Banach spaces with projectional skeletons, II

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Plichko spaces

- A Banach space X is Plichko if there are a linearly dense set $G \subseteq X$ and a norming space $D \subseteq X^*$ such that

$$|\{x \in G \colon \langle x, x^* \rangle \neq 0\}| \leqslant \aleph_0$$

for every $x^* \in D$.
- $\langle X, D \rangle$ will be called a Plichko pair.

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- X is weakly Lindelöf determined (WLD) if $\langle X, X^* \rangle$ is a Plichko pair.
- X is weakly compactly generated (WCG) if $X = \operatorname{cl} \operatorname{lin} K$ for some weakly compact set K.

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Projectional skeletons

Let X be a Banach space. A projectional skeleton in X is a family $\{P_s\}_{s\in\Gamma}$ of projections on X such that

1. Γ is an up-directed partially ordered set.
2. P_sX is separable for every $s \in \Gamma$.
3. $X = \bigcup_{s\in\Gamma} P_sX$.
4. If $s_0 < s_1 < s_2 < \dots$ then $t = \sup_{n\in\omega} s_n$ exists in Γ and $P_tX = \mathrm{cl}(\bigcup_{n\in\omega} P_{s_n}X)$.

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Claim

Let $\{P_s\}_{s\in\Gamma}$ be a projectional skeleton in X. Then there exists a closed and cofinal subset Π of Γ such that

$$\sup_{s\in\Pi} \|P_s\| < +\infty.$$

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Let X be a Banach space with a projectional skeleton. Then every separable subspace is contained in a complemented separable space.

Lemma

Let $\mathfrak{s} = \{P_s\}_{s \in \Gamma}$ be a projectional skeleton in X and let $S \subseteq \Gamma$ be an up-directed set. Then the formula

$$P_S x = \lim_{s \in S} P_s x \qquad (x \in X)$$

well defines a projection on X whose range is

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Projectional resolutions of the identity

Theorem

Let X be a Banach space with a 1-projectional skeleton $\{P_s\}_{s\in\Gamma}$. Then X has a projectional resolution of the identity $\{P_\alpha\}_{\alpha\leqslant\kappa}$ such that $P_\alpha = P_{S_\alpha}$ for some up-directed set $S_\alpha \subseteq \Gamma$ ($\alpha \leqslant \kappa$).

Recall that a PRI on a Banach space X is a sequence of projections $\{P_\alpha\}_{\alpha\leqslant\kappa}$, where $\kappa = \operatorname{dens} X$ and

1. $\|P_\alpha\| = 1$, $P_\kappa = \mathrm{id}_X$ and $P_\alpha X$ has density $\leqslant \kappa + \aleph_0$,
2. $\alpha < \beta \implies P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$,
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Theorem

Let $\{P_\alpha\}_{\alpha<\kappa}$ be a projectional sequence in a Banach space X and let $D \subseteq X^*$ be a norming space such that

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and $\langle P_\alpha X, P_\alpha^* D\rangle$ is a Plichko pair for each $\alpha < \kappa$. Then $\langle X, D\rangle$ is a Plichko pair.

Corollary

Let X be a Banach space. The following properties are equivalent.

(a) *X has a commutative projectional skeleton.*

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Retractional skeletons

Let K be a compact, let Γ be an up-directed poset. An r-skeleton in K is a family of retractions $\{r_s\}_{s\in\Gamma}$ satisfying

1. $s \leqslant t \implies r_s \circ r_t = r_t \circ r_s = r_s$;
2. each $r_s[K]$ is metrizable;
3. if $s_0 < s_1 < \dots$ in Γ then $t = \sup_{n\in\omega} s_n$ exists and
$$r_t(x) = \lim_{n\to\infty} r_{s_n}(x)$$
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4. $x = \lim_{s\in\Gamma} r_s(x)$ for every $x \in K$.

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Let $\{r_s\}_{s\in\Gamma}$ be an r-skeleton in a compact K. Then $\{r_s^*\}_{s\in\Gamma}$ is a projectional skeleton in $\mathcal{C}(K)$.

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Example

Claim

Given an ordinal λ, the compact space $\lambda + 1$ has an r-skeleton.

Theorem (O. Kalenda 2002)

The space $\mathcal{C}(\omega_2 + 1)$ is not Plichko.

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Elementary substructures

Reflection Principle

Let $\varphi(x_1, \ldots, x_n)$ be a formula and let $a_1, \ldots, a_n$ be fixed sets such that $\varphi(a_1, \ldots, a_n)$ is true. Then there exists a regular cardinal χ such that

$$\langle H(\chi), \in \rangle \models \varphi(a_1, \ldots, a_n).$$

Löwenheim-Skolem Theorem

Assume $A \subseteq H(\chi)$. Then there exists $M \subseteq H(\chi)$ such that $A \subseteq M$, $|M| = |A| + \aleph_0$ and $\langle M, \in \rangle \preceq \langle H(\chi), \in \rangle$, i.e.

$$\forall\, \varphi(x_1, \ldots, x_n)\ \forall\, a_1, \ldots, a_n \in M,$$
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Let X be a Banach space, let $D \subseteq X^*$ be r-norming and let χ be a big enough regular cardinal. Further, let $M \preceq H(\chi)$ be such that $D \in M$. Then

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Moreover, the canonical projection $P_M\colon \mathrm{cl}(X \cap M) \oplus {}^{\perp}(D \cap M) \to \mathrm{cl}(X \cap M)$ *has norm* $\leqslant r$.

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Let X be a Banach space and let $D \subseteq X^*$ be a norming set. TFAE:

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Bandlow's Property Ω

Theorem (I. Bandlow 1994)

Let K be a compact. The following are properties are equivalent:

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Claim

$\langle X, D\rangle$ has Property $\Omega \iff \langle X, D\rangle$ generates projections.

Corollary

A Banach space X is weakly Lindelöf determined if and only if $\langle X, X^*\rangle$ has Property Ω.

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