Shape sensitivity analysis of time-dependent flows of incompressible fluids

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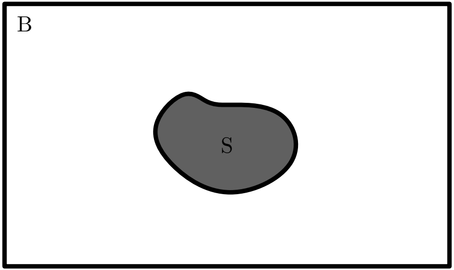
IFIP 2011, Berlin, September 12–16, 2011

1. Introduction
2. Formal results
3. Overview of mathematical theory for non-Newtonian fluids
4. Justification of the material and shape derivatives
5. Conclusion

Introduction

We study the shape differentiability of a given functional that depends on the flow of an incompressible fluid. The fluid is contained in a bounded domain $\Omega := B \setminus S \subset \mathbb{R}^2$ which surrounds an obstacle S.



Equations of motion

Motion of the fluid is described by the following system:

$$\partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbb{S}(\mathbb{D}\mathbf{v}) + \nabla p + \mathbb{C}\mathbf{v} = \mathbf{f} \qquad \text{in } Q, \tag{1a}$$

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } Q, \tag{1b}$$

$$\mathbf{v} = \mathbf{0} \qquad \text{on } \Sigma, \tag{1c}$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0 \qquad \text{in } \Omega. \tag{1d}$$

Q	$(0, T) \times \Omega$
Σ	$(0, T) \times \partial\Omega$
Ω	bounded domain in $\mathbb{R}^2$
$\mathbb{S}$	traceless part of the Cauchy stress
$\mathbb{D}\mathbf{v}$	symmetric part of $\nabla \mathbf{v}$
$\mathbb{C}$	Coriolis force (constant skew symmetric matrix)

Shape functional

The shape of the obstacle S is to be optimized subject to the drag functional

$$J(\Omega) := \int_0^T \int_{\partial S} (\mathbb{S}(\mathbb{D}\mathbf{v}) - p\mathbb{I})\mathbf{n} \cdot \mathbf{d},$$

where $\mathbf{d}$ is a given unit vector.

Our aim is to

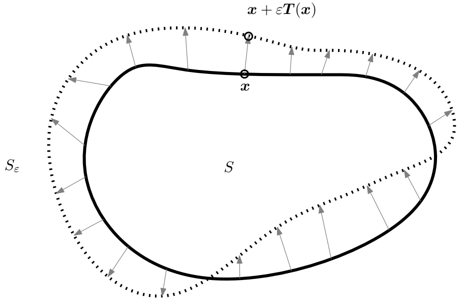
- show that J is differentiable;
- find the shape gradient of J.

Deformation of the shape of S

We choose a vector field $\mathbf{T} \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^2)$ vanishing in the vicinity of ∂B and define the mapping

$$\mathbf{y} = \mathbf{x} + \varepsilon \mathbf{T}(\mathbf{x}),$$

which describes the perturbation of the boundary ∂S. For small $\varepsilon > 0$ the mapping $\mathbf{x} \mapsto \mathbf{y}$ takes diffeomorphically the region Ω onto $\Omega_\varepsilon = B \setminus S_\varepsilon$ where $S_\varepsilon = \mathbf{y}(S)$.



Shape and material derivatives of solutions

On the perturbed domain we consider the problem

$$\begin{aligned} \partial_t \bar{\mathbf{v}}_\varepsilon + \operatorname{div}(\bar{\mathbf{v}}_\varepsilon \otimes \bar{\mathbf{v}}_\varepsilon) - \operatorname{div}\mathbb{S}(\mathbb{D}\bar{\mathbf{v}}_\varepsilon) + \nabla \bar{p}_\varepsilon + \mathbb{C}\bar{\mathbf{v}}_\varepsilon &= \mathbf{f} && \text{in } Q_\varepsilon, \\ \operatorname{div}\bar{\mathbf{v}}_\varepsilon &= 0 && \text{in } Q_\varepsilon, \\ \bar{\mathbf{v}}_\varepsilon &= \mathbf{0} && \text{on } \Sigma_\varepsilon, \\ \bar{\mathbf{v}}_\varepsilon(0,\cdot) &= \mathbf{v}_0 && \text{in } \Omega_\varepsilon. \end{aligned}$$

The shape and material derivatives are formally defined as follows:

$$\mathbf{v}' := \lim_{\varepsilon \to 0} \frac{\bar{\mathbf{v}}_\varepsilon - \mathbf{v}}{\varepsilon}, \quad \dot{\mathbf{v}} := \lim_{\varepsilon \to 0} \frac{\bar{\mathbf{v}}_\varepsilon \circ \mathbf{y} - \mathbf{v}}{\varepsilon}.$$

It will be usefull to work with a modified material derivative

$$\tilde{\mathbf{v}} := \lim_{\varepsilon \to 0} \frac{\det(\mathbb{I} + \varepsilon \nabla \mathbf{T})(\mathbb{I} + \varepsilon \nabla \mathbf{T})^{-1}(\bar{\mathbf{v}}_\varepsilon \circ \mathbf{y}) - \mathbf{v}}{\varepsilon},$$

which has the property $\operatorname{div}\tilde{\mathbf{v}} = 0$.

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$$

which has the property $\operatorname{div} \tilde{\mathbf{v}} = 0$.

Formal results for the shape and material derivatives

The shape derivative of J has the form:

$$dJ(\Omega;\mathbf{T}) := \frac{dJ}{d\varepsilon}\bigg|_{\varepsilon=0} = \int_0^T \int_{\partial S} (\mathbb{S}'(\mathbb{D}\mathbf{v})\mathbb{D}\mathbf{v}' - p'\mathbb{I}) : \mathbf{d}\otimes\mathbf{n} - (\mathbf{f}\cdot\mathbf{d})\mathbf{T}\cdot\mathbf{n}$$

where the shape derivatives $(\mathbf{v}', p')$ are determined through the linearized system

$$\begin{aligned}
\partial_t\mathbf{v}' + \operatorname{div}(\mathbf{v}'\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{v}') - \operatorname{div}(\mathbb{S}'(\mathbb{D}\mathbf{v})\mathbb{D}\mathbf{v}')& \\
+\nabla p' + \mathbb{C}\mathbf{v}' &= \mathbf{0} && \text{in } Q,\\
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Formal results involving adjoint states

To avoid the shape derivatives (which depend implicitly on the direction $\mathbf{T}$) we introduce the adjoint problem

$$
\begin{aligned}
-\partial_t \mathbf{w} - 2(\mathbb{D}\mathbf{w})\mathbf{v} - \operatorname{div}\left(\mathbb{S}'(\mathbb{D}\mathbf{v})^\top \mathbb{D}\mathbf{w}\right) + \nabla s - \mathbb{C}\mathbf{w} &= \mathbf{0} && \text{in } Q,\\
\operatorname{div} \mathbf{w} &= 0 && \text{in } Q,\\
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\mathbf{w}(T,\cdot) &= \mathbf{0} && \text{in } \Omega.
\end{aligned}
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Consequently dJ has the form

$$
dJ(\Omega;\mathbf{T}) = -\int_0^T \int_{\partial S} \left[\left(\mathbb{S}'(\mathbb{D}\mathbf{v})^\top \mathbb{D}\mathbf{w} - s\mathbb{I} \right) : \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \otimes \mathbf{n} + \mathbf{f}\cdot\mathbf{d} \right] \mathbf{T}\cdot\mathbf{n}.
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Validity of formal results

Questions

Existence and uniqueness of state variables?
$\rightarrow$ mathematical theory of non-Newtonian fluids

Existence and uniqueness of shape derivatives and adjoints?
$\rightarrow$ existence theory of generalized Stokes system
$\rightarrow$ regularity of non-Newtonian fluids

Regularity of adjoints and existence of shape gradient dJ?
$\rightarrow$ L^p theory for generalized Stokes system

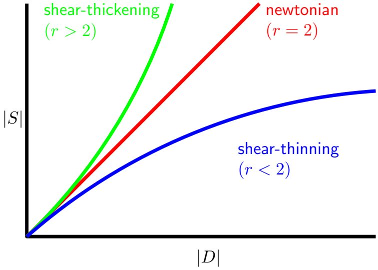
Fluids with shear rate-dependent viscosity

Non-newtonian fluids have applications in many areas of sciences and industry, e.g.:

- hemodynamics, biomechanics, mechanics of geomaterials;
- mechanical engineering, polymer chemistry, food industry...

Essentially, we deal with stress tensors of the type

$$\mathbb{S}(\mathbb{D}\mathbf{v}) \approx (1 + |\mathbb{D}\mathbf{v}|^{r-2})\mathbb{D}\mathbf{v}, \ r > 1.$$



Structural assumptions

We assume that there exist constants $C_1, C_2, C_3 > 0$, and $r \in [2, 4)$ s.t.

$$C_1(1 + |\mathbb{A}|^{r-2})|\mathbb{B}|^2 \leq \mathbb{S}'(\mathbb{A}) :: (\mathbb{B} \otimes \mathbb{B}) \leq C_2(1 + |\mathbb{A}|^{r-2})|\mathbb{B}|^2,$$

$$|\mathbb{S}''(\mathbb{A})| \leq C_3(1 + |\mathbb{A}|^{r-3})$$

for any $0 \neq \mathbb{A}, \mathbb{B} \in \mathbb{R}^{2\times 2}$.
Consequently

- $\mathbb{S}$ is strongly monotone;
- The mapping $\mathbb{D} \mapsto \mathbb{S}(\mathbb{D})$ is continuous from L^r to L^{r-1};
- The mapping $\mathbb{D} \mapsto \mathbb{S}'(\mathbb{D})$ is continuous from L^r to L^{r-2}.

Mathematical theory for non-Newtonian fluids in 2D

$$\partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbb{S}(\mathbb{D}\mathbf{v}) + \nabla p = \mathbf{f} \quad \text{in } Q, \tag{3a}$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } Q, \tag{3b}$$

$$\mathbf{v} = \mathbf{0} \quad \text{on } \Sigma, \tag{3c}$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0 \quad \text{in } \Omega. \tag{3d}$$

Theorem (Ladyzhenskaya [1967], Lions [1969])

For $r \geq 2$, $\mathbf{f} \in L^2(0, T; \mathbf{W}^{1,2}_{0,div}(\Omega)^*)$ *and* $\mathbf{v}_0 \in \mathbf{W}^{1,r}_{0,div}(\Omega)$ *there is a unique weak solution of* (3) *that satisfies the energy inequality*

$$\frac{1}{2}\|\mathbf{v}(t)\|^2_{2,\Omega} + \int_0^t \int_\Omega \mathbb{S}(\mathbb{D}\mathbf{v}) : \mathbb{D}\mathbf{v} = \int_0^t \int_\Omega \mathbf{f} \cdot \mathbf{v} + \frac{1}{2}\|\mathbf{v}_0\|^2_{2,\Omega}, \quad \textit{for a.a. } t \in (0, T).$$

Remark: Existence theory has been extended also to $3D$ and $r < 2$ e.g. by Frehse et al. [2000], Bulíček et al. [2007].

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Theorem (Kaplický [2005])

If $r \in [2,4)$ and additionally for some $\tilde{q} > 2$

$$\mathbf{f} \in L^\infty(0,T;\mathbf{L}^{\tilde{q}}(\Omega)), \partial_t \mathbf{f} \in L^{\tilde{q}}(0,T;\mathbf{W}^{-1,\tilde{q}}(\Omega)) \text{ and } \mathbf{v}_0 \in \mathbf{W}^{2,2}(\Omega)$$

then there exists $q > 2$ and $\alpha > 0$ such that for arbitrary $\epsilon > 0$

$$\mathbf{v} \in L^\infty(\epsilon,T;\mathbf{W}^{2,q}(\Omega)),$$

$$\nabla \mathbf{v} \in \mathcal{C}^{0,\alpha}([\epsilon,T] \times \overline{\Omega}),$$

$$\partial_t \mathbf{v} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{W}^{1,2}_{0,div}(\Omega)).$$

Well-posedness of the linearized system, L^p-estimates

$$
\begin{aligned}
\partial_t \mathbf{u} - \operatorname{div}(\mathbb{A}\nabla\mathbf{u}) + \nabla q &= \mathbf{f} && \text{in } Q, && \text{(4a)}\\
\operatorname{div}\mathbf{u} &= 0 && \text{in } Q, && \text{(4b)}\\
\mathbf{u} &= \mathbf{0} && \text{on } \Sigma, && \text{(4c)}\\
\mathbf{u}(0,\cdot) &= \mathbf{u}_0 && \text{in } \Omega && \text{(4d)}
\end{aligned}
$$

Let $\mathbb{A} \in L^\infty(Q, \mathbb{R}^{2^4})$ be symmetric and positive definite, $\mathbf{f} \in L^2(Q)$, $\mathbf{u}_0 \in L^2_{0,\mathrm{div}}(\Omega)$. Then the generalized Stokes system (4) has a unique weak solution.

Theorem (Solonnikov [2001])

Let $\Omega \in \mathcal{C}^3$, $\mathbb{A} \in \mathcal{C}(\overline{Q}, \mathbb{R}^{2^4})$ be symmetric and positive definite, $\mathbf{f} \in L^p(Q)$, $\mathbf{u}_0 \in \mathbf{W}^{1,p}_{0,div}(\Omega) \cap \mathbf{W}^{2-2/p,p}(\Omega)$. Then the generalized Stokes problem has a unique weak solution which satisfies:

$$\mathbf{u} \in L^p(0,T;\mathbf{W}^{2,p}(\Omega)) \cap W^{1,p}(0,T;\mathbf{L}^p(\Omega)),$$

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$$\mathbf{u} \in L^p(0,T;\mathbf{W}^{2,p}(\Omega)) \cap W^{1,p}(0,T;\mathbf{L}^p(\Omega)),$$

$$q \in L^p(0,T;W^{1,p}(\Omega)).$$

Boundedness of $\mathbb{D}\mathbf{v}$

We will require the linearized problem with $\mathbb{A} \approx \mathbb{S}'(\mathbb{D}\mathbf{v})$ to have a unique solution. For this reason we need

$$\mathbb{D}\mathbf{v} \in L^{\infty}(Q),$$

however the result by Kaplický yields this only starting at $t = \epsilon > 0$. To overcome this technical issue, we follow Wachsmuth and Roubíček [2010] and assume that

- the process has started already at some time $-\epsilon < 0$,
- the initial condition at $t = -\epsilon$ and the body force satisfy assumptions of the Theorem by Kaplický so that $\mathbb{D}\mathbf{v} \in \mathcal{C}(\overline{Q})$,
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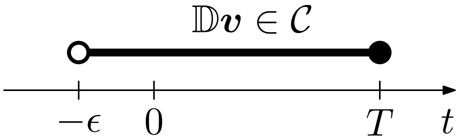

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Formulation in the fixed domain

Introducing the notation

$$\mathbb{M} := \mathbb{I} + \varepsilon\nabla\mathbf{T}^\top, \ \mathfrak{g} := \det\mathbb{M}, \ \mathbb{N} := \mathfrak{g}\mathbb{M}^{-1},$$

$$\mathbf{v}_\varepsilon := \mathbb{N}^\top(\bar{\mathbf{v}}_\varepsilon \circ \mathbf{y}), \quad p_\varepsilon := \bar{p}_\varepsilon \circ \mathbf{y}, \quad \mathbf{v}_{0\varepsilon} := \mathbb{N}^\top(\mathbf{v}_0 \circ \mathbf{y}),$$

we formulate the problem for $(\bar{\mathbf{v}}_\varepsilon, \bar{p}_\varepsilon)$ on the fixed domain:

$$\begin{aligned}
\mathfrak{g}\mathbb{N}^{-1}\mathbb{N}^{-\top}\partial_t\mathbf{v}_\varepsilon + \operatorname{div}(\mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon) - \mathbb{N}^{-1}\operatorname{div}(\mathbb{N}^\top\mathbb{S}(\mathbb{D}_\varepsilon\mathbf{v}_\varepsilon)) & \\
+\nabla p_\varepsilon + \mathbb{C}\mathbf{v}_\varepsilon = \mathbf{f} + \mathbf{A}_\varepsilon^1 & \quad \text{in } Q, \\
\operatorname{div}\mathbf{v}_\varepsilon = 0 & \quad \text{in } Q, \\
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Here $\mathbb{D}_\varepsilon\mathbf{v}_\varepsilon := \mathfrak{g}^{-1}(\mathbb{N}\nabla(\mathbb{N}^{-\top}\mathbf{v}_\varepsilon))_{sym}$, and $\mathbf{A}_\varepsilon^1 \in L^2(0,T;\mathbf{W}^{1,2}_{0,div}(\Omega)^*)$ is a term of order ε, defined by

$$\begin{aligned}
\mathbf{A}_\varepsilon^1 = {} & \operatorname{div}(\mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon) - \mathbb{N}^{-1}\operatorname{div}(\mathbf{v}_\varepsilon \otimes \mathbb{N}^{-\top}\mathbf{v}_\varepsilon) \\
& + (\mathbb{C} - \mathfrak{g}\mathbb{N}^{-1}\mathbb{C}\mathbb{N}^{-\top})\mathbf{v}_\varepsilon + \mathfrak{g}\mathbb{N}^{-1}(\mathbf{f}\circ\mathbf{y}) - \mathbf{f}.
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Note: We do not require any additional regularity of $\mathbf{v}_\varepsilon$.

Uniform estimates and convergence of $\mathbf{v}_\varepsilon$

Using the energy inequality we find that

$$\{\mathbf{v}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^r(0,T;\mathbf{W}^{1,r}_{0,div}(\Omega)) \cap \mathbf{L}^{2r}(Q),$$

$$\{\mathbb{N}^{-1}\mathrm{div}\,(\mathbb{N}^\top\mathbb{S}(\mathbb{D}_\varepsilon\mathbf{v}_\varepsilon))\}_{\varepsilon>0} \text{ is bounded in } L^r(0,T;\mathbf{W}^{1,r}_{0,div}(\Omega))^*,$$

$$\{\mathfrak{g}\mathbb{N}^{-1}\mathbb{N}^{-\top}\partial_t\mathbf{v}_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^r(0,T;\mathbf{W}^{1,r}_{0,div}(\Omega))^*.$$

Hence there is a weak limit $\bar{\mathbf{v}}$ of the sequence $\{\mathbf{v}_\varepsilon\}$ in the above spaces. Using the strong monotonicity of $\mathbb{S}$ we obtain:

$$\mathbb{D}\mathbf{v}_\varepsilon \to \mathbb{D}\mathbf{v} \text{ strongly in } L^r(Q),$$

which implies

$$\mathbb{N}^{-1}\mathrm{div}\,(\mathbb{N}^\top\mathbb{S}(\mathbb{D}_\varepsilon\mathbf{v}_\varepsilon)) \to \mathrm{div}\,\mathbb{S}(\mathbb{D}\bar{\mathbf{v}}) \text{ in } L^r(0,T;\mathbf{W}^{1,r}_{0,div}(\Omega))^*$$

and thus $\bar{\mathbf{v}} = \mathbf{v}$ is the solution to (1).

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System for the differences

We are going to estimate the differences

$$(\mathbf{u}_\varepsilon, q_\varepsilon) := \left(\frac{\mathbf{v}_\varepsilon - \mathbf{v}}{\varepsilon}, \frac{p_\varepsilon - p}{\varepsilon}\right).$$

which solve the problem:

$$\begin{aligned} \mathfrak{g}\mathbb{N}^{-1}\mathbb{N}^{-\top}\partial_t \mathbf{u}_\varepsilon + \operatorname{div}(\mathbf{v}_\varepsilon \otimes \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon \otimes \mathbf{v}) + \nabla q_\varepsilon + \mathbb{C}\mathbf{u}_\varepsilon & \\ -\mathbb{N}^{-1}\operatorname{div}\left(\mathbb{N}^\top \frac{\mathbb{S}(\mathbb{D}_\varepsilon \mathbf{v}_\varepsilon) - \mathbb{S}(\mathbb{D}_\varepsilon \mathbf{v})}{\varepsilon}\right) &= \frac{1}{\varepsilon}\mathbf{A}_\varepsilon && \text{in } Q, \\ \operatorname{div}\mathbf{u}_\varepsilon &= 0 && \text{in } Q, \\ \mathbf{u}_\varepsilon &= \mathbf{0} && \text{on } \Sigma, \\ \mathbf{u}_\varepsilon(0,\cdot) &= \frac{\mathbf{v}_{0\varepsilon} - \mathbf{v}_0}{\varepsilon} && \text{in } \Omega. \end{aligned}$$

The term $\mathbf{A}_\varepsilon$ on the r.h.s. is given by

$$\mathbf{A}_\varepsilon = \mathbf{A}_\varepsilon^1 + (\mathbb{I} - \mathfrak{g}\mathbb{N}^{-1}\mathbb{N}^{-\top})\partial_t \mathbf{v} + \operatorname{div}\mathbb{S}(\mathbb{D}\mathbf{v}) - \mathbb{N}^{-1}\operatorname{div}\left(\mathbb{N}^\top \mathbb{S}(\mathbb{D}_\varepsilon \mathbf{v})\right).$$

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Uniform estimates of the differences $\mathbf{u}_\varepsilon$

Due to the regularity of $\mathbf{v}$, we have that

$$\left\{\frac{1}{\varepsilon}\mathbf{A}_\varepsilon\right\}_{\varepsilon>0} \quad \text{is bounded in } L^2(0,T;\mathbf{W}^{1,2}_{0,div}(\Omega)^*).$$

Testing the momentum equation by $\mathbf{u}_\varepsilon$ we obtain:

$$\|\mathbf{u}_\varepsilon(t)\|_2^2+\int_0^t\|\mathbb{D}\mathbf{u}_\varepsilon\|_2^2 \le C\Big(\int_0^T\|\nabla\mathbf{v}_\varepsilon\|_2\|\nabla\mathbf{u}_\varepsilon\|_2\|\mathbf{u}_\varepsilon\|_2^2+\|\mathbf{v}_0\|_2^2+\frac{1}{\varepsilon}\int_0^T\|\mathbf{A}_\varepsilon\|^2\Big)$$

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$$\left\{\frac{\mathbb{S}(\mathbb{D}_\varepsilon\mathbf{v}_\varepsilon)-\mathbb{S}(\mathbb{D}_\varepsilon\mathbf{v})}{\varepsilon}\right\}_{\varepsilon>0} \quad \text{is bounded in } L^{\frac{2r}{3r-4}}(Q),$$

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Convergence to the material derivatives

Anyway, with help of the strong convergence of $\mathbb{D}\mathbf{v}_\varepsilon$ and the above estimates we can show that

$$\mathbf{u}_\varepsilon \rightharpoonup \tilde{\mathbf{v}},$$
$$\frac{1}{\varepsilon}\mathbf{A}_\varepsilon \rightharpoonup \mathbf{A}_0', \text{ in some weak sense,}$$

where $\tilde{\mathbf{v}}$ is a solution of the linearized problem

$$\begin{aligned}
\partial_t \tilde{\mathbf{v}} + \operatorname{div}(\tilde{\mathbf{v}} \otimes \mathbf{v} + \mathbf{v} \otimes \tilde{\mathbf{v}}) - \operatorname{div}(\mathbb{S}'(\mathbb{D}\mathbf{v})\mathbb{D}\tilde{\mathbf{v}}) + \nabla \tilde{p} + \mathbb{C}\tilde{\mathbf{v}} = \mathbf{A}_0' \quad & \text{in } Q, \\
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\tilde{\mathbf{v}}(0,\cdot) = \tilde{\mathbf{v}}_0 \quad & \text{in } \Omega,
\end{aligned}$$

where $\tilde{\mathbf{v}}_0 := \mathbb{N}'^\top \mathbf{v}_0 + (\nabla \mathbf{v}_0)\mathbf{T}$. Since this problem has a unique weak solution, we have justified the shape differentiability of the state equation.

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Shape derivative of J

The original definition

$$J(\Omega) = \int_0^T \int_{\partial S} (\mathbb{S}(\mathbb{D}\mathbf{v}) - p\mathbb{I})\mathbf{n} \cdot \mathbf{d}$$

is not suitable for weak solutions, hence we rewrite it integrating by parts and using the state equation:

$$J(\Omega) = \int_\Omega (\mathbf{v}(T) - \mathbf{v}_0) \cdot \xi + \int_Q [(\mathbb{C}\mathbf{v} - \mathbf{f}) \cdot \xi + (\mathbb{S}(\mathbb{D}\mathbf{v}) - \mathbf{v} \otimes \mathbf{v})] : \nabla \xi.$$

Here $\xi \in C^\infty(\overline{\Omega}, \mathbb{R}^2)$ is arbitrary function supported in the vicinity of S such that $\xi_{|\partial S} = \mathbf{d}$ and $\operatorname{div} \xi = 0$. Similarly, we obtain:

$$\begin{aligned} J(\Omega_\varepsilon) = &\int_\Omega \mathfrak{g}\mathbb{N}^{-1}\mathbb{N}^{-\top}(\mathbf{v}_\varepsilon(T) - \mathbf{v}_{0\varepsilon}) \cdot \xi \\ &+ \int_Q \Big[\mathfrak{g}\left(\mathbb{N}^{-1}\mathbb{C}\mathbb{N}^{-\top}\mathbf{v}_\varepsilon - \mathbb{N}^{-1}(\mathbf{f} \circ \mathbf{y})\right) \cdot \xi \\ &+ \left(\mathbb{N}^\top\mathbb{S}(\mathbb{D}_\varepsilon\mathbf{v}_\varepsilon) - \mathbf{v}_\varepsilon \otimes (\mathbb{N}^{-\top}\mathbf{v}_\varepsilon)\right) : \nabla(\mathbb{N}^{-\top}\xi)\Big]. \end{aligned}$$

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Shape derivative of J

Using the available convergence of $\mathbf{u}_\varepsilon$ we can show that

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = J_D(\tilde{\mathbf{v}}, \tilde{p}) + J_G(\mathbf{T}),$$

where J_D and J_G are linear functions of $\tilde{\mathbf{v}}$, $\tilde{p}$ and $\mathbf{T}$, respectively. Since the pair $(\tilde{\mathbf{v}}, \tilde{p})$ depends continuously on the $\mathcal{C}^2$ norm of $\mathbf{T}$, we conclude that the map

$$\mathbf{T} \mapsto dJ(\Omega; \mathbf{T})$$

is a bounded linear functional on $\mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^2)$.

Shape gradient of J

The initial and boundary condition for the adjoint problem

$$\begin{aligned} -\partial_t \mathbf{w} - 2(\mathbb{D}\mathbf{w})\mathbf{v} - \operatorname{div}\left(\mathbb{S}'(\mathbb{D}\mathbf{v})^\top \mathbb{D}\mathbf{w}\right) + \nabla s - \mathbb{C}\mathbf{w} &= \mathbf{0} && \text{in } Q, \\ \operatorname{div}\mathbf{w} &= 0 && \text{in } Q, \\ \mathbf{w} &= \mathbf{d} && \text{on } \Sigma, \\ \mathbf{w}(T,\cdot) &= \mathbf{0} && \text{in } \Omega. \end{aligned}$$

are incompatible, hence we cannot expect $\mathbb{D}\mathbf{w}, s \in L^1(\Sigma)$.

Nevertheless, on the time interval $(0, T-\delta)$ with any small δ we can apply the L^p theory for the linearized system and obtain

$$dJ(\Omega;\mathbf{T}) = -\lim_{\delta \searrow 0} \int_0^{T-\delta} \int_{\partial S} \left[\left(\mathbb{S}'(\mathbb{D}\mathbf{v})^\top \mathbb{D}\mathbf{w} - s\mathbb{I} \right) : \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \otimes \mathbf{n} + \mathbf{f}\cdot\mathbf{d} \right] \mathbf{T}\cdot\mathbf{n}.$$

Shape gradient of J

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are incompatible, hence we cannot expect $\mathbb{D}\mathbf{w}, s \in L^1(\Sigma)$.
Nevertheless, on the time interval $(0, T-\delta)$ with any small δ we can apply the L^p theory for the linearized system and obtain

$$dJ(\Omega;\mathbf{T}) = -\lim_{\delta \searrow 0} \int_0^{T-\delta} \int_{\partial S} \left[\left(\mathbb{S}'(\mathbb{D}\mathbf{v})^\top \mathbb{D}\mathbf{w} - s\mathbb{I} \right) : \frac{\partial \mathbf{v}}{\partial \mathbf{n}} \otimes \mathbf{n} + \mathbf{f}\cdot\mathbf{d} \right] \mathbf{T}\cdot\mathbf{n}.$$

Conclusion

- Under appropriate regularity assumptions we have shown the existence of material derivatives and differentiability of the shape functional.
- Recent results in the regularity theory indicate that the differentiability might be expected for $r \geq 4$ (Beirão da Veiga et al. [2010]).
- Unfortunately, the cases $r < 2$ as well as higher space dimensions remain open as long as $\nabla \mathbf{v} \notin L^\infty$.

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