Shape stability of incompressible fluids subject to Navier's slip

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Introduction

We consider the time evolution of an incompressible fluid with shear-rate–dependent viscosity:

$$\operatorname{div} \vec{v} = 0, \tag{1a}$$

$$\partial_t \vec{v} + \operatorname{div}(\vec{v} \otimes \vec{v}) = -\nabla p + \operatorname{div}\left[\nu(|\mathbb{D}(\vec{v})|^2)\mathbb{D}(\vec{v})\right] + \vec{f}, \tag{1b}$$

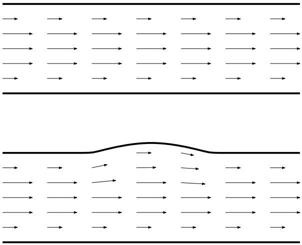
in a bounded 3D domain, completed by the Navier slip boundary condition

$$(\mathbb{T}\vec{n})_\tau = -a\vec{v}_\tau, \ \ \vec{v} \cdot \vec{n} = 0. \tag{1c}$$

Our primary aim is to study the domain dependence of solutions $(\vec{v}, p)$.

What is shape stability

Problem is shape-stable, if small change in the geometry of the domain leads to small change in the solution(s).



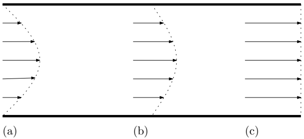
In other words: If $\Omega_n \to \Omega$, do solutions $(\vec{v}_n, p_n)$ converge to a solution of the same problem on Ω?

- Indicator of model robustness
- Keystone for existence of optimal shapes

Slip vs. no slip

In general it is not clear what is the right boundary condition for walls.

- Often it is reasonable to prescribe no slip: $\vec{v}_{|\partial\Omega} = 0$. This condition is known to be stable under quite general boundary perturbations (e.g. equi-Lipschitz setting).
- In case of e.g. rough or chemically patterned surfaces some kind of slip condition is more suitable.



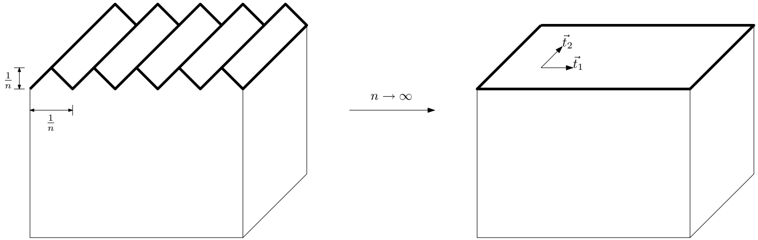
Velocity profiles: (a) no-slip, (b) partial slip, (c) complete slip.

We study the shape stability problem for Navier's slip condition

$$(\mathbb{T}\vec{n})_\tau = -a\vec{v}_\tau, \ \vec{v}\cdot\vec{n} = 0, \ a \geq 0.$$

Examples of instability of Navier's slip (1/2)

Equi-Lipschitz domains, stationary problem



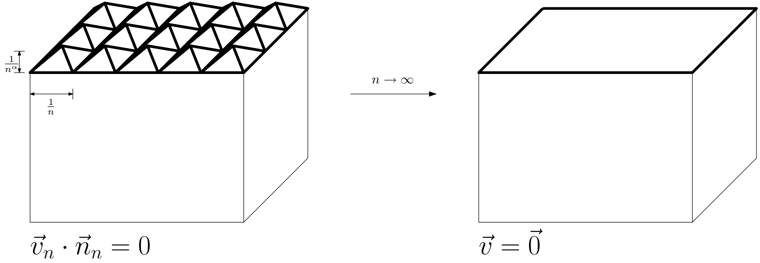
$$\vec{v}_n \cdot \vec{n}_n = 0$$
$$(\mathbb{T}\vec{n}_n)_{\tau_n} = -a(\vec{v}_n)_{\tau_n}$$

$$\vec{v} \cdot \vec{n} = 0$$
$$\vec{v} \cdot \vec{t}_1 = 0$$
$$\mathbb{T}\vec{n} \cdot \vec{t}_2 = -\tilde{a}\vec{v} \cdot \vec{t}_2, \ \tilde{a} > a$$

(Bucur, Feireisl, and Nečasová [2008])

Examples of instability of Navier's slip (2/2)

Domains in $\mathcal{C}^{1,\alpha-1}$, $\alpha < \frac{2r}{r+1}$
arbitrary $\{\vec{v}_n\}$ s.t. $\vec{v}_n \rightharpoonup \vec{v}$ in $W^{1,r}$



(Březina [2009])

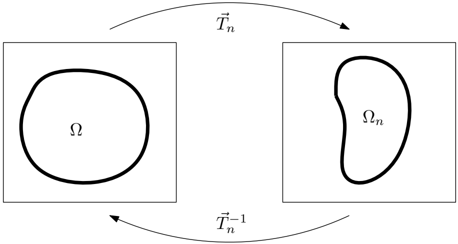
Convergence of domains

Let $D \subset \mathbb{R}^3$ be bounded and $\mathcal{O} := \{\Omega \in \mathcal{C}^{1,1};\ \Omega \subset D\}$.

Definition

We say that $\Omega_n \to \Omega$ in $\mathcal{O}$ if there exist $\mathcal{C}^{1,1}$-diffeomorphisms $\vec{T}_n : \overline{D} \to \overline{D}$ such that

- $\vec{T}_n(\Omega) = \Omega_n$,
- $\vec{T}_n \to \vec{Id}$ and $\vec{T}_n^{-1} \to \vec{Id}$ in $\mathcal{C}^{1,1}(\overline{D}, \overline{D})$,
- $|\nabla^2 \vec{T}_n|$ and $|\nabla^2 \vec{T}_n^{-1}|$ are bounded independently of n.



Convergence of functions

Let

$$W_N^{1,q}(\Omega) := \{\vec{\varphi} \in W^{1,q}(\Omega);\ \vec{\varphi}\cdot\vec{n} = 0 \text{ on } \partial\Omega\},$$
$$\vec{X}^q := L^q(D,\mathbb{R}^3) \times L^q(D,\mathbb{R}^{3\times 3}).$$

For $\Omega \in \mathcal{O}$ we define

$$E_\Omega : W^{1,q}(\Omega,\mathbb{R}^3) \to \vec{X}^q;\ \vec{\varphi} \mapsto (\chi_\Omega\vec{\varphi}, \chi_\Omega\nabla\vec{\varphi}).$$

Definition

Let $\Omega_n \to \Omega$ in $\mathcal{O}$. We say that

$$\vec{\varphi}_n \to \vec{\varphi} \text{ in } \vec{X}^q \text{ iff } E_{\Omega_n}\vec{\varphi}_n \to E_\Omega\vec{\varphi};$$

$$\vec{\varphi}_n \rightharpoonup \vec{\varphi} \text{ in } \vec{X}^q \text{ iff } E_{\Omega_n}\vec{\varphi}_n \rightharpoonup E_\Omega\vec{\varphi}.$$

If $\vec{\varphi}_n \in W_0^{1,q}(\Omega_n)$ then

$$\vec{\varphi}_n \to \vec{\varphi} \text{ in } \vec{X}^q \quad \Leftrightarrow \quad \chi_{\Omega_n}\vec{\varphi}_n \to \chi_\Omega\vec{\varphi} \text{ in } W_0^{1,q}(D).$$

Structural assumptions

Assumptions on the extra stress $\mathbb{S}(\mathbb{D}(\vec{v})) := \nu(|\mathbb{D}(\vec{v})|^2)\mathbb{D}(\vec{v})$:

1. $\mathbb{S} \in \mathcal{C}^1(\mathbb{R}^{3\times 3}, \mathbb{R}^{3\times 3})$, $\mathbb{S}(0) = 0$;
2. There exist constants $C_1, C_2 > 0$, $\kappa \in \{0, 1\}$ and $r > 1$ s.t.

$$C_1(\kappa + |\mathbb{A}|^{r-2})|\mathbb{B}|^2 \leq \frac{\partial \mathbb{S}(\mathbb{A})}{\partial \mathbb{A}} :: (\mathbb{B} \otimes \mathbb{B}) \leq C_2(\kappa + |\mathbb{A}|^{r-2})|\mathbb{B}|^2$$

   for any $0 \neq \mathbb{A}, \mathbb{B} \in \mathbb{R}^{3\times 3}$.

Assumptions on the body force and initial datum:

3. $\vec{f} \in L^{r'}(0, T; \vec{X}^{r'})$;
4. $\vec{v}_0 \in L^2(D)^3$, $\operatorname{div} \vec{v}_0 = 0$.

Definition of weak solution

We say that $(\vec{v}, p)$ is a weak solution of problem $(P(\Omega))$, iff

- $\vec{v} \in L^\infty(0, T; L^2(\Omega)) \cap L^r(0, T; W^{1,r}_{N,div}(\Omega))$;
  $\partial_t \vec{v} \in L^\sigma(0, T; W_N^{-1,\sigma}(\Omega))$ and $p \in L^\sigma(0, T; L^\sigma_0(\Omega))$;
- for every $\vec{\varphi} \in L^{\sigma'}(0, T; W_N^{1,\sigma'}(\Omega))$, $\sigma = \begin{cases} r' & \text{if } r \geq \frac{11}{5} \\ \frac{5r}{6} & \text{if } r < \frac{11}{5} \end{cases}$:

$$\int_0^T \Big[\langle \partial_t \vec{v}, \vec{\varphi} \rangle - (\vec{v} \otimes \vec{v}, \nabla \vec{\varphi}) + (\mathbb{S}(\mathbb{D}(\vec{v})), \mathbb{D}(\vec{\varphi})) - (p, \operatorname{div} \vec{\varphi}) + a \int_{\partial\Omega} \vec{v} \cdot \vec{\varphi} \Big] = \int_0^T \langle \vec{f}, \vec{\varphi} \rangle;$$

- for a.a. $t \in (0, T)$ the energy inequality holds:

$$\frac{1}{2} \|\vec{v}(t)\|_2^2 + \int_0^t (\mathbb{S}(\mathbb{D}(\vec{v})), \mathbb{D}(\vec{v})) + a \int_0^t \|\vec{v}\|^2_{2,\partial\Omega} \leq \frac{1}{2} \|\vec{v}_0\|_2^2 + \int_0^t \langle \vec{f}, \vec{v} \rangle;$$

- $\lim_{t \to 0+} \|\vec{v}(t) - \vec{v}_0\|_2^2 = 0$.

Existence of solutions

Theorem (Bulíček, Málek, and Rajagopal [2007])

Let $T > 0$, $\Omega \in \mathcal{C}^{1,1}$, $\Omega \subset \mathbb{R}^3$ *and* $r > \frac{8}{5}$. *Then*

- *there exists a weak solution* $(\vec{v}, p)$ *to* $(P(\Omega))$;
- *if moreover* $r > \frac{5}{2}$, *then the weak solution is unique.*

Shape stability

Theorem (Main result)

- *Let* $r > \frac{8}{5}$, $\Omega_n \to \Omega$ *and* $(\vec{v}_n, p_n)$ *be solutions to* $(P(\Omega_n))$. *Then*

$$\begin{aligned} \vec{v}_n &\rightharpoonup \vec{v} && \text{in } L^r(0,T;\vec{X}^r), \\ \chi_{\Omega_n}\vec{v}_n &\rightharpoonup^* \chi_\Omega \vec{v} && \text{in } L^\infty(0,T;L^2(D)), \\ \chi_{\Omega_n} p_n &\rightharpoonup \chi_\Omega p && \text{in } L^\sigma(0,T;L^\sigma_0(D)). \end{aligned}$$

where $(\vec{v}, p)$ *is a solution to* $(P(\Omega))$.

Outline of the proof

1. Uniform estimate:
   - Korn's inequality for $\Omega \in \mathcal{C}^{0,1}$
   - L^q-regularity for Laplace eq. with Neumann b.c. ($\Omega \in \mathcal{C}^{1,1}$)
2. Limit passage $\Omega_n \to \Omega$:
   - Mosco's conditions:

$$\begin{array}{rcl} \vec{\varphi}_n \cdot \vec{n} = 0 \text{ and } \vec{\varphi}_n \rightharpoonup \vec{\varphi} & \Rightarrow & \vec{\varphi} \cdot \vec{n} = 0 \\ \exists \{\vec{\varphi}_n\},\ \vec{\varphi}_n \cdot \vec{n} = 0 \text{ and } \vec{\varphi}_n \to \vec{\varphi} & \Leftarrow & \vec{\varphi} \cdot \vec{n} = 0 \end{array}$$

   - Weak convergence of $(\vec{v}_n, p_n)$ satisfies to pass in the terms

$$\int_0^T \left[\langle \partial_t \vec{v}_n, \vec{\varphi}_n \rangle_{\Omega_n} - (p_n, \operatorname{div} \vec{\varphi}_n)_{\Omega_n} + a \int_{\partial\Omega_n} \vec{v}_n \cdot \vec{\varphi}_n \right];$$

   - Aubin-Lions lemma gives strong convergence enabling limit in convective term;
   - Nonlinear term $\mathbb{S}(\mathbb{D}(\vec{v}_n))$ handled by strong monotonicity and Vitali's lemma ($r > \frac{11}{5}$) or by L^∞-truncation method ($r < \frac{11}{5}$).

Mosco's conditions (1/4)

Lemma (Mosco's conditions)

Let $\Omega_n \to \Omega$.

1. *For every* $\{\vec{\varphi}_n\}$ *s.t.* $\vec{\varphi}_n \in L^q(0,T;W^{1,s}_N(\Omega_n))$ *and*

$$\vec{\varphi}_n \rightharpoonup \vec{\varphi} \text{ in } L^q(0,T;\vec{X}^s)$$

   it holds that $\vec{\varphi} \in L^q(0,T;W^{1,s}_N(\Omega))$;

2. *For any* $\vec{\varphi} \in L^q(0,T;W^{1,s}_N(\Omega))$ *there exists a sequence* $\{\vec{\varphi}_n\}$, $\vec{\varphi}_n \in L^q(0,T;W^{1,s}_N(\Omega_n))$ *such that*

$$\vec{\varphi}_n \to \vec{\varphi} \text{ in } L^q(0,T;\vec{X}^s).$$

Mosco's conditions (2/4)

Proof of (i): We rewrite the impermeability condition as follows:

$$\int_{\Omega_n} (\vec{\varphi}_n(t) \cdot \nabla\psi - \psi \operatorname{div} \vec{\varphi}_n(t)) = 0 \qquad \forall \psi \in \mathcal{C}^\infty(\overline{D})$$

and pass to the limit with $n \to \infty$ to conclude that

$$\int_{\Omega} (\vec{\varphi} \cdot \nabla\psi - \psi \operatorname{div} \vec{\varphi}) = 0$$

a.a. in $(0, T)$, thus $\vec{\varphi} \cdot \vec{n}_\Omega = 0$.

Mosco's conditions (3/4)

Proof of (ii): (idea)
Let $\vec{T}_n$ be the transformation which maps Ω onto Ω_n and assume first that $\vec{T}_n, \vec{T}_n^{-1} \to \vec{Id}$ in $\mathcal{C}^2$. If we define

$$\vec{\varphi}_n := \left[(\nabla T_n)\vec{\varphi}\right] \circ T_n^{-1}$$

then $\vec{\varphi}_n \cdot \vec{n}_{\Omega_n} = \vec{\varphi} \cdot \vec{n}_\Omega = 0$, so that $\vec{\varphi}_n \in L^q(0,T; W_N^{1,s}(\Omega_n))$.
The convergence of $\vec{T}_n$ and $\vec{T}_n^{-1}$ is then sufficient to prove

$$E_{\Omega_n}\vec{\varphi}_n \to E_\Omega\vec{\varphi} \text{ in } L^q(0,T;\vec{X}^s).$$

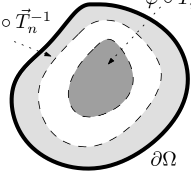
Roughly speaking,

$$\nabla\left[(\nabla\vec{T}_n)\vec{\varphi}\right] \approx \underbrace{(\nabla\vec{T}_n)}_{\to\mathbb{I}}\nabla\vec{\varphi} + \underbrace{(\nabla^2\vec{T}_n)}_{\to 0}\vec{\varphi}.$$

Mosco's conditions (4/4)

Proof of (ii): (continued)
If $\vec{T}_n, \vec{T}_n^{-1}$ converge in $\mathcal{C}^{1,1}$ only, then we modify the construction using a cut-off function $\eta_\epsilon = \begin{cases} 1 & \text{near } \partial\Omega \\ 0 & \text{if dist}_{\partial\Omega} > \epsilon \end{cases}$:

$$\vec{\varphi}_n := \left\{ \left[\eta_\epsilon (\nabla T_n) + (1 - \eta_\epsilon) \mathbb{I} \right] \vec{\varphi} \right\} \circ T_n^{-1}.$$

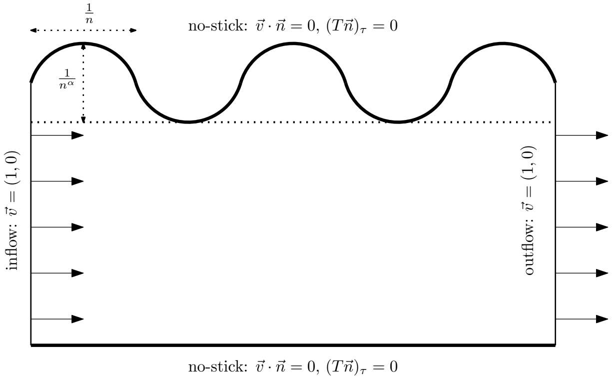


Then

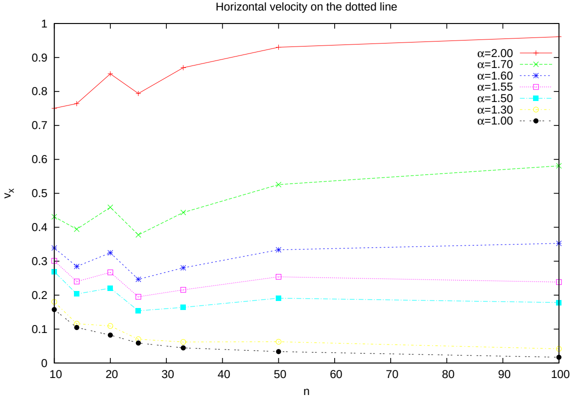
$$\nabla \left[\eta_\epsilon (\nabla \vec{T}_n - \mathbb{I}) \vec{\varphi} + \vec{\varphi} \right] \approx \underbrace{(\nabla^2 \vec{T}_n)}_{\text{bdd.}} \underbrace{\eta_\epsilon \vec{\varphi}}_{\text{small}} + \underbrace{(\nabla \vec{T}_n - \mathbb{I})}_{\to 0} (\eta_\epsilon \nabla \vec{\varphi} + \nabla \eta_\epsilon \vec{\varphi}) + \nabla \vec{\varphi}.$$

Numerical example

Stationary Stokes problem in a chanel with curved top:

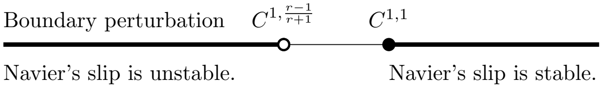


Numerical example - results



Limit boundary condition: no-slip

Conclusion



- Numerical and rigorous arguments show that Navier's slip condition is unstable under boundary perturbations of low regularity, while for smooth deformations it remains stable.
- The minimal regularity of boundary that preserves slip is optimal for unsteady flows of power law fluids.

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D. Bucur, E. Feireisl, and Š. Nečasová. Influence of wall roughness on the slip behaviour of viscous fluids. *Proc. Roy. Soc. Edinburgh Sect. A*, 138(5):957–973, 2008.

M. Bulíček, J. Málek, and K. R. Rajagopal. Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity. *Indiana Univ. Math. J.*, 56(1): 51–85, 2007. ISSN 0022-2518.