Shape stability of incompressible fluids subject to Navier's slip

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Introduction

We consider the time evolution of an incompressible fluid with shear-rate–dependent viscosity:

$$\operatorname{div} \vec{v} = 0, \tag{1a}$$

$$\partial_t \vec{v} + \operatorname{div} (\vec{v} \otimes \vec{v}) = -\nabla p + \operatorname{div} \left[\nu(|\mathbb{D}(\vec{v})|^2)\mathbb{D}(\vec{v})\right] + \vec{f}, \tag{1b}$$

in a bounded 3D domain, completed by the Navier slip boundary condition

$$(\mathbb{T}\vec{n})_\tau = -a\vec{v}_\tau, \ \ \vec{v} \cdot \vec{n} = 0, \ \ a \geq 0. \tag{1c}$$

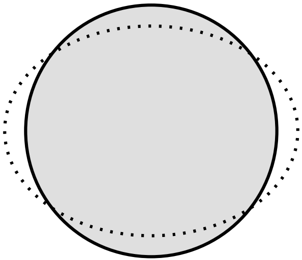
Our primary aim is to prove the stability of solutions $(\vec{v}, p)$ with respect to perturbations of the boundary.

Navier's slip condition

- In general it is not clear what is the right boundary condition.
- Slip conditions seem suitable in case of rough or chemically reacting surfaces.
- Mathematically easier to handle than no-slip (pressure estimates, collision of bodies in a fluid).

Why shape stability

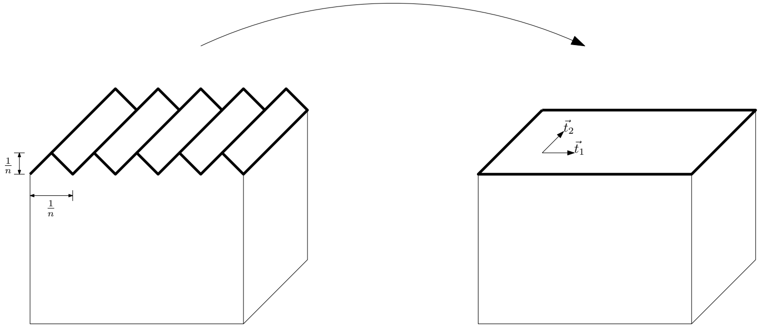
Problem is shape-stable, if small change in the geometry of the domain leads to small change in the solution(s).



- First step in shape optimization
- Important itself - robustness of model

Examples of instability of Navier's slip (1/2)

Equi-Lipschitz domains, stationary problem, $\tilde{a} > a$



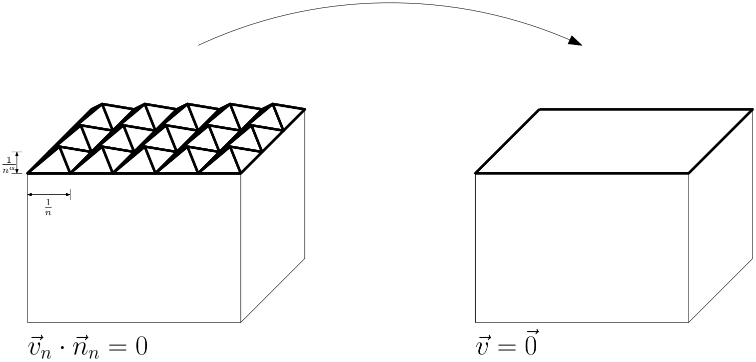
$$\vec{v}_n \cdot \vec{n}_n = 0$$
$$\vec{v}_n \times \vec{n}_n + a\mathbb{T}\vec{n}_n \times \vec{n}_n = 0$$

$$\vec{v} \cdot \vec{n} = 0$$
$$\vec{v} \cdot \vec{t}_1 = 0$$
$$\vec{v} \cdot \vec{t}_2 + \tilde{a}\mathbb{T}\vec{n} \cdot \vec{t}_2 = 0$$

(Bucur, Feireisl, and Nečasová [2008])

Examples of instability of Navier's slip (2/2)

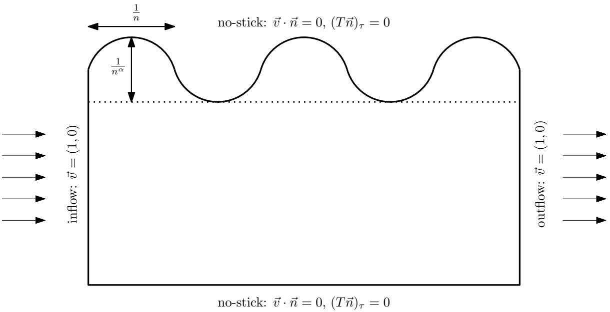
$\alpha < \frac{2r}{r+1} (< 2)$, $\vec{v}_n \rightharpoonup \vec{v}$ in $W^{1,r}(\widehat{\Omega})$ arbitrary



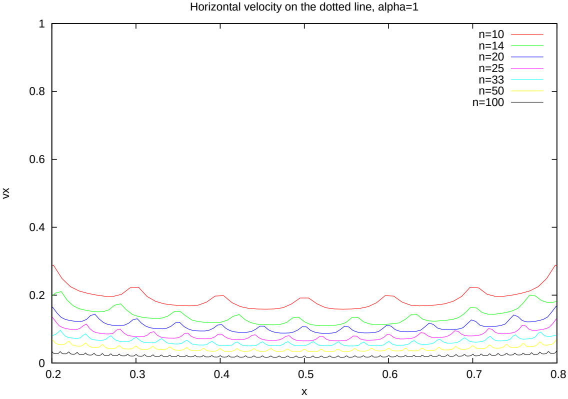
(Březina [2009])

Numerical example

Stationary Stokes problem in a chanel with curved top:

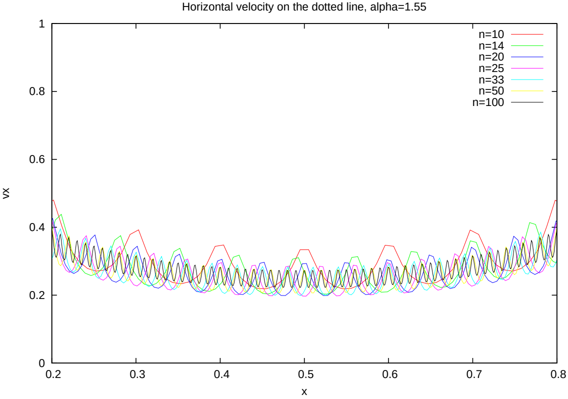


Numerical example - $\alpha = 1$



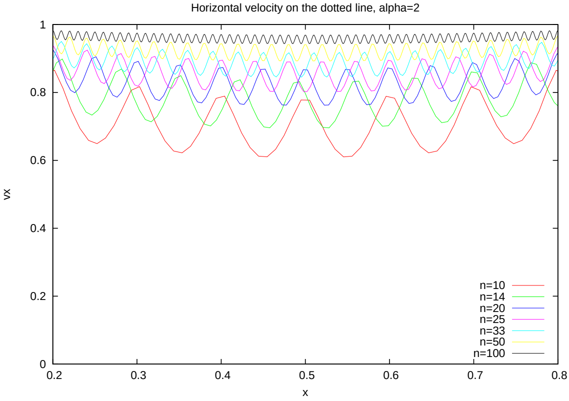
Limit boundary condition: no-slip

Numerical example - $\alpha = 1.55$



Limit boundary condition: ?

Numerical example - $\alpha = 2$

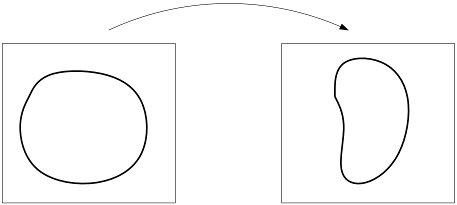


Limit boundary condition: no-stick

Convergence of domains

Let $\alpha \geq 1$ be given. We say that $\Omega_n \to \Omega$ if

- there is a bounded domain $\widehat{\Omega} \supset \Omega_n, \Omega$;
- there exist $\mathcal{C}^\alpha$-diffeomorphisms $T_n : \widehat{\Omega} \to \widehat{\Omega}$ such that
  - $T_n(\Omega) = \Omega_n$,
  - $T_n \to Id$ in $\mathcal{C}^\alpha$.



$\mathcal{O}$ will denote a system of domains Ω which are uniformly in $\mathcal{C}^\alpha$.

Extension operator and convergence of functions

Extension operator

For $\Omega \in \mathcal{O}$ there exists a bounded linear mapping

$$E_\Omega \in \mathcal{L}(W^{1,r}(\Omega), W^{1,r}(\widehat{\Omega}))$$

such that $(E_\Omega \varphi)_{|\Omega} = \varphi$. Moreover, if $\Omega_n \to \Omega$ then $E_{\Omega_n}(\varphi_{|\Omega_n}) \to E_\Omega(\varphi_{|\Omega})$ for every $\varphi \in W^{1,r}(\widehat{\Omega})$.

Convergence of functions We say that

$$\varphi_n \xrightarrow{\Omega_n \to \Omega} \varphi \text{ in } L^q(W^{1,s})$$

iff $E_{\Omega_n}\varphi_n \to E_\Omega \varphi$ in $L^q(0, T; W^{1,s}(\widehat{\Omega}))$.

Convergence of functionals

$$W_N^{1,r}(\Omega) := \{\vec{\varphi} \in W^{1,r}(\Omega);\ \vec{\varphi} \cdot \vec{n} = 0\}$$

Convergence of functionals

Let $\vec{g}_n \in L^{q'}(0,T; W_N^{-1,s'}(\Omega_n))$ and $\vec{g} \in L^{q'}(0,T; W_N^{-1,s'}(\Omega))$. We say that

$$\vec{g}_n \xrightharpoonup{\Omega_n \to \Omega} \vec{g} \text{ in } L^{q'}(W_N^{-1,s'})$$

iff $\int_0^T \langle \vec{g}_n, \vec{\varphi}_n \rangle \to \int_0^T \langle \vec{g}, \vec{\varphi} \rangle$ whenever $\vec{\varphi}_n \xrightarrow{\Omega_n \to \Omega} \vec{\varphi}$ in $L^q(W_N^{1,s})$.

Convergence of domains and Navier's slip (1/3)

Lemma (Mosco's conditions)

Let $\alpha \geq 2$ and $\Omega_n \to \Omega$.

1. *For every $\vec{\varphi}_n \in L^q(0,T;W^{1,s}_N(\Omega_n))$ s.t.*

$$\vec{\varphi}_n \xrightarrow{\Omega_n\to\Omega} \vec{\varphi} \text{ in } L^q(W^{1,s})$$

it holds that $\vec{\varphi} \in L^q(0,T;W^{1,s}_N(\Omega))$;

2. *For any $\vec{\varphi} \in L^q(0,T;W^{1,s}_N(\Omega))$ there exists a sequence $\{\vec{\varphi}_n\}$, $\vec{\varphi}_n \in L^q(0,T;W^{1,s}_N(\Omega_n))$ such that*

$$\vec{\varphi}_n \xrightarrow{\Omega_n\to\Omega} \vec{\varphi} \text{ in } L^q(W^{1,s}).$$

Convergence of domains and Navier's slip (2/3)

Proof of (i): We rewrite the impermeability condition as follows:

$$\int_{\Omega_n} (\vec{\varphi}_n(t) \cdot \nabla \psi - \psi \operatorname{div} \vec{\varphi}_n(t)) = 0 \qquad \forall \psi \in \mathcal{C}^\infty(\overline{\widehat{\Omega}}).$$

Using the fact that $\chi_{\Omega_n} \to \chi_\Omega$ in $L^q(\widehat{\Omega})$ for all $q \in [1, \infty)$, we can pass to the limit with $n \to \infty$ and obtain:

$$0 = \int_{\widehat{\Omega}} \chi_{\Omega_n} (E_{\Omega_n}\vec{\varphi}_n \cdot \nabla \psi - \psi \operatorname{div} E_{\Omega_n}\vec{\varphi}_n) \\ \to \int_{\widehat{\Omega}} \chi_\Omega \left(\widehat{\vec{\varphi}} \cdot \nabla \psi - \psi \operatorname{div} \widehat{\vec{\varphi}}\right) = \int_\Omega (\vec{\varphi} \cdot \nabla \psi - \psi \operatorname{div} \vec{\varphi})$$

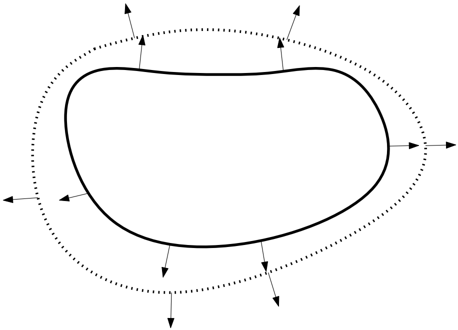
a.a. in $(0, T)$, thus $\vec{\varphi} \cdot \vec{n}_\Omega = 0$.

Convergence of domains and Navier's slip (3/3)

Proof of (ii): (idea)

- Let $T_n \in \mathcal{C}^\alpha(\widehat{\Omega}, \widehat{\Omega})$ be the transformation which maps Ω onto Ω_n;
- Define $\vec{\varphi}_n := (\nabla T_n)(\vec{\varphi} \circ T_n^{-1})$. Then $\vec{\varphi}_n \in W_N^{1,r}(\Omega_n)$;
- To prove $E_{\Omega_n}\vec{\varphi}_n \to E_\Omega \vec{\varphi}$ in $W^{1,r}(\widehat{\Omega})$ it is sufficient if

$$T_n \to Id \text{ and } T_n^{-1} \to Id \text{ in } \mathcal{C}^2.$$



Definition of weak solution

Assumptions on the extra stress $\mathbb{S}(\mathbb{D}(\vec{v})) := \nu(|\mathbb{D}(\vec{v})|^2)\mathbb{D}(\vec{v})$:

1. $\mathbb{S} \in \mathcal{C}^1(\mathbb{R}^{3\times 3}, \mathbb{R}^{3\times 3})$, $\mathbb{S}(0) = 0$;
2. There exist constants $C_1, C_2 > 0$, $\kappa \in \{0, 1\}$ and $r > 1$ s.t.

$$C_1(\kappa + |\mathbb{A}|^{r-2})|\mathbb{B}|^2 \leq \frac{\partial \mathbb{S}(\mathbb{A})}{\partial \mathbb{A}} :: (\mathbb{B} \otimes \mathbb{B}) \leq C_2(\kappa + |\mathbb{A}|^{r-2})|\mathbb{B}|^2$$

   for any $0 \neq \mathbb{A}, \mathbb{B} \in \mathbb{R}^{3\times 3}$.

Assumptions on the body force and initial datum:

3. $\vec{f} \in L^{r'}(0, T; L^{r'}(\widehat{\Omega})^3)$;
4. $\vec{v}_0 \in L^2(\widehat{\Omega})^3$, $\operatorname{div} \vec{v}_0 = 0$.

Definition of weak solution

We say that $(\vec{v}, p)$ is a weak solution of problem $(P(\Omega))$, iff

- $\vec{v} \in L^\infty(0,T;L^2(\Omega)) \cap L^r(0,T;W^{1,r}_{N,div}(\Omega))$;
  $\partial_t \vec{v} \in L^\sigma(0,T;W_N^{-1,\sigma}(\Omega))$ and $p \in L^\sigma(0,T;L^\sigma_0(\Omega))$;
- for every $\vec{\varphi} \in L^{\sigma'}(0,T;W_N^{1,\sigma'}(\Omega))$, $\sigma = \begin{cases} r' & \text{if } r \geq \frac{11}{5} \\ \frac{5r}{6} & \text{if } r < \frac{11}{5} \end{cases}$:

$$\int_0^T \Big[\langle \partial_t \vec{v}, \vec{\varphi}\rangle - (\vec{v} \otimes \vec{v}, \nabla \vec{\varphi}) + (\mathbb{S}(\mathbb{D}(\vec{v})), \mathbb{D}(\vec{\varphi})) - (p, \operatorname{div} \vec{\varphi}) + a \int_{\partial\Omega} \vec{v} \cdot \vec{\varphi}\Big] = \int_0^T \langle \vec{f}, \vec{\varphi}\rangle;$$

- for a.a. $t \in (0,T)$ the energy inequality holds:

$$\frac{1}{2}\|\vec{v}(t)\|_2^2 + \int_0^t (\mathbb{S}(\mathbb{D}(\vec{v})), \mathbb{D}(\vec{v})) + a\int_0^t \|\vec{v}\|^2_{2,\partial\Omega} \leq \frac{1}{2}\|\vec{v}_0\|_2^2 + \int_0^t \langle \vec{f}, \vec{v}\rangle;$$

- $\lim_{t\to 0+} \|\vec{v}(t) - \vec{v}_0\|_2^2 = 0$.

Existence of solutions

Theorem (Bulíček, Málek, and Rajagopal [2007])

Let $r > \frac{8}{5}$, $T > 0$ *and* $\Omega \in \mathcal{C}^{1,1}$. *Then*

- *there exists a weak solution* $(\vec{v}, p)$ *to* $(P(\Omega))$;
- *if moreover* $r > \frac{5}{2}$, *then the weak solution is unique.*

Shape stability

Theorem (Main result)

- *Let* $r > \frac{8}{5}$, $\alpha \geq 2$, $\Omega_n \to \Omega$ *and* $(\vec{v}_n, p_n)$ *be solutions to* $(P(\Omega_n))$. *Then there is a solution* $(\vec{v}, p)$ *to* $(P(\Omega))$ *s.t.*

$$\begin{aligned} \vec{v}_n &\rightharpoonup \vec{v} && \text{in } L^r(0,T;W^{1,r}(\widehat{\Omega})),\\ \vec{v}_n &\rightharpoonup^* \vec{v} && \text{in } L^\infty(0,T;L^2(\widehat{\Omega})),\\ p_n &\rightharpoonup p && \text{in } L^\sigma(0,T;L^\sigma(\widehat{\Omega})), \end{aligned}$$

provided that all functions were extended to $\widehat{\Omega}$.

Outline of the proof

1. Uniform estimate:
   - Korn's inequality
   - L^q-regularity for Laplace equation with Neumann b.c.
2. Limit passage $\Omega_n \longrightarrow \Omega$:
   - Take $\vec{\varphi} \in L^{\sigma'}(0, T; W_N^{1,\sigma'}(\Omega))$ and approximate by $\vec{\varphi}_n \longrightarrow \vec{\varphi}$;
   - Weak convergence satisfies to pass in the terms

$$\int_0^T \left[\langle \partial_t \vec{v}_n, \vec{\varphi}_n \rangle_{\Omega_n} - (p_n, \operatorname{div} \vec{\varphi}_n)_{\Omega_n} + a \int_{\partial\Omega_n} \vec{v}_n \cdot \vec{\varphi}_n \right];$$

   - Aubin-Lions lemma gives strong convergence enabling limit in convective term;
   - Nonlinear term $\mathbb{S}(\mathbb{D}(\vec{v}_n))$ handled by strong monotonicity and Vitali's lemma ($r > \frac{11}{5}$) or by L^∞-truncation method ($r < \frac{11}{5}$).

Uniform energy estimate (1/3)

Lemma (Korn's inequality)

Let $\Omega \in \mathcal{C}^{1,1}$ and $q \in (1, \infty)$. Then the inequality

$$C_{Korn} \|\vec{v}\|_{1,q,\Omega} \leq \|\mathbb{D}(\vec{v})\|_{q,\Omega} + \|\vec{v}\|_{2,\partial\Omega}.$$

holds for all $\vec{v} \in W^{1,q}(\Omega)$, $\operatorname{tr} \vec{v} \in L^2(\partial\Omega)$ with a constant $C_{Korn} := C_{Korn}(q) > 0$ independent of Ω.

Uniform energy estimate (2/3)

For any $z \in L^q_0(\Omega)$ we denote by $\mathcal{N}_\Omega^{-1}(z)$ the unique solution of the Neumann problem

$$\Delta u = z \text{ in } \Omega \qquad \nabla u \cdot \vec{n} = 0 \text{ on } \partial\Omega, \qquad \int_\Omega u = 0.$$

Lemma (L^q-regularity for Laplace equation with Neumann b.c.)

There exists a constant $C_{reg} := C_{reg}(q) > 0$ independent of $\Omega \in \mathcal{C}^{1,1}$ such that

$$\|\mathcal{N}_\Omega^{-1}(z)\|_{2,q,\Omega} \leq C_{reg}\|z\|_{q,\Omega}.$$

Uniform energy estimate (3/3)

Lemma (Uniform energy estimate)

There is a constant $c > 0$ *independent of* $\Omega \in \mathcal{C}^{1,1}$ *such that every weak solution* $(\vec{v}, p)$ *of* $(P(\Omega))$ *satisfies:*

$$\sup_{t\in(0,T)} \|\vec{v}(t)\|_{2,\Omega}^2 + \int_0^T \|\vec{v}\|_{1,r,\Omega}^r + \int_0^T \|\vec{v}\|_{2,\partial\Omega}^2 + \int_0^T \|p\|_{\sigma,\Omega}^\sigma \le c.$$

Proof: Energy inequality + Korn's inequality + regularity of $\mathcal{N}_\Omega^{-1}(\cdot)$

Limit passage $\Omega_n \to \Omega$ (1/10)

1. **Weak limits**

   Existence of

$$\vec{v} \in L^\infty(0,T;L^2(\Omega)) \cap L^r(0,T;W^{1,r}(\Omega)),$$
$$p \in L^\sigma(0,T;L^\sigma_0(\Omega))$$

   follows from energy estimate.
   Mosco's property (i) yields that $\vec{v} \cdot \vec{n} = 0$.

Limit passage $\Omega_n \to \Omega$ (2/10)

2. **Convergence of r.h.s.**
   Take $\vec{\varphi} \in L^{\sigma'}(0,T;W^{1,\sigma'}_N(\Omega))$. Mosco's property (ii) yields existence of $\vec{\varphi}_n \in L^{\sigma'}(0,T;W^{1,\sigma'}_N(\Omega_n))$ strongly converging to $\vec{\varphi}$.

$$\int_0^T \int_{\Omega_n} \vec{f} \cdot \vec{\varphi}_n = \int_0^T \int_{\widehat{\Omega}} \underbrace{\chi_{\Omega_n}}_{\to \chi_\Omega \text{ in } L^q(\widehat{\Omega})} \underbrace{\vec{f}}_{\in L^{r'}(L^{r'})} \cdot \underbrace{E_{\Omega_n}\vec{\varphi}_n}_{\to E_\Omega \vec{\varphi} \text{ in } L^r(L^{r+\varepsilon})}$$

3. **Convergence of pressure term** – analogous

$$\int_0^T \int_{\Omega_n} p_n \operatorname{div} \vec{\varphi}_n \to \int_0^T \int_\Omega p \operatorname{div} \vec{\varphi}$$

Limit passage $\Omega_n \to \Omega$ (3/10)

4. **Convergence of $\int_0^T \langle \partial_t \vec{v}_n, \vec{\varphi}_n \rangle_{\Omega_n}$**

   We define auxiliary functionals $\vec{g}_n \in L^\sigma(0,T; W^{-1,\sigma}(\widehat{\Omega}))$:

$$\int_0^T \langle \vec{g}_n, \vec{\varphi} \rangle_{\widehat{\Omega}} := \int_0^T \Big[(\vec{v}_n \otimes \vec{v}_n, \nabla \vec{\varphi})_{\Omega_n} - (\mathbb{S}(\mathbb{D}(\vec{v}_n)), \mathbb{D}(\vec{\varphi}))_{\Omega_n} - a \int_{\partial\Omega_n} \vec{v}_n \cdot \vec{\varphi} + (p_n, \operatorname{div} \vec{\varphi})_{\Omega_n} + \langle \vec{f}, \vec{\varphi} \rangle_{\Omega_n} \Big]$$

   so that for every $\vec{\varphi} \in L^{\sigma'}(0,T; W_N^{1,\sigma'}(\Omega_n))$ it holds:

$$\int_0^T \langle \vec{g}_n, \vec{\varphi} \rangle_{\widehat{\Omega}} = \int_0^T \langle \partial_t \vec{v}_n, \vec{\varphi} \rangle_{\Omega_n}$$

   Energy estimates $\Rightarrow \{\vec{g}_n\}$ is bounded $\Rightarrow \vec{g}_n \rightharpoonup \vec{g}$.

Limit passage $\Omega_n \longrightarrow \Omega$ (4/10)

To identify $\vec{g}$ we use definition of $\partial_t \vec{v}_n$: Let $\psi \in \mathcal{D}(0,T)$ and $\vec{\phi}_n \in W_N^{1,\sigma'}(\Omega_n)$.

$$\int_0^T \langle \vec{g}_n, \vec{\phi}_n \rangle_{\widehat{\Omega}} \psi = \int_0^T \langle \partial_t \vec{v}_n, \vec{\phi}_n \rangle_{\Omega_n} \psi = -\int_0^T (\vec{v}_n, \vec{\phi}_n)_{\Omega_n} \psi' = -\int_0^T (\vec{v}_n, \chi_{\Omega_n} \vec{\phi}_n)_{\widehat{\Omega}} \psi'.$$

If $\vec{\phi}_n \xrightarrow{\Omega_n \to \Omega} \vec{\phi}$ in $W_N^{1,\sigma'}$ then we can pass to the limit:

$$\int_0^T \langle \vec{g}, \vec{\phi} \rangle_{\widehat{\Omega}} \psi = -\int_0^T (\vec{v}, \chi_\Omega \vec{\phi})_{\widehat{\Omega}} \psi' = \int_0^T \langle \partial_t \vec{v}, \vec{\phi} \rangle_\Omega \psi,$$

and thus $\vec{g} = \partial_t \vec{v}$.

Limit passage $\Omega_n \to \Omega$ (5/10)

5. **Convergence of traces**

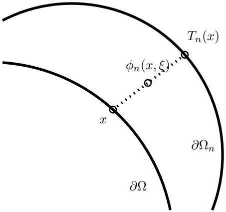
   We first shift the boundary integral to $\partial\Omega$:

$$\int_0^T \int_{\partial\Omega_n} \vec{v}_n \cdot \vec{\varphi}_n = \int_0^T \int_{\partial\Omega} (\vec{v}_n \circ T_n) \cdot \underbrace{(\vec{\varphi}_n \circ T_n)}_{\to\vec{\varphi} \text{ in } L^{\sigma'}} \underbrace{\det \nabla T_n}_{\to 1 \text{ in } L^\infty} .$$

For almost all $\vec{x} \in \partial\Omega$ we can write:

$$\vec{v}_n(T_n(\vec{x})) = \vec{v}_n(\vec{x}) + \int_0^1 \partial_\xi [\vec{v}_n(\vec{\phi}_n(\vec{x}, \xi))] \, d\xi$$

where $\vec{\phi}_n(\vec{x}, \xi) := (1-\xi)\vec{x} + \xi T_n$, $\xi \in [0, 1]$.



Limit passage $\Omega_n \to \Omega$ (6/10)

For n large enough, T_n is close to Id and so is $\vec{\phi}_n$. We can thus assume that $|\det \nabla \vec{\phi}_n^{-1}| \leq c$. Integrating over $(0,T) \times \partial\Omega$ we obtain:

$$\begin{aligned}
\int_0^T \int_{\partial\Omega} |\vec{v}_n \circ T_n - \vec{v}_n|^r \\
&= \int_0^T \int_{\partial\Omega} \int_0^1 \left|\nabla[\vec{v}_n \circ \vec{\phi}_n] \cdot (T_n - Id)\right|^r \\
&\leq \|T_n - Id\|_{\infty,\widehat{\Omega}}^r \int_0^T \int_{\vec{\phi}_n(\partial\Omega \times (0,1))} |\nabla \vec{v}_n|^r \det \nabla \vec{\phi}_n^{-1} \\
&\leq c\|T_n - Id\|_{\infty,\widehat{\Omega}}^r \int_0^T \int_{\widehat{\Omega}} |\nabla \vec{v}_n|^r.
\end{aligned}$$

Limit passage $\Omega_n \to \Omega$ (7/10)

6. **Convergence of** $\int_0^T (\vec{v}_n \otimes \vec{v}_n, \nabla\vec{\varphi}_n)_{\Omega_n}$
   It is sufficient to show that

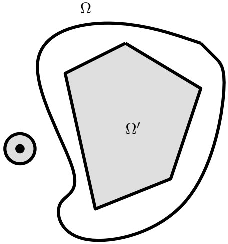
$$\chi_{\Omega_n}\vec{v}_n \to \chi_\Omega \vec{v} \text{ a.a. in } (0,T)\times\widehat{\Omega}.$$

   Let Ω' be a compact subset of Ω and $\xi \in \mathcal{D}(\Omega)$, $\xi_{|\Omega'} \equiv 1$.
   Then $\xi\vec{v}_n \in L^r(W_0^{1,r}(\Omega))$ and $\partial_t[\xi\vec{v}_n] \in L^\sigma(W^{1,\sigma'}(\Omega)^*)$.

Aubin-Lions $\Rightarrow$ $\vec{v}_n \to \vec{v}$ in $L^z(0,T;L^z(\Omega'))$
for some $z \geq 1$, consequently

$$\chi_{\Omega_n}\vec{v}_n \to \chi_\Omega \vec{v} \text{ pointwise a.a. in } (0,T)\times\Omega.$$

On the other hand, for $\vec{x} \in \widehat{\Omega}\setminus\overline{\Omega}$ there exists a small neighborhood of $\vec{x}$ on which χ_{Ω_n} vanishes as n is sufficiently large.



Limit passage $\Omega_n \to \Omega$ (8/10)

7. **Limit passage in the viscous term**
We use the method of L^∞-test function to prove that
$$\mathbb{D}(\vec{v}_n) \to \mathbb{D}(\vec{v}) \text{ a.e. in } \Omega. \tag{2}$$
Once this is proved, we can easily pass to the limit in
$$\int_0^T \int_{\Omega_n} \mathbb{S}(\mathbb{D}(\vec{v}_n)) : \mathbb{D}(\vec{\varphi}_n) = \int_0^T \int_{\widehat{\Omega}} \chi_{\Omega_n} \mathbb{S}(\mathbb{D}(\vec{v}_n)) : \mathbb{D}(\vec{\varphi}_n).$$
We show that for every $\varepsilon > 0$ and for some $\theta \in (\frac{1}{r}, 1)$ there is a subsequence of $\{\vec{v}_n\}$ such that
$$\lim_{n\to\infty} \int_0^T \int_{\Omega'} |(\mathbb{S}(\mathbb{D}(\vec{v}_n)) - \mathbb{S}(\mathbb{D}(\vec{v}))) : \mathbb{D}(\vec{v}_n - \vec{v})|^\theta \leq \varepsilon. \tag{3}$$
Then we can take $\varepsilon_m \searrow 0$, $\Omega'_m \to \Omega$ uniformly, and for each $m \in \mathbb{N}$ select subsequences so that the Cantor diagonal sequence fulfils (2).

Limit passage $\Omega_n \to \Omega$ (9/10)

Due to convergence of $\vec{v}_n$ in measure it is sufficient to show that

$$\int_{Q_n} \xi(\mathbb{S}(\mathbb{D}(\vec{v}_n)) - \mathbb{S}(\mathbb{D}(\vec{v}))) : \mathbb{D}(\vec{v}_n - \vec{v}) \leq C\varepsilon,$$

where $Q_n := \{(t,\vec{x}) \in (0,T) \times \Omega';\ |\vec{v}_n - \vec{v}| < L\}$ for some ε and L. We define

$$\vec{\varphi}_n := \xi(\vec{v}_n - \vec{v}) \left(1 - \min\left(\frac{|\vec{v}_n - \vec{v}|}{L}, 1\right)\right),$$

$$\vec{\psi}_n := \vec{\varphi}_n - \nabla \mathcal{N}_\Omega^{-1}(\vec{\varphi}_n).$$

Then $\vec{\psi} \to 0$ in $L^s(L^s(\Omega))$ for all $s \in [1,\infty)$ and weakly in $L^r(W_0^{1,r}(\Omega))$.

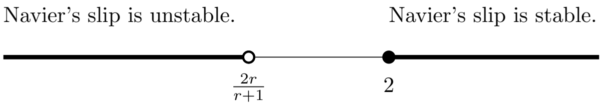
Limit passage $\Omega_n \to \Omega$ (10/10)

It is possible to show that

$$\int_{Q_n} \xi(\mathbb{S}(\mathbb{D}(\vec{v}_n)) - \mathbb{S}(\mathbb{D}(\vec{v}))) : \mathbb{D}(\vec{v}_n - \vec{v}) \\ = \int_0^T \int_{\Omega'} (\mathbb{S}(\mathbb{D}(\vec{v}_n)) - \mathbb{S}(\mathbb{D}(\vec{v}))) : \mathbb{D}(\vec{\psi}_n) + \text{l.o.t.} \quad (4)$$

Term on the r.h.s. vanishes for $n \to \infty$ as follows from the weak formulation of $(P(\Omega_n))$.

Conclusion



- Numerical and rigorous arguments show that Navier's slip condition is unstable under boundary perturbations of low regularity, while for smooth deformations it remains stable.
- The question is, what happens in the region in between.

J. Březina. Asymptotic properties of solutions to the equations of incompressible fluid mechanics. Preprint 1, Nečas Center for Mathematical Modeling, 2009.

D. Bucur, E. Feireisl, and Š. Nečasová. Influence of wall roughness on the slip behaviour of viscous fluids. *Proc. Roy. Soc. Edinburgh Sect. A*, 138(5):957–973, 2008.

M. Bulíček, J. Málek, and K. R. Rajagopal. Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity. *Indiana Univ. Math. J.*, 56(1): 51–85, 2007. ISSN 0022-2518.