cases are similar and left to the reader.

First, assume that F is t-pseudoextensible, E, B_+, B_-, B_i, S are as in (34), and $w_i\in F$, $i<n$, witness that the conclusion of (34) fails, i.e., $w_i \vDash \bigwedge_{\varphi\in B_+} \boxdot\varphi \wedge \bigwedge_{\psi\in B_i} \neg\boxdot\psi$. Let $\{u_e : e\in E\}$ be a $\langle \circ, E\rangle$-tpp of $\{w_i : i<n\}$ wrt Σ. Put $D = \{\psi\in B_+ : u_e \nvDash \Box\psi\}$, where $e\in E$. By (30), this definition is independent of e, and for every $\psi\in D$, there exists $f(\psi)\in E$ such that $u_{f(\psi)} \nvDash \psi$. This defines a function $f\colon D\to E$. Putting $e = S(f)$, inspection shows that u_e refutes the premise of (34) corresponding to D and f.

Conversely, let $X = \{w_i : i < n\} \subseteq F$ and $E \subseteq \mathbf{2}^P$ be such that $|E| \le_0 k$, and X has no $\langle \circ, E\rangle$-tpp wrt Σ. Put $B_i = \{\psi \in B : w_i \nvDash \boxdot\psi\}$, $B_- = \bigcup_{i<n} B_i$, $B_+ = B \smallsetminus B_-$.

Let $D \subseteq B_+$ and $f\colon D \to E$. If there were $\{u_e : e \in E\} \subseteq F$ such that

$$u_e \vDash P^e \wedge \bigwedge_{\varphi \in B_+ \smallsetminus D} \boxdot\varphi \wedge \bigwedge_{\psi \in B_- \cup D} \neg\Box\psi \wedge \bigwedge_{f(\psi)=e} \neg\psi,$$

then $\{u_e : e \in E\}$ would be a $\langle \circ, E\rangle$-tpp of X, with $\{\psi \in B_+ : u_e \nvDash \Box\psi\} = D$. Thus, there must exist $S(f) := e \in E$ such that

$$F \vDash P^e \wedge \bigwedge_{\varphi \in B_+ \smallsetminus D} \boxdot\varphi \to \bigvee_{\psi \in B_- \cup D} \Box\psi \vee \bigvee_{f(\psi)=e} \psi. \tag{35}$$

This defines a function $S\colon \dot{\bigcup}_{D \subseteq B_+} E^D \to E$ for which all premises of (34) hold in F by (35). However, the definition of B_+ and B_i ensures that the ith conclusion of (34) is false in w_i, hence (34) does not hold in F. $\Box$

Theorem 5.22 *Let $L \supseteq$ **K4** have fmp, T be a set of extension conditions such that L is T-extensible, and Σ a finite set of formulas closed under subformulas. The following are equivalent for any rule $\Gamma \,/\, \Delta$ such that $\Gamma \subseteq \Sigma$.*

(i) $L + \mathrm{Ext}_T^{\mathrm{Par}}$ *proves* $\Gamma \,/\, \Delta$.

(ii) $\mathrm{PExt}_T^{\Sigma} \vdash_L \Gamma \,/\, \Delta$.

(iii) $\Gamma \,/\, \Delta$ *holds in every finite L-model, T-pseudoextensible wrt Σ.*

Proof: (iii) $\to$ (ii) $\to$ (i) follows from Lemmas 5.21 and 2.4.

(i) $\to$ (iii): Let F_0 be a finite T-pseudoextensible L-model such that $F_0 \nvDash \Gamma \,/\, \Delta$, we will find a locally finite T-extensible L-model $F \nvDash \Gamma \,/\, \Delta$.

Similarly to the proof of Theorem 5.16, we will construct a sequence of finite L-models $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ such that F_k is a generated submodel of F_{k+1}, while maintaining the property

$$\forall v \in F_k \, \exists u \in F_0 \; u \equiv_\Sigma v. \tag{36}$$

As in Theorem 5.16, we assume that F_0 is included in a countable set Z, and we will define F_k so that their underlying sets are also included in Z. Let $\{\langle t_k, X_k\rangle : k \in \omega\}$ be an enumeration of all pairs of $t_k = \langle *_k, n_k, E_k\rangle$ and $X_k \subseteq Z$, where $|X_k| = n_k$, $E_k \subseteq \mathbf{2}^{P_k}$ for some finite $P_k \subseteq \mathrm{Par}$, $P_k \supseteq P := \Sigma \cap \mathrm{Par}$, $|E_k| = 1$ if $*_k = \bullet$, $\langle C_k, n_k\rangle \preceq \langle C, n\rangle$ for some $\langle C, n\rangle \in T$, where $C_k = \bullet$ if $*_k = \bullet$, and $C_k = ⓜ$ if $*_k = \circ$ and $|E_k| = m$, and each pair occurs infinitely many times in the enumeration.

Assuming F_k is already defined, let $F_{k+1} = F_k$ if $X_k \nsubseteq F_k$, or if X_k has a $\langle *_k, E_k\rangle$-tp in F_k. Otherwise, fix $\langle C, n\rangle \in T$ such that $\langle C_k, n_k\rangle \preceq \langle C, n\rangle$, and let $E = \{e \restriction P : e \in E_k\}$. Write $X_k = \{z_i : i < n_k\}$, and for every $i < n_k$, find $w_i \in F_0$ such that $z_i \equiv_\Sigma w_i$ by the induction hypothesis. Since F_0 is $\langle C, n\rangle$-pseudoextensible, there exists a $\langle *_k, E\rangle$-tpp $\{u_e : e \in E\} \subseteq F_0$ of $\{w_i : i < n\}$ wrt Σ. We choose distinct elements $\{v_e : e \in E_k\} \subseteq Z \smallsetminus F_k$, and define

$F_{k+1} = F_k \cup \{v_e : e \in E_k\}$, where the accessibility relation and valuation of parameters in v_e is defined so that $\{v_e : e \in E_k\}$ is a $\langle *_k, E_k\rangle$-tp of X_k, and valuation of variables is defined by

$$F_{k+1}, v_e \vDash x \Leftrightarrow F_0, u_{e\restriction P} \vDash x.$$

By induction on the complexity of ψ, we can prove $v_e \vDash \psi$ iff $u_{e\restriction P} \vDash \psi$ for every $\psi \in \Sigma$ as in Lemma 4.6, which shows that (36) holds for F_{k+1}. Moreover, F_{k+1} is based on an L-frame, as L is $\langle C, n\rangle$-extensible, and therefore $\langle C_k, n_k\rangle$-extensible.

When the construction is completed, we put $F = \bigcup_k F_k$. Then F is a locally finite model based on an L-frame, it is T-extensible by construction, and (36) implies $F \vDash \Gamma$. On the other hand, F_0 is a generated submodel of F, hence $F \nvDash \psi$ for every $\psi \in \Delta$, thus $F \nvDash \Gamma \mathbin{/} \Delta$. $\Box$

In view of Theorem 5.17, Theorem 5.22 provides a description of admissible rules in Par-extensible logics. We will state it explicitly for cluster-extensible logics, as we can give explicit bounds in this case. Recall Definition 4.20.

Theorem 5.23 *Let L be a clx logic, and $\Gamma \mathbin{/} \Delta$ a rule. The following are equivalent.*

(*i*) $\Gamma \mathrel{|\!\sim}_L \Delta$.

(*ii*) $L + \mathrm{Ext}^{\mathrm{Par}}_{\mathrm{bas}(L)}$ *proves* $\Gamma \mathbin{/} \Delta$.

(*iii*) $\Gamma \mathbin{/} \Delta$ *holds in every* (*countable, locally finite, Kripke*) $\mathrm{bas}(L)$*-extensible L-frame.*

(*iv*) $\mathrm{PExt}^{\Sigma}_{\mathrm{bas}(L)} \vdash_L \Gamma \mathbin{/} \Delta$, *where* $\Sigma = \mathrm{Sub}(\Gamma)$.

(*v*) $\Gamma \mathbin{/} \Delta$ *holds in every L-model that is* $\mathrm{bas}(L)$*-pseudoextensible wrt Σ, and has size at most 4^n, where $n = \sum_{\varphi\in\Gamma\cup\Delta}|\varphi|$.*

More precisely, let b be the number of boxed subformulas of $\Gamma \cup \Delta$, and m the cardinality of

$$(\Sigma \cap \mathrm{Par}) \cup \{\psi, \Box\psi : \Box\psi \in \Sigma\}.$$

If $\Gamma \mathrel{|\!\not\sim}_L \Delta$, there exists an L-model $\mathrm{bas}(L)$*-pseudoextensible wrt Σ and refuting $\Gamma \mathbin{/} \Delta$, of size at most*

$$3 \cdot 2^b(2^m + |\Delta|).$$

Proof: The equivalence of the conditions follows from Theorems 5.17 and 5.22, except for the size bound. Assume $\Gamma \mathrel{|\!\not\sim}_L \Delta$, and let us estimate the size of the countermodel F to (iv) constructed using Lemma 2.4. Let

$$\Theta = (\Sigma \cap \mathrm{Par}) \cup \{\psi, \Box\psi : \Box\psi \in \Sigma\},$$

and denote by $B(\Theta)$ its Boolean closure. Notice that premises and conclusions of all rules from $\mathrm{PExt}^{\Sigma}_{\mathrm{bas}(L)}$ are in $B(\Theta)$. In the proof of Lemma 2.4, we find a suitable partition $B(\Theta) \cup \Gamma \cup \Delta = X \mathbin{\dot\cup} Y$ such that $\Gamma \subseteq X$ and $\Delta \subseteq Y$, for every $\psi \in Y$ we fix an L-model $F_\psi \vDash X$, $F_\psi \nvDash \psi$, and we define F as the disjoint union of all the F_ψ's.

Since all boxed subformulas of $B(\Theta) \cup \Gamma \cup \Delta$ are already subformulas of $\Gamma \cup \Delta$, we can make $|F_\psi| < 3 \cdot 2^b$ for every $\psi \in Y$ by Theorem 4.38. We have $|\Delta|$ models F_ψ for $\psi \in \Delta$. As for the rest, the number of nonequivalent formulas in $Y \cap B(\Theta)$ may be as large as $2^{2^{|\Theta|}} = 2^{2^m}$, however we will not need so many models. Every $\psi \in B(\Theta)$ can be expressed in full conjunctive normal form as $\psi = \bigwedge_i \psi_i$, where each ψ_i is a clause of the form

$$\bigvee_{\vartheta \in \Theta} \vartheta^{e(\vartheta)} \tag{37}$$

for some $e \in \mathbf{2}^\Theta$. Since X is closed under $\vdash_L$, $\psi \in Y$ iff $\psi_i \in Y$ for some i, and if we include in F a model $F_{\psi_i} \nvDash \psi_i$, we will automatically have $F \nvDash \psi$. Thus, it suffices to include in F models F_ψ only for $\psi \in \Delta$ or $\psi \in Y$ of the form (37), and there are at most $|\Delta| + 2^m$ such formulas ψ, which gives $|F| < 3 \cdot 2^b(2^m + |\Delta|)$. We can also estimate $2^b(2^m + |\Delta|) \le 2^{2n-2}$, hence $|F| < 4^n$. □

5.1 Single-conclusion bases

Theorems 5.17, 5.22, and 5.23 provide a description of multiple-conclusion admissible rules of Par-extensible logics. Clearly, the description also applies to single-conclusion rules as its special case, however it does not provide *bases* of single-conclusion admissible rules. We will construct such bases in this section; it amounts to axiomatization of single-conclusion fragments of consequence relations generated by extension rules.

We will distinguish two cases, depending on the properties of the logic. A logic $L \supseteq$ **K4** is called *linear*, if it is complete wrt a class of general frames $\langle W, <, A\rangle$ such that the induced relation $\le$ is a linear preorder; equivalently, a logic is linear iff it has width 1 iff it extends **K4.3**. Notice that a Par-extensible logic is *not* linear iff it is $\langle C, 2\rangle$-extensible for some $C \in \{\bullet, ①\}$.

If L is a linear Par-extensible logic, the multiple-conclusion basis given in Theorem 5.17 consists of rules with at most one conclusion, hence we can easily fix it up to obtain a single-conclusion basis.

Definition 5.24 For any rule $\Gamma \mathbin{/} \Delta$, we define $(\Gamma \mathbin{/} \Delta)^\perp = \Gamma \mathbin{/} \Delta, \perp$. If B is a set of rules, let $B^\perp := \{\varrho^\perp : \varrho \in B\}$.

Lemma 5.25 *Let $L \supseteq$ **K4**, X a set of rules, and $\Gamma \mathbin{/} \Delta$ a rule. Then*

$$\Gamma \vdash_{L+X^\perp} \Delta \quad \textit{iff} \quad \Delta \neq \varnothing \textit{ and } \Gamma \vdash_{L+X} \Delta.$$

Proof: The set of rules with nonempty conclusion is closed under cut, hence the right-hand side defines a consequence relation. Let us call it $\vdash_1$. On the one hand, $\vdash_1$ includes L and all rules from $X^\perp$, hence $\vdash_{L+X^\perp} \subseteq \vdash_1$. On the other hand, $\{\varrho : \varrho^\perp \in L + X^\perp\}$ defines a consequence relation including $L + X$, therefore $\Gamma \vdash_1 \Delta$ implies $\Gamma \vdash_{L+X^\perp} \Delta, \perp$. If $\Delta \neq \varnothing$, we can use cut on $\perp \vdash_L \psi$ for any $\psi \in \Delta$ to obtain $\Gamma \vdash_{L+X^\perp} \Delta$. □

Remark 5.26 Obviously, the only property of L we used in the proof is $\perp \vdash_L x$. If L is an arbitrary logic, an analogous lemma holds where we use a variable not appearing in $\Gamma \cup \Delta$ instead of $\perp$ in Definition 5.24.

Corollary 5.27 *If L is a linear* Par*-extensible logic, then single-conclusion L-admissible rules have a basis consisting of the rules*

$$\begin{cases} \left(\mathrm{Ext}^{\mathrm{Par}}_{C,0}\right)^{\perp}, & n = 0, \\ \mathrm{Ext}^{\mathrm{Par}}_{C,1}, & n = 1 \end{cases}$$

for every $\langle C, n\rangle \in EC$ such that L has a type-$\langle C, n\rangle$ frame, and $|C| \leq 2^{|\mathrm{Par}|}$. $\square$

We now turn to non-linear Par-extensible logic. Any such logic admits the *disjunction property* rules $\mathrm{DP} = \{\mathrm{DP}_n : n \in \omega\}$, where DP_n is

$$\bigvee_{i<n} \Box x_i \Big/ \{x_i : i < n\}.$$

Notice that $\mathbf{K4} + \mathrm{DP} = \mathbf{K4} + \mathrm{DP}_0 + \mathrm{DP}_2$: the rule DP_1 is a substitution instance of DP_2, and then we can prove DP_n in $\mathbf{K4} + \mathrm{DP}_0 + \mathrm{DP}_2$ by induction on n:

$$\bigvee_{i<n+1} \Box x_i \vdash_{\mathbf{K4}} \Box x_n \vee \Box \bigvee_{i<n} \Box x_i \vdash_{\mathrm{DP}_2} x_n, \bigvee_{i<n} \Box x_i,$$

hence

$$\bigvee_{i<n+1} \Box x_i \vdash_{\mathbf{K4}+\mathrm{DP}_2+\mathrm{DP}_n} \{x_i : i < n+1\}$$

by a cut. Since every rule $\Gamma \ / \ \Delta$ is equivalent over $\mathbf{K4} + \mathrm{DP}$ to a single-conclusion rule (namely $\Gamma \ / \ \bigvee_{\psi\in\Delta} \Box\psi$), we are left with the question for which sets B of single-conclusion rules is $L + B + \mathrm{DP}$ conservative over $L + B$ in the sense that it proves the same single-conclusion rules. We will use the following general result.

Lemma 5.28 *For every single-conclusion consequence relation $\vdash$, there exists the largest multiple-conclusion consequence relation $\vdash_m$ whose single-conclusion fragment is $\vdash$, and it can be described explicitly by*

$$\Gamma \vdash_m \Delta \quad \textit{iff} \quad \forall \Theta, \varphi, \sigma \left(\forall \psi \in \Delta \, (\sigma(\psi), \Theta \vdash \varphi) \Rightarrow \sigma(\Gamma), \Theta \vdash \varphi\right), \tag{38}$$

where the quantification is over all finite sets of formulas Θ, formulas φ, and substitutions σ.

Proof: The right-hand side of (38) defines a consequence relation. For example, we verify that $\vdash_m$ is closed under cut. Assume that $\Gamma \vdash_m \Delta, \chi$, $\Gamma, \chi \vdash_m \Delta$, and $\sigma(\psi), \Theta \vdash \varphi$ for every $\psi \in \Delta$. Using $\Gamma, \chi \vdash_m \Delta$, we have $\sigma(\Gamma), \sigma(\chi), \Theta \vdash \varphi$, hence $\sigma(\psi), \sigma(\Gamma), \Theta \vdash \varphi$ for every $\psi \in \Delta \cup \{\chi\}$. Since $\Gamma \vdash_m \Delta, \chi$, we obtain $\sigma(\Gamma), \Theta \vdash \varphi$.

Since $\vdash$ is closed under cuts and substitutions, $\vdash_m$ extends $\vdash$. On the other hand, $\Gamma \vdash_m \psi$ implies $\Gamma \vdash \psi$ by taking $\Theta = \varnothing$, $\varphi = \psi$, and $\sigma = \mathrm{id}$.

Let $\vdash'$ be another consequence relation whose single-conclusion fragment is $\vdash$, and assume $\Gamma \vdash' \Delta$. If Θ, φ, σ are such that $\sigma(\psi), \Theta \vdash \varphi$ for every $\psi \in \Delta$, we have $\sigma(\psi), \Theta \vdash' \varphi$ for every $\psi \in \Delta$ as $\vdash' \supseteq \vdash$, hence $\sigma(\Gamma), \Theta \vdash' \varphi$ by a cut with $\sigma(\Gamma) \vdash' \sigma(\Delta)$. This is a single-conclusion rule, hence $\sigma(\Gamma), \Theta \vdash \varphi$. Thus, $\Gamma \vdash_m \Delta$. $\square$

Lemma 5.29 *Let $L \supseteq \mathbf{K4}$, and B be a set of single-conclusion rules. Then the following are equivalent.*

(*i*) *$L + B$ is the single-conclusion fragment of $L + B + \mathrm{DP}$.*

(*ii*) *For every φ, ψ, and χ: $\psi \vdash_{L+B} \varphi$ and $\chi \vdash_{L+B} \varphi$ implies $\Box\psi \vee \Box\chi \vdash_{L+B} \varphi$.*

(*iii*) *$\Box x \vdash_{L+B} x$, and for every φ, ψ, and ξ: $\psi \vdash_{L+B} \varphi$ implies $\Box\xi \vee \Box\psi \vdash_{L+B} \Box\xi \vee \Box\varphi$.*

(*iv*) *$\Box x \vdash_{L+B} x$, and for every $(\Gamma \,/\, \varphi) \in B$,*

$$\Box x \vee \bigwedge_{\psi \in \Gamma} \Box\psi \vdash_{L+B} \Box x \vee \Box\varphi,$$

where x is a variable not occurring in $\Gamma \cup \{\varphi\}$.

Proof: Let $\vdash$ denote the consequence relation $L + B$.

(ii) $\to$ (i): By Lemma 5.28, it suffices to show that $\mathrm{DP} \subseteq {\vdash_m}$. For DP_0, this amounts to $\bot, \Theta \vdash \varphi$. As for DP_2, assume $\psi, \Theta \vdash \varphi$ and $\chi, \Theta \vdash \varphi$. We have $\psi \wedge \vartheta \vdash \varphi$ and $\chi \wedge \vartheta \vdash \varphi$, where $\vartheta = \bigwedge \Theta$, hence

$$\Box\psi \vee \Box\chi, \Theta \vdash_{\mathbf{K4}} \Box(\psi \wedge \vartheta) \vee \Box(\chi \wedge \vartheta) \vdash \varphi$$

using (ii).

(iii) $\to$ (ii): If $\psi \vdash \varphi$ and $\chi \vdash \varphi$, then

$$\Box\psi \vee \Box\chi \vdash \Box\varphi \vee \Box\chi \vdash \Box\varphi \vee \Box\varphi \vdash \varphi$$

by (iii).

(iv) $\to$ (iii): Define

$$\Gamma \vdash_1 \varphi \quad \text{iff} \quad \Box\xi \vee \Box \bigwedge_{\psi \in \Gamma} \psi \vdash \Box\xi \vee \Box\varphi \quad \text{for every } \xi.$$

Then $\vdash_1$ is a single-conclusion consequence relation including L. We will verify closure under cut: assume $\Gamma \vdash_1 \chi$ and $\Gamma, \chi \vdash_1 \varphi$. Put $\gamma = \bigwedge \Gamma$. We have

$$\Box\xi \vee \Box\gamma \vdash \Box\xi \vee \Box\chi,$$
$$\Box\xi \vee \Box\gamma \vdash \Box\xi \vee \Box\gamma,$$

hence

$$\Box\xi \vee \Box\gamma \vdash \Box\xi \vee \Box(\gamma \wedge \chi) \vdash \Box\xi \vee \Box\varphi.$$

Clearly, $\vdash_1$ includes B by (iv) and substitution, hence it includes $\vdash$, which gives (iii).

(i) $\to$ (iv): $\Box x \vdash x$ is a special case of DP. For any $\Gamma \,/\, \varphi$ in B, we have

$$\Box x \vee \Box \bigwedge \Gamma \vdash_{\mathrm{DP}} x, \bigwedge \Gamma,$$
$$x \vdash_{\mathbf{K4}} \Box x \vee \Box\varphi,$$
$$\bigwedge \Gamma \vdash_B \varphi \vdash_{\mathbf{K4}} \Box x \vee \Box\varphi,$$

hence

$$\Box x \vee \Box \bigwedge \Gamma \vdash_{L+B+\mathrm{DP}} \Box x \vee \Box\varphi. \qquad \Box$$

Lemma 5.29 suggests that we can turn a basis into a single-conclusion basis by taking the single-conclusion rules $\Gamma \mathbin{/} \bigvee_{\psi\in\Delta} \Box\psi$ equivalent to rules from the basis over DP, and adding "side variables" in the spirit of (iv). The way we do it below (unboxing the side variable in the conclusion) also ensures the property $\Box x \vdash_{L+B} x$.

Definition 5.30 For any rule $\Gamma \mathbin{/} \Delta$, we define $(\Gamma \mathbin{/} \Delta)^\vee$ to be the rule

$$\Box x \lor \bigwedge_{\varphi\in\Gamma} \Box\varphi \Big/ x \lor \bigvee_{\psi\in\Delta} \Box\psi,$$

where x is a variable not occurring in $\Gamma\cup\Delta$. If B is a set of rules, we put $B^\vee := \{\varrho^\vee : \varrho \in B\}$.

We note that for the specific case of extension rules, $\mathrm{Ext}^\vee_{\bullet,n,\{e\}}$ is

$$\Box z \lor \Box\Big(P^e \land \Box y \to \bigvee_{i<n} \Box x_i\Big) \Big/ z \lor \bigvee_{i<n} \Box(\boxdot y \to x_i),$$

and $\mathrm{Ext}^\vee_{\circ,n,E,e_0}$ is

$$\frac{\Box z \lor \Box\Big[P^{e_0} \land \boxdot\Big(y \to \bigvee_{e\in E} \Box(P^e \to y)\Big) \land \bigwedge_{e\in E} \boxdot\big(\Box(P^e \to \Box y) \to y\big) \to \bigvee_{i<n} \Box x_i\Big]}{z \lor \bigvee_{i<n} \Box(\boxdot y \to x_i)}.$$

Theorem 5.31 *Let L be a non-linear* Par*-extensible logic, and*

$$T = \big\{\langle C, n\rangle : L \textit{ has a type-}\langle C, n\rangle \textit{ frame, and } |C| \le 2^{|\mathrm{Par}|}\big\}.$$

Then $\big\{\big(\mathrm{Ext}^{\mathrm{Par}}_{C,n}\big)^\vee : \langle C, n\rangle \in T\big\}$ is a basis of L-admissible single-conclusion rules.

Proof: Clearly, $(\Gamma \mathbin{/} \Delta)^\vee$ is derivable in $L + (\Gamma \mathbin{/} \Delta) + \mathrm{DP}$, hence all rules in

$$B := \big\{\big(\mathrm{Ext}^{\mathrm{Par}}_{C,n}\big)^\vee : \langle C, n\rangle \in T\big\}$$

are L-admissible.

On the other hand, since L is consistent, it has a type-$\langle C, 0\rangle$ frame for some C, and $\big(\mathrm{Ext}^{\mathrm{Par}}_{C,0}\big)^\vee$ derives $\Box z \mathbin{/} z$, i.e.,

$$\Box x \vdash_{L+B} x.$$

Moreover, if $\Box z \lor \Box\varphi \mathbin{/} z \lor \bigvee_{i<n} \Box\psi_i$ is one of the rules $\big(\mathrm{Ext}^{\mathrm{Par}}_{C,n}\big)^\vee \in B$, we can derive

$$\begin{aligned}
\Box u \lor \Box(\Box z \lor \Box\varphi) &\vdash_{\mathbf{K4}} \Box\big(\Box(\Box u \lor \Box z) \lor \Box\varphi\big)\\
&\vdash_{L+B} \Box(\Box u \lor \Box z) \lor \Box\varphi\\
&\vdash_{L+B} \Box u \lor \Box z \lor \bigvee_{i<n} \Box\psi_i\\
&\vdash_{\mathbf{K4}} \Box u \lor \Box\Big(z \lor \bigvee_{i<n} \Box\psi_i\Big).
\end{aligned}$$

Thus, $L + B$ derives all single-conclusion rules provable in $L + B + \mathrm{DP}$ by Lemma 5.29. However, if $\varphi \mathbin{/} \{\psi_i : i < n\}$ is again one of the rules $\mathrm{Ext}^{\mathrm{Par}}_{C,n}$, $\langle C, n\rangle \in T$, we have

$$\varphi \vdash_{\mathbf{K4}} \Box\bot \vee \Box\varphi \vdash_B \bot \vee \bigvee_{i<n} \Box\psi_i \vdash_{\mathrm{DP}} \{\psi_i : i < n\},$$

hence $L + B + \mathrm{DP}$ includes the basis of $\vdash_L$ from Theorem 5.17. Thus, $L + B$ derives all single-conclusion L-admissible rules. □

5.2 Finite and independent bases

In this section, we investigate whether Par-extensible logics have finite or independent bases of admissible rules with parameters. Recall that a basis of admissible rules B is *independent* if for every $\varrho \in B$, $B \smallsetminus \{\varrho\}$ is not a basis. Clearly, a finite basis can be made independent by successively omitting redundant rules, but this may not be possible for infinite bases.

First, we observe that finite bases are out of question in the presence of infinitely many parameters, apart from the trivial cases of inconsistent logics (which have the empty set for a basis) and logics with no tautologies (which have a multiple-conclusion basis $\{x \mathbin{/} \varnothing\}$, and a single-conclusion basis $\{x \mathbin{/} y\}$; of course, modal logics always have some tautologies anyway).

Proposition 5.32 *If L is a consistent logic with at least one tautology, then every basis of L-admissible multiple-conclusion or single-conclusion rules has to involve all parameters.*

In particular, if Par *is infinite, then L has no finite basis of multiple-conclusion or single-conclusion admissible rules.*

Proof: Assume for contradiction that B is a basis, and $p \in \mathrm{Par}$ does not appear in B. We have $p \vdash_L x$, hence the rule $p \mathbin{/} x$ is derivable in $L + B$. Since L is closed under atomic substitutions, and p does not appear in B, we can substitute another variable y for p in the derivation. Thus, $L + B$ derives $y \mathbin{/} x$, i.e., $y \vdash_L x$. If L has a tautology, we can substitute it for y, and we can substitute an arbitrary formula for x, hence L is inconsistent. □

Nevertheless, logics can have independent bases of admissible rules in infinitely many parameters in a nontrivial way. Consider the simplest example of the classical logic **CPC**. One can check easily that it has a basis B consisting of the rules

$$\neg P^e \mathbin{/} \bot, \qquad e \in \mathbf{2}^P, P \subseteq \mathrm{Par} \text{ finite} \tag{39}$$

(together with the rule $\bot \mathbin{/} \varnothing$ in the multiple-conclusion case). This basis is not independent, since $\neg P'^{e'} \vdash_{\mathbf{CPC}} \neg P^e$ when e' is the restriction of e to $P' \subseteq P$. However, we can make it independent by splitting (39) into rules that do away with one parameter at a time: namely, if we bijectively enumerate $\mathrm{Par} = \{p_n : n \in \omega\}$, and put $P_n = \{p_i : i < n\}$, the set of rules

$$\neg P^e_{n+1} \mathbin{/} \neg P_n^{e\restriction P_n}, \qquad n \in \omega, e \in \mathbf{2}^{P_{n+1}} \tag{40}$$

is equivalent to B, and independent over **CPC** (if we fix a rule ϱ as in (40), the set of formulas implied by $\neg P^e_{n+1}$ is closed under the rules of **CPC**, and rules of the form (40) other than ϱ, but it is not closed under ϱ).

This kind of transformation works well for variable-free rules such as (39), but it is rather unclear how to adapt it to extension rules. We thus leave it as an open problem whether clx logics have independent bases of admissible rules with infinitely many parameters.

We now turn to the case of finitely many parameters. We define a transitive modal logic L to have *bounded branching* if it includes $\mathbf{K4BB}_k$ for some $k \in \omega$. If L has fmp, this is equivalent to the condition that all finite L-frames have branching at most k.

Lemma 5.33 *If* Par *is finite and L is a* Par*-extensible logic with bounded branching, then L has a finite basis (and a fortiori an independent basis) of either multiple-conclusion or single-conclusion admissible rules.*

Proof: There are only finitely many $\langle C, n\rangle$ such that $|C| \le 2^{|\mathrm{Par}|}$ and L has a type-$\langle C, n\rangle$ frame, as well as finitely many choices for $e_0 \in E \subseteq \mathbf{2}^{\mathrm{Par}}$, hence the bases given in Theorem 5.17, Corollary 5.27, and Theorem 5.31 are finite. $\Box$

For logics with unbounded branching, the bases we constructed earlier are infinite, and we will show that in fact no finite bases exist, nevertheless the logics do have independent bases. The main source of non-independence in the bases given in Theorems 5.17 and 5.31 is that $L + \mathrm{Ext}_{C,n}$ derives $\mathrm{Ext}_{C,m}$ when $m \le_0 n$, as an m-element set X whose tight predecessor we are seeking can be (non-injectively) enumerated as $\{w_i : i < n\}$. We get around this problem in a similar way as in [17] by considering variants of extension rules that express the existence of tight predecessors of antichains of size exactly n.

Definition 5.34 Assume that Par is finite. For any $n \in \omega$ and $e \in \mathbf{2}^{\mathrm{Par}}$, $\mathrm{Ext}^{=}_{\bullet,n,\{e\}}$ denotes the rule

$$P^e \wedge \Box y \to \bigvee_{i<n} \Box x_i \Big/ \Big\{ \boxdot\Big(y \wedge \bigwedge_{j\ne i} x_j\Big) \to x_i : i < n \Big\}.$$

Note that the big conjunction is empty if $n \le 1$. If $n \ne 1$ and $\varnothing \ne E \subseteq \mathbf{2}^{\mathrm{Par}}$, $\mathrm{Ext}^{=}_{\circ,n,E}$ is defined as

$$\frac{\boxdot\Big(y \to \bigvee_{e\in E} \Box(P^e \to y)\Big) \wedge \bigwedge_{e\in E} \boxdot\big(\Box(P^e \to \Box y) \to y\big) \to \bigvee_{i<n} \Box x_i}{\Big\{ \boxdot\Big(y \wedge \bigwedge_{j\ne i} x_j\Big) \to x_i : i < n \Big\}}.$$

Lemma 5.35 *Assume that* Par *is finite, and let W be a descriptive or Kripke parametric frame.*

(*i*) *If $* \in \{\bullet, \circ\}$, $n \in \omega$, and $\varnothing \ne E \subseteq \mathbf{2}^{\mathrm{Par}}$, where $|E| = 1$ if $* = \bullet$, and $n \ne 1$ if $* = \circ$, then $W \vDash \mathrm{Ext}^{=}_{*,n,E}$ iff every antichain $X \subseteq W$ of size n has a $\langle *, E\rangle$-tp.*

(*ii*) *If $e \in E \subseteq \mathbf{2}^{\mathrm{Par}}$, then $W \vDash \mathrm{Ext}_{\circ,1,E,e}$ iff $\{w\}$ has a $\langle \circ, E\rangle$-tp whenever $w \in W$ is irreflexive, or no point of $\mathrm{cl}(w)$ satisfies Par^e.*

Proof: By a straightforward modification of the proofs of Theorems 5.3 and 5.4, which we leave to the reader. Recall that the hard part of (ii) was already noted in Remark 5.11. $\Box$

Lemma 5.36 *Assume that* Par *is finite. Let L be a* Par*-extensible logic, and put*

$$T = \{\langle C, n\rangle \in EC : L \text{ has a type-}\langle C, n\rangle \text{ frame, and } |C| \le 2^{|\mathrm{Par}|}\}.$$

For every $\langle C, n\rangle \in T$, we include in B the following rules.

(*i*) *If $C = \bullet$:* $\mathrm{Ext}^{=}_{\bullet,n,E}$ *for every $E = \{e\}$, $e \in \mathbf{2}^{\mathrm{Par}}$.*

(*ii*) *If $C = ⓚ$ and $n \ne 1$:* $\mathrm{Ext}^{=}_{\circ,n,E}$ *for every $E \subseteq \mathbf{2}^{\mathrm{Par}}$ such that $|E| = k$.*

(*iii*) *If $C = ⓚ$ and $n = 1$:* $\mathrm{Ext}_{\circ,1,E,e}$ *for every $e \in E \subseteq \mathbf{2}^{\mathrm{Par}}$ such that $|E| = k$, unless* $\mathrm{Par} = \varnothing$ *and $L \supseteq$* **S4**.

Then B is an independent basis of L-admissible rules.

Proof: Let W be a descriptive L-frame. For every finite $X \subseteq W$, there exists an antichain $Y \subseteq X$ (nonempty if X is nonempty) such that $X{\upharpoonleft} = Y{\upharpoonleft}$. Then it follows from Corollary 5.5 and Lemma 5.35 that $W \vDash B$ iff $W \vDash \mathrm{Ext}^{\mathrm{Par}}_t$ for every $t \in T$; the only thing to note is that the rule omitted in the exceptional case of (iii) is valid in W automatically (in other words, it is derivable in **S4**), because in the absence of parameters, a reflexive point is its own reflexive tight predecessor. Thus, $L + B$ is equivalent to $L + \{\mathrm{Ext}^{\mathrm{Par}}_t : t \in T\}$, and B is a basis of L-admissible rules by Theorem 5.17.

It remains to show that B is independent. Fix $\varrho \in B$ corresponding to C, n, E, and (in case (iii)) e, we will construct a parametric Kripke L-frame where $B \smallsetminus \{\varrho\}$ is valid, while ϱ is not. Let F_0 be a finite L-frame generated by an antichain $X = \{w_i : i < n\}$ (notice that $\langle C, n\rangle \in T$ implies that L is consistent). If $n = 1$, we can arrange that w_0 is irreflexive if $C = \bullet$ or $\mathrm{Par} = \varnothing$, or otherwise that $\mathrm{cl}(w_0)$ realizes $\mathrm{Par}^{e'}$ for every $e' \in E \smallsetminus \{e\}$, but not Par^{e}.

As in the proof of Theorem 5.16, we will construct by induction on k a sequence of finite L-frames $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ whose underlying sets are included in a countable set Z. Let $\{\langle C_k, n_k, E_k, X_k\rangle : k \in \omega\}$ be an enumeration of all quadruples where $\langle C_k, n_k\rangle \in T$, $E_k \subseteq \mathbf{2}^{\mathrm{Par}}$, $|E_k| = |C_k|$, $X_k \subseteq Z$, $|X_k| = n_k$, such that every quadruple appears infinitely many times in the enumeration. Assume that F_k has already been constructed. If $X_k \nsubseteq F_k$, X_k is not an antichain, or X_k has a $\langle *_k, E_k\rangle$-tp in F_k (where $*_k \in \{\bullet, \circ\}$ is the reflexivity of C_k), or if $X_k{\upharpoonleft} = X{\upharpoonleft}$, $C_k = C$, and $E_k = E$, we put $F_{k+1} = F_k$. Otherwise, F_{k+1} consists of F_k together with a $\langle *_k, E_k\rangle$-tp of X_k whose elements are taken from $Z \smallsetminus F_k$. Let $F = \bigcup_{k\in\omega} F_k$.

Since L is $\langle C_k, n_k\rangle$-extensible, we obtain by induction that every F_k is an L-frame, hence also F is an L-frame. If $*$ denotes the reflexivity of C, then X has no $\langle *, E\rangle$-tp in F_0, and we never added one in the later stages, hence it has no $\langle *, E\rangle$-tp in F. In case (iii), we also ensured that the root cluster of X is irreflexive, or avoids e. Thus, $F \nvDash \varrho$ by Lemma 5.35. On the other hand, we have $F \models B \smallsetminus \{\varrho\}$, since we added all the required tight predecessors. The only potential problem is with rules $\mathrm{Ext}_{\circ,1,E,e'}$ for $e' \ne e$ if $n = 1$ and C is reflexive, as we did not include a $\langle *, E\rangle$-tp of X in F. However, in this case $\mathrm{Par}^{e'}$ is satisfied in $\mathrm{rcl}(X)$. $\square$

Lemma 5.37 *Assume that* Par *is finite. Let L be a non-linear* Par*-extensible logic, and B the basis from Lemma* 5.36*. Define B_1 as the set of rules $\varrho^{\vee}$ for $\varrho \in B$, except that we omit*

$(\mathrm{Ext}^{=}_{\circ,0,E})^{\vee}$ *if* $\mathrm{Par} = \varnothing$ *and* $L \supseteq \mathbf{D4.1}$. *Then* B_1 *is an independent basis of* L*-admissible single-conclusion rules.*

Proof: First, assume that $\mathrm{Par} = \varnothing$ and $L \supseteq \mathbf{D4.1}$. If $L \not\supseteq \mathbf{S4}$, B_1 includes $(\mathrm{Ext}^{=}_{\bullet,1,E})^{\vee}$ or $(\mathrm{Ext}_{\circ,1,E,e})^{\vee}$, where $E = \mathbf{2}^{\varnothing}$ and e is its unique element, hence it derives $\Box z \;/\; z \vee \Box(\boxdot y \to x_0)$. Substituting $y \mapsto \top$ and $x_0 \mapsto \bot$, we obtain $\Box z \;/\; z \vee \Box\bot$, hence $\Box z \;/\; z$ as $L \supseteq \mathbf{D4}$. (If $L \supseteq \mathbf{S4}$, the rule $\Box z \;/\; z$ is outright derivable in L.) Now, the omitted rule $(\mathrm{Ext}^{=}_{\circ,0,E})^{\vee}$ is equivalent to $\Box z \vee \Box\neg\boxdot(y \leftrightarrow \Box y) \;/\; z$, and $\vdash_{\mathbf{D4.1}} \neg\Box\neg\boxdot(y \leftrightarrow \Box y)$, hence $(\mathrm{Ext}^{=}_{\circ,0,E})^{\vee}$ is derivable in $\mathbf{D4.1} + \Box z \;/\; z \subseteq L + B_1$.

Thus, $L + B_1$ is equivalent to $L + B^{\vee}$. The same argument as in Theorem 5.31 shows that $L + B^{\vee} + \mathrm{DP} = L + B$ is conservative over $L + B^{\vee}$, hence $B^{\vee}$ and B_1 are bases of L-admissible single-conclusion rules by Lemma 5.36. In order to show that B_1 is independent, let $\varrho^{\vee} \in B_1$ with $\varrho \in B$, we need to construct a frame $F \vDash L + (B_1 \smallsetminus \{\varrho^{\vee}\})$ where $\varrho^{\vee}$ fails. Let $C, n, E, e, *$ be the data associated with ϱ as in the proof of Lemma 5.36.

If $\mathrm{Par} = \varnothing$, $n = 0$, and $\vdash_L \diamondsuit\Box\bot$ (whence $C = \bullet$), we can take for F the irreflexive one-element model: the rule $\Box\bot \;/\; \bot$, implied by $(\mathrm{Ext}^{=}_{\bullet,0,E})^{\vee}$, fails in F, whereas any other rule in B_1 corresponds to $n' > 0$, thus its conclusion includes a boxed disjunct, and as such the rule holds in any model satisfying $\Box\bot$.

Otherwise, we construct F as in the proof of Lemma 5.36 with the following modification: when defining F_{k+1} from F_k, we also include as a disjoint part of the model a fresh copy of a fixed finite L-frame G. The original argument remains valid, hence $F \vDash L + (B \smallsetminus \{\varrho\})$ and $F \nvDash \varrho$, as long as G does not include a $\langle *, E\rangle$-tp of X. This can only happen if $n = 0$, and we avoid it by choosing G more carefully in this case: if $\mathrm{Par} \neq \varnothing$, we take for G a one-element model satisfying $\mathrm{Par}^{e'}$, where $E \neq \{e'\}$. If $\mathrm{Par} = \varnothing$, we take a one-element model of different reflexivity than $*$, if possible. Since we already dealt with the case when $\circ \nvDash L$ (i.e., $\vdash_L \diamondsuit\Box\bot$), the only remaining possibility is that $\bullet \nvDash L$. If the two-element cluster is an L-frame, we take it for G. If not, we have $L \supseteq \mathbf{D4.1}$, but then B_1 does not include any rule with $n = 0$, so this case cannot happen.

Since B_1 includes at least one rule ϱ' with $n' = 2$, F is downward directed: if $u, v \in F_k$, there is $w \in F_{k+1}$ incomparable with u, v in F_{k+1}, hence due to tight predecessors added for ϱ', there are $w', w'' \in F$ such that $w' < u, w$ and $w'' < v, w'$, in particular $w'' < u, v$. Consequently, $F \vDash \mathrm{DP}$. Since ϱ is equivalent to $\varrho^{\vee}$ over $\mathbf{K4} + \mathrm{DP}$, this implies $F \vDash B_1 \smallsetminus \{\varrho^{\vee}\}$ and $F \nvDash \varrho^{\vee}$. □

Theorem 5.38 *Assume that* Par *is finite, and* L *is a* Par*-extensible logic. Then* L *has an independent basis of either multiple-conclusion or single-conclusion admissible rules. Moreover,* L *has a finite basis if and only if* L *has bounded branching.*

Proof: If L has bounded branching, it has finite and independent bases by Lemma 5.33, thus let us assume that L has unbounded branching. L has an independent basis B (or B_1 in the single-conclusion case) by Lemma 5.36 (Lemma 5.37, resp.). This basis is infinite, since T includes $\langle *, n\rangle$ for arbitrarily large n for some $* \in \{\bullet, ①\}$. If we assume for contradiction that L has a finite basis, then all its elements are derivable over L from a finite fragment of B (B_1, resp.), which is then also a basis. But this contradicts the independence of B (B_1, resp.). □

6 Conclusion

As we have demonstrated in this paper, essential parts of the theory of admissible rules and unification in transitive modal logics satisfying extension properties appropriately generalize to admissibility and unification with parameters: this includes the semantic description of projective formulas and the existence of projective approximations (which implies finitary unification type and decidability of admissibility), the existence of relatively transparent bases of admissible rules reflecting extension conditions satisfied by finite frames for the logic, basic properties of bases (their finite or independent axiomatizability), and semantic correspondence of the consequence relation given by admissible rules to frames having suitable tight predecessors.

On the one hand, the results and methods used are similar to the parameter-free case on the most basic level. On the other hand, there are also significant differences that mostly boil down to the fact that admissibility and unification without parameters is insensitive to sizes of clusters in the models involved, whereas we have to take proper clusters seriously when working with parameters since elements of a finite cluster can be distinguished by a valuation of parameters. This makes the analysis technically more complicated, but more importantly, it shows up in the statements of some of the results: for example, the Löwenheim substitutions serving as building blocks of projective unifiers are no longer based on a simple choice between two formulas $\boxdot\varphi \to x$ and $\boxdot\varphi \wedge x$. (While it was not formulated that way in Definition 3.1, we can view the functions D parametrizing the substitutions as follows: we fix the parametric frame consisting of a large cluster with one point satisfying each possible combination of the parameters, and we consider all models D based on this frame.)

Perhaps even more striking is the effect on bases of admissible rules: in the parameter-free case, the bases consist of very simple rules in the form of a "relativized disjunction property"; with parameters, the rules corresponding to irreflexive tight predecessors look quite similar, but the rules for reflexive tight predecessors get more complex, as they need to express exactly the composition of the tight predecessor cluster in terms of its valuation of parameters.

We will see other interesting instances of this phenomenon in the planned sequel to this paper, which will deal with the computational complexity of admissible rules. We know from [16] that in the parameter-free case, admissibility in the logics in question can be only coNP-complete or coNEXP-complete according to whether the logic is linear or not. In contrast, there will be several more possibilities for admissibility with parameters, and they will depend on sizes of clusters occurring in frames of the logic.

Finally, let us recall that the following question was left unresolved in Section 5.2:

Problem 6.1 *Do clx logics have independent bases of admissible rules with infinitely many parameters?*

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