Boundary conditions for fluids with pressure and shear-rate dependent viscosity

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Introduction – description of the problem

Example 1

Consider Navier-Stokes equations with Dirichlet boundary condition:

$$\begin{aligned} \operatorname{div}(\vec{v} \otimes \vec{v}) - \operatorname{div} \mathbb{D}(\vec{v}) + \nabla p &= \vec{f} && \text{in } \Omega, \\ \operatorname{div} \vec{v} &= 0 && \text{in } \Omega, \\ \vec{v} &= \vec{0} && \text{on } \partial\Omega. \end{aligned}$$

- For reasonable domain $\Omega \subset \mathbb{R}^d$ there exists a solution $(\vec{v}, p) \in W^{1,2}(\Omega)^d \times L^2_0(\Omega)$.
- For any $p_0 \in \mathbb{R}$ the pair $(\vec{v}, p + p_0)$ is again a solution.
- The value p_0 is irrelevant, one can choose $p_0 := 0$.

Introduction – description of the problem II.

Example 2

Consider the equations for fluids with the viscosity $\nu(p, |\mathbb{D}|^2)$:

$$\begin{aligned} \operatorname{div}(\vec{v} \otimes \vec{v}) - \operatorname{div} \mathbb{S}(p, |\mathbb{D}|^2) + \nabla p &= \vec{f} && \text{in } \Omega, \\ \operatorname{div} \vec{v} &= 0 && \text{in } \Omega, \\ \vec{v} &= \vec{0} && \text{on } \partial\Omega, \\ \int_\Omega p &= p_0 \in \mathbb{R}, \end{aligned}$$

where $\mathbb{S}(p, |\mathbb{D}(\vec{v})|^2) := \nu(p, |\mathbb{D}(\vec{v})|^2)\mathbb{D}(\vec{v})$.

- Under some assumptions there exists a solution $(\vec{v}, p) \in W^{1,r}(\Omega)^d \times L^{r'}(\Omega)$.
- The value p_0 is a necessary input parameter – how to choose it?

Introduction - known existence results

- Dirichlet b.c., steady-state case (Franta et al. [2005], Lanzendörfer [2009])
- Navier's b.c., unsteady case (Bulíček et al. [2007], Bulíček and Fišerová [2009])

All results consider viscosity which satisfies

(A1) $\frac{\partial\nu(p,|\mathbb{D}|^2)}{\partial|\mathbb{D}|^2} \approx (1+|\mathbb{D}|^2)^{\frac{r-4}{2}}$, $r \in (1,2)$;

(A2) $\left|\frac{\partial\nu(p,|\mathbb{D}|^2)}{\partial p}\right| \leq C(1+|\mathbb{D}|^2)^{\frac{r-4}{4}}$;

e.g.

$$\nu(p,|\mathbb{D}|^2) = (A+|\mathbb{D}|^2+(1+(\alpha p)^2)^{\frac{1}{r-2}})^{\frac{r-2}{2}}.$$

Inflow/outflow boundary conditions

Let $\partial\Omega$ be divided into Γ_D (wall) and Γ (inflow/outflow). Prescribing suitable boundary conditions of the type

$$\begin{aligned} \vec{v} &= 0 && \text{on } \Gamma_D, \\ p\vec{n} - \mathbb{S}\vec{n} &= \vec{b}(\vec{v}) && \text{on } \Gamma, \end{aligned}$$

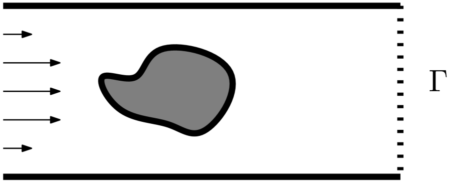
eventually

$$p - \mathbb{S}\vec{n}\cdot\vec{n} = b(\vec{v}),\ \ \vec{v}\times\vec{n} = 0 \text{ on } \Gamma,$$

the pressure mean value will be **uniquely** determined. and the constant p_0 cannot be prescribed!

Examples of inflow/outflow b.c.

Free outflow



1. Nonreflecting conditions of the type

$$p\vec{n} - \mathbb{S}\vec{n} = \vec{h}(\vec{x}) + \frac{1}{2}(\vec{v}\cdot\vec{n})^{-}\vec{v}$$

2. Conditions on the Bernoulli pressure

$$\left(p + \frac{1}{2}|\vec{v}|^2\right)\vec{n} - \mathbb{S}\vec{n} = \vec{h}(\vec{x})$$

Examples of inflow/outflow b.c. II.

Porous wall/membrane

3. Filtration conditions of the type

$$p - \mathbb{S}\vec{n} \cdot \vec{n} = p_{out} + (c_1 + c_2|\vec{v} \cdot \vec{n}| + c_3|\vec{v} \cdot \vec{n}|^2)\vec{v} \cdot \vec{n},$$
$$\vec{v} \times \vec{n} = \vec{0}$$

p_{out} ...... given pressure at the outlet
c_1, c_2, c_3 ... coefficients from the generalized Darcy law

Weak formulation

$$\begin{aligned} \operatorname{div}(\vec{v} \otimes \vec{v}) - \operatorname{div} \mathbb{S}(p, |\mathbb{D}|^2) + \nabla p &= \vec{f} && \text{in } \Omega, \\ \operatorname{div} \vec{v} &= 0 && \text{in } \Omega, \\ \vec{v} &= \vec{0} && \text{on } \Gamma_D, \\ p\vec{n} - \mathbb{S}\vec{n} &= \vec{b}(\vec{v}) && \text{on } \Gamma. \end{aligned}$$

Definition

A pair $(\vec{v}, p) \in W^{1,r}_{\Gamma_D, div}(\Omega)^d \times L^r(\Omega)$ *is called weak solution iff for every* $\vec{\varphi} \in W^{1,r}_{\Gamma}(\Omega)^d$

$$\int_\Omega \left[div(\vec{v} \otimes \vec{v}) \cdot \vec{\varphi} + \mathbb{S}(p, |\mathbb{D}(\vec{v})|^2) : \mathbb{D}(\vec{\varphi}) - p div \vec{\varphi} \right] + \int_\Gamma \vec{b}(\vec{v}) \cdot \vec{\varphi} = \langle \vec{f}, \vec{\varphi} \rangle.$$

Main result

(A1) $\frac{\partial \mathbb{S}(p,|\mathbb{D}|^2)}{\partial \mathbb{D}} \approx (1+|\mathbb{D}|^2)^{\frac{r-2}{2}}$, $r \in (1,2)$;

(A2) $\left|\frac{\partial \mathbb{S}(p,|\mathbb{D}|^2)}{\partial p}\right| \le C(1+|\mathbb{D}|^2)^{\frac{r-2}{4}}$;

(A3) for every $\vec{\varphi} \in L^\gamma(\Gamma)$:

$$\int_\Gamma \vec{b}(\vec{\varphi}) \cdot \vec{\varphi} \ge -\frac{1}{2}\int_\Gamma (\vec{\varphi}\cdot\vec{n})|\vec{\varphi}|^2.$$

Theorem

(i) *Let (A1)–(A3). Then there exists a weak solution $(\vec{v}, p)$. Moreover p is determined uniquely by $\vec{v}$.*

(ii) *For small data there is exactly one weak solution.*

For the mentioned examples of b.c. the assumption $(A3)$ holds true.

Key arguments of the proof

1. A priori estimate of the convective term

$$\int_\Omega \operatorname{div}(\vec{v} \otimes \vec{v}) \cdot \vec{v} = \int_\Omega \underbrace{\operatorname{div} \vec{v}}_{=0} |\vec{v}|^2 + \int_\Omega \vec{v} \cdot \nabla \left(\frac{|\vec{v}|^2}{2} \right)$$

$$\overset{Green}{=} \frac{1}{2} \int_\Gamma (\vec{v} \cdot \vec{n}) |\vec{v}|^2$$

$$\int_\Gamma \vec{b}(\vec{v}) \cdot \vec{v} \geq -\frac{1}{2} \int_\Gamma (\vec{v} \cdot \vec{n}) |\vec{v}|^2$$

Key arguments of the proof II.

2. Uniform presure estimate
   The Bogovskii operator (div^{-1})

$$\vec{\mathcal{B}} : L^q_0(\Omega) \to W^{1,q}_0(\Omega)^d$$

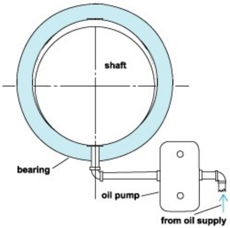
can be extended to

$$\vec{\mathcal{B}}_\Gamma : L^q(\Omega) \to W^{1,q}_{\Gamma_D}(\Omega);$$

Application

Journal bearing

- Due to extreme pressure differences the dependence of the lubricant viscosity on the pressure is observed.
- By regulating the amount of the lubricant through a small hole in the surface one can reduce the wear-out.



M. Bulíček, J. Málek, and K. R. Rajagopal. Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity. *Indiana Univ. Math. J.*, 56(1): 51–85, 2007. ISSN 0022-2518.

Miroslav Bulíček and Veronika Fišerová. Existence theory for steady flows of fluids with pressure and shear rate dependent viscosity, for low values of the power-law index. *Zeitschrift für Analysis und ihre Anwendungen*, 28:349–371, 2009.

M. Franta, J. Málek, and K. R. Rajagopal. On steady flows of fluids with pressure- and shear-dependent viscosities. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 461(2055):651–670, 2005. ISSN 1364-5021.

Martin Lanzendörfer. On steady inner flows of an incompressible fluid with the viscosity depending on the pressure and the shear rate. *Nonlinear Analysis: Real World Applications*, 10(4): 1943–1954, 2009.