Control of Distributed Systems
- Tutorial and Overview

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September 9, 2011

Distributed systems consist of an interconnection of two or more subsystems. Control of such systems is structured by two or more controllers, each receiving an observation stream from a local subsystem and providing an input to the local subsystem. Coordinated distributed systems are defined for linear systems, for Gaussian systems, and for discrete-event systems and an algebraic-geometric characterization is provided. Coordination control of distributed systems requires a specific control synthesis procedure which is presented. Distributed control with communication between controllers is formulated and discussed.

2000 Mathematics Subject Classification: 93A14, 93A15, 93C65, 93E20.

Keywords Distributed system, coordination control, hierarchical control, distributed control, distributed control with communication.

1 Introduction

The reader will find in this paper an exposition on control of distributed and of hierarchical systems. The presentation includes problems, a description of the case studies, concepts of distributed systems, control architectures, expositions of coordination control and distributed control with communication, and theorems. The character of the paper is tutorial and, in addition, it provides an overview of ongoing research. The paper is partly based on the results of the project *Control for Coordination of Distributed Systems* (C4C Project) which was sponsored in part by the European Commission.

Most of the large-scale engineering systems in use today are composed of many smaller systems. Such a system will be referred to as a distributed system. Control of distributed systems is in need of much deeper development than is so far available in the literature. Decentralized control has been developed during several decades but the results are of limited use in control engineering so far. Most large-scale systems have a hierarchical structure. Control of hierarchical systems has been discussed much but there is no full fledged body of results.

In this paper the approach of coordination control is described for distributed systems. In a coordinated system there are two or more subsystems and a coordinator. The role of the coordinator is to coordinate the joint actions of the subsystems so that the full system can meet the overall control objectives. The underlying concept is that of conditional independent subsystems. Coordinated systems have been defined for linear systems, Gaussian systems, and discrete-event systems. Research issues of coordination control include: (1) The construction of a coordinator. (2) The system theoretic properties of a coordinator including minimality, controllability, and observability. (3) Control synthesis of a coordinated system. Besides coordination control, the paper presents a discussion of distributed control with communication between controllers for distributed systems. The backpressure algorithm for control of communication networks was proposed by L. Tassiulas and A. Epremides, see [31], and this form of distributed control with communication is used in various forms of communication networks. In addition, distributed control with communication has been studied in control of discrete-event systems. More details are provided later in the paper.

A summary of the paper follows. Five case studies of control of distributed systems are briefly described in Section 2. System and control architectures are described in Section 3. Coordination control concepts are treated in Section 4 while their control synthesis is described in Section 5. Distributed control with communication is treated in Section 6.

2 Case Studies of Control of Distributed Systems

An overview of the five case studies of control of distributed systems follows; the case studies are from the C4C Project. The case studies motivate the research on control of distributed systems.

The case studies of control of distributed systems were selected for the C4C Project and are described below. Each of the case studies represents a problem of control engineering which is the focus of current research. The case studies are far from being trivial and the research for the case studies will continue after the life time of the project. The strength of the C4C Project is partly in the technologically-advanced character of its case studies. Per case study the aim of the research, the owner of the problem, the motivation, and the main research issues are discussed.

Underwater Vehicles The aim of the case study is to develop and to demonstrate control algorithms for autonomous underwater vehicles (AUVs). The motivation for the case study is the need for control algorithms for the vehicles to operate autonomously, and to coordinate the activities of two or more vehicles.

The main research issues include: (1) Coordination control of multiple AUVs. (2) Autonomous operation of the AUV individually. (3) Communication between AUVs and surface vehicles. (4) A demonstration of AUVs and other vehicles at a review meeting. These research issues lead to coordination control, to distributed control with communication, and to hierarchical control of distributed systems.

The leader of this case study is the Department FEUP of the University of Porto. There is a laboratory at this university with both underwater vehicles and aerial vehicles under the joint guidance of Fernando Lobo Pereira and João Sousa. The company Oceanscan - Marine Systems Technology with Alexandre Sousa is also involved.

Aerial vehicles The aim of the case study is to develop theory for coordination of Uninhabited Aerial Vehicles (UAVs). The motivation of the case study is the use of aerial vehicles for environmental monitoring like for forest fires or for oil spills. This motivation requires coordination between the vehicles during monitoring missions to organize searches in such a way that the object is located as fast as possible.

The main research issues of the case study include: (1) Control of search missions. (2) Coordination of multiple vehicles during search missions. These research issues lead to distributed control with communication of distributed systems.

The leader of this case study is the Department ECE of the University of Cyprus. The department does not have a laboratoria with vehicles. The research leader, Marios Polycarpou, had gained experience in modeling the vehicles before the start of the C4C Project.

Road Networks The aim of the case study is to develop measures for the computer architecture and for the control of road traffic in a hierarchically-structured road network. The motivations of the case study are the goals of the government of The Netherlands to facilitate the overall traffic flow, to decrease the environmental costs of the damage of road traffic to humans and to the environment, and to decrease the economic costs. In The Netherlands there are now five traffic control centers for monitoring and control of road networks. There are similar networks for provincial and urban roads. A provincial-urban network in Belgium is also part of the case study.

The main research issues are: (1) How to structure the system and the control architecture of these networks? (2) How to develop coordination controllers of the many levels of these hierarchical systems? These research issues lead to hierarchical control of hierarchical systems.

The leader of the case study is the Department of Technology, Management, and Policy of the Delft University of Technology. The leader of that group was involved in the computer architecture of the online monitoring and control of the traffic control centers for motorways in The Netherlands. The company Trinité Automation B.V. is a software house which developed most of the software of the traffic control centers for motorways.

Automated Guided Vehicles The aim of the case study is to develop algorithms for control of automated guided vehicles on a container terminal. The company partner operates a container terminal in the harbor of Antwerp, Belgium. The motivation is the fully automatic transportation of containers from the quay to a yard and from there to a truck for further transportation, and conversely. Currently the transportation with straddle carriers is executed by human drivers. The objectives are to organize the movements of the carriers such that they operate fully independently, interact with the other vehicles, and adjust their routes if necessary.

The main research issues are: (1) Coordination control of the routing of vehicles, including the interaction at intersections. (2) Coordination control to achieve nonblockingness. These research issues lead to distributed control with communication and to hierarchical control of distributed systems.

The leader of this case study is the Department Mechanical Engineering of the Eindhoven University of Technology. The company with the container terminal is Hesse Noord-Natie in Antwerp of which the official name changed in 2010 to PSA Antwerp.

Complex Machines The aim of the case study is to develop coordination control of complex machines consisting of many different sensors, actuators, and local control computers. High-speed printers are a typical example of such machines but other technological machines are also relevant.

The main research issues include: (1) Hierarchical modeling of the operation of the complex machine as an automaton to be controlled. (2) Development of control synthesis of the machines, using supervisory control of discrete-event systems and control of hybrid systems. These research issues lead to hierarchical control and to coordination control of distributed systems.

The leader of the case study is the Department of Mechanical Engineering of the Eindhoven University of Technology. The company problems which produces the printers is Océ Technologies B.V. established in the town of Venlo, The Netherlands.

Further Reading The reader may find information of the case studies in the C4C Deliverables. These are available at the C4C Web Site with address http://www.c4c-project.eu

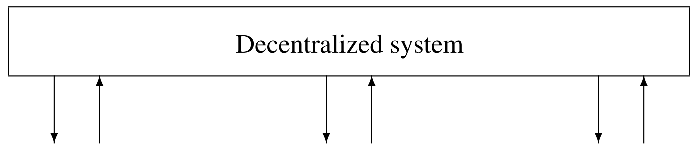

Figure 1: Diagram of a decentralized system.

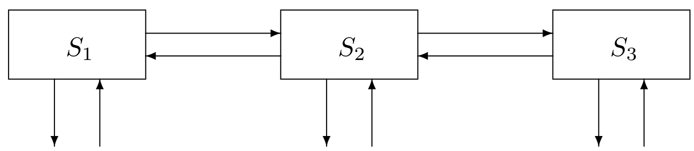

Figure 2: Diagram of a distributed system.

3 System and Control Architectures

In the literature of control and system theory there is neither a standard for classification of system architectures nor for control architectures. Therefore one is introduced below. For the development of control theory it is better to distinguish the theory on the basis of the system architecture. It should be clear that the classification proposed in this paper is preliminary, it may have to be modified later on. Yet, for the purpose of this paper it is better to have a temporary classification than not to have one. References on earlier work on distributed systems include the books by Siljak, [35, 36, 37, 38].

3.1 System Architectures

A distributed system is the interconnection of two or more subsystems, see Fig. 2. The interconnection has then two or more connections for controllers. Each controller receives an observation stream from the distributed system, often from the local subsystem. Each controller can supply an input to the interconnected system, again, to the local subsystem. In the past one used the term decentralized control system. A decentralized control system is a monolithic control system with connections for two or more controllers as described above for distributed systems, see Fig. 1. The difference between a distributed and a decentralized control system is then in the distinction of the interconnected system into local subsystems. Most decentralized control systems used in the past were in fact distributed control systems. The expression distributed or decentralized control system will mostly be abbreviated to distributed system or decentralized system respectively. The term distributed system is much used in computer science and therefore the term is preferred by the authors.

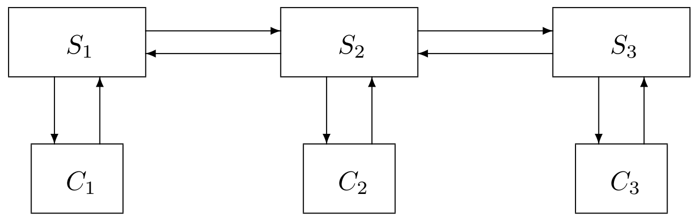

Figure 3: Diagram of the control architecture of distributed control.

Examples A description of the different system architectures used for the C4C Project follows. For the underwater vehicles and the aerial vehicles the system architecture is such that the network is constantly changing. Each vehicle is modeled as a vehicle system. Any vehicle can interact with other vehicles via a communication channel. In certain situations the vehicles are also physically connected. For the communication of underwater vehicles sonar communication is used and the range with limited energy is typically 500 m. to 800 m. at a particular energy used. For the communication of aerial vehicles the range of communication is much larger. The configuration of these networks will in general change quickly over time because new vehicles appear or present vehicles disconnect from the network.

The system architecture of traffic control centers for road networks is rather static, it is in general fixed and changes only in case new investments in roads are made. The system architecture used in the C4C case study for the provincial road network in North-Holland is strictly hierarchical. Distinguish the layers, from the bottom up, the section level, the link level of several sections, the ring level, a focus area, a provincial network, etc. Each subsystem is connected to two or more subsystems at the next-lower layer. In addition, it is connected to one parent subsystem at the next higher layer. In the considered semi-urban road network the successive intersections are connected both physically as roads and in regard to the control processors. The network structure of these systems is rather fixed.

The system structure of the automated guided vehicles (AGVs) on a container terminal is similar to that of the underwater vehicles and the aerial vehicles discussed above. But the interaction of the vehicles is more intense. If the density of the vehicles on the container roads is too high then a form of traffic control has to be imposed. The system architecture thus involves local controllers to coordinate the flow of the vehicles. The system structure changes dynamically.

The system architecture of the complex machines like high-speed printers is rather static, it is fixed with the construction of the machine. The different sensors, actuators, and controllers are connected in a fixed network. The system architecture is best described as hierarchical but the form of the hierarchy will depend on the purpose of the machine and has to be formulated for each individual machine.

3.2 Control Architectures

For control of distributed control systems one may distinguish a variety of control architectures. The main guideline for the classification of control architectures are distinctions in terms of the degree of coordination of the subsystems and of the degree of hierarchy.

The classification consists of the following subclasses: (1) Distributed control. (2) Distributed control with communication between controllers. (3) Coordinated control. (4) Hierarchical control. The subclasses are listed in the order of an increasing form of centralization. Coordination control and distributed control with communication are discussed in more detail in subsequent sections.

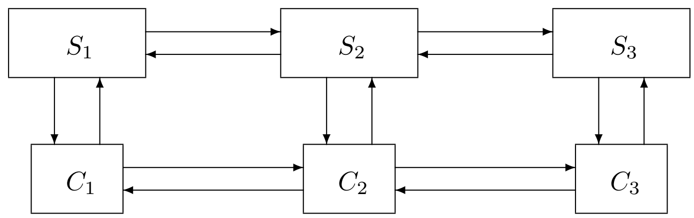

Figure 4: Diagram of the control architecture of distributed control with communication. In general there can also be direct communication between C_1 and C_3 which is not displayed in the diagram.

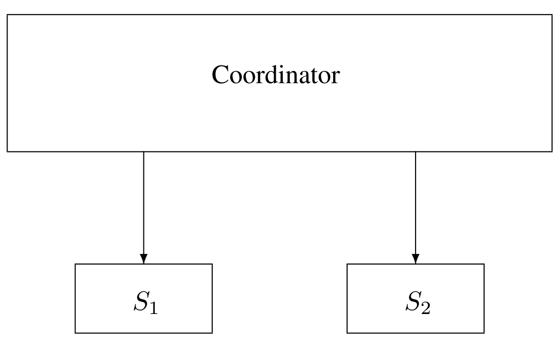

Figure 5: Diagram of the control architecture of coordination control.

Distributed Control In this subclass of control architectures there are two or more control laws or controllers. Each controller receives an observation stream directly from the system or the subsystem and it produces an input to the distributed system. There is no online communication whatsoever between different controllers, see Fig. 3.

Control synthesis of distributed control is difficult. Consider a cost function of the tuple of control laws. The optimal control problem is to determine a tuple of control laws which achieves the lowest cost. There is no method for the determination of such a tuple. Dynamic programming is not directly applicable because it is based on complete state observation. Attention is often restricted to a tuple which forms a person-by-person equilibrium which is derived from the concept of a Nash equilibrium of game theory. If it can be proven that such an equilibrium is also a minimal tuple then the problem becomes to determine and to compute that equilibrium.

Distributed Control with Communication In this subclass of control architectures there are two or more control laws or controllers. Each controller receives an observation stream directly from the distributed system but also receives one or more observation streams from other controllers, see Fig. 4. The observation stream from other controllers is either nonpermanent or not the full state information, meaning the information is not sent after every observation or it is a strict subset of the state information. The latter condition distinguishes this subclass from the next.

Major research issues for this control architecture are the economic costs and the risks of control. The communication requires costs for investment and for energy. But those costs may be smaller than the loss in performance if no communication is used. A framework and case studies are missing for distributed control with communication.

Distributed control with communication is used for many control engineering problems. The backpressure algorithm for routing of messages in a communication network is a major example. The algorithm and the study of its properties are due to L. Tassiulas and A. Ephremides, see [31]. The algorithm is used in current communication networks. The alternating bit protocol is another example of distributed control with communication in the form of a discrete-event system. In this protocol a message is sent from a sender to a receiver via a communication channel. The receiver sends an acknowledgement of the received message to the sender in the form of one bit. After receipt of the acknowledgement, the sender proceeds to the transmission of the next message. Control of adjacent intersections in a C4C Case Study is another example. The backpressure algorithm is a special case of nearest neighbor control discussed elsewhere in the paper.

Control synthesis of distributed control with communication is even more difficult than distributed control without communication. First a communication law has to be determined when extra observations are to be requested or are to be sent by a controller. Next the additional observations have to be integrated into a state estimator. Finally a tuple of control laws has to be determined. Because of these difficulties, there is no substantial theory for this case, except for the case of distributed control with communication of discrete-event systems. Yet, in practice, researchers formulate control laws of which the performance is satisfactory. Guidelines for the control design will be useful for this control architecture.

Coordination Control In this subclass of control architectures one considers a coordinated system and a corresponding coordinated control architecture. Recall from the discussion of system architectures that a coordinated system consists of a coordinator and two or more subsystems such that conditioned on the coordinator the subsystems are independent. In a coordinated control architecture there is a control law or controller for the coordinator and a controller for each of the subsystems, see Fig. 5. The communication in a coordination control architecture is permanent which distinguishes it from the distributed control with communication architecture formulated above.

Coordination control is motivated by the fact that the control objectives can in general not be met by distributed control with or without communication. It is then necessary to impose a degree of centralized control here called coordination control. In a coordinated system there is communication between the coordinator and the subsystems.

Control synthesis of a distributed control system with the coordination control architecture is a little involved. The overall control structure consists of a controller for the coordinator and for each of the subsystems. See Section 5 for an exposition.

Hierarchical Control In this subclass of control architectures one considers a hierarchical control system and a corresponding hierarchical control architecture. A hierarchical control system consists of two or more layers in which each subsystem of a layer is related to one or more subsystems at the next-lower layer and related to one subsystem at the next-higher layer. In a hierarchical control architecture there is one control law or controller per subsystem where the controllers per subsystem interact with each other as in the hierarchical system. Thus a controller of one subsystem is related to those of the subsystems at the next-lower layer to which the subsystem it is related and is related to the controller of the subsystem of the next-higher layer to which the subsystem is related. A hierarchical control architecture is a generalization of the coordination control architecture defined above.

Hierarchical control for particular systems has been investigated. For discrete-event systems hierarchical control has been treated but it seems to concern mostly the case of one subsystem at each level. More theory is needed for hierarchical control.

Major references on optimization and control of hierarchical systems include [1, 8, 10, 22, 29].

Concluding Remarks on Architectures The control architectures defined above are listed in the order of increasing degree of dependence between the controllers. Which control architecture is appropriate for a particular distributed system in combination with control objectives, has to be decided on a case by case basis.

For further reading on system and control architectures of distributed systems see [35, 36, 34].

4 Coordination Control - Concepts and Characterizations

4.1 Introduction

Consider a distributed control system consisting of two or more subsystems, with control objectives of which at least one relates to two or more subsystems. The control problem is to construct a coordinator and controllers for each of the subsystems such that the closed-loop system meets the control objectives. Coordination control of distributed stochastic systems is motivated by the experience that often a distributed control cannot meet the control objectives. Therefore a form of coordination between the subsystems is necessary.

In coordination control one distinguishes a coordinator which guides all other subsystems in their control tasks. There is only one coordinator in coordination control of distributed systems (In a hierarchical system there are in general two or more coordinators). The coordinator may receive information from the other subsystems and may issue tracking reference signals to the other subsystems.

A list of three examples of coordination control follows. (1) Control of underwater vehicles for a mission. In this case the coordinator is the mission control or headquarters but it could also be a surface vessel or a particular underwater vehicle. (2) Control of a link in a motorway, say about 8 km. The subsystems to be controlled are the sections of the link of the motorway and the coordinator is the control law of the link. (3) Control of a set of several dams or reservoirs for purposes of power generation. The subsystems to be controlled are the individual reservoirs while the coordinator is the controller of the regional network of reservoirs.

Examples of control tasks of control engineering problems include: (1) The assignment of two or more subsystems to a particular task. For example, underwater vehicles for plume tracking. Other examples include the assignment of pumps on an oil platform and generators in a power plant or in a power network. (2) Avoiding direct conflicts between vehicles. For examples, vehicles should not collide. (3) Control of power networks with many local controllers and actuators.

The reader may find various forms of coordination control in the literature. At a general level the forms of coordination used in different papers are related but in the details the forms may be different. In this paper another form of coordination control is used. It consists of a coordinator and two or more subsystems which interact such that the coordinator issues signals to the subsystems but the subsystems do neither directly send signals to the coordinator nor to each other. Coordination control of linear systems is described in [26, 13] and of discrete-event systems in [16].

Problem issues of coordination control. The following problem issues are partly resolved but others are open research issues. (1) Formulation of the concept of a coordinated system. (2) Characterization of a coordinated system. (3) Decomposition of a coordinated system. (4) Controllable and observable canonical forms of coordinated systems. (5) Control synthesis with locally complete observations. (6) Optimal control of coordinated systems. (7) Communication issues in coordinated systems. (8) Filtering and state estimation. (9) Control synthesis with partial observations.

4.2 The Concept of a Gaussian Coordinated System

The concept of a linear coordinated system will be defined. The coordinator subsystem is in its dynamics not affected by the other local subsystems. The dynamics of each subsystem (different from the coordinator) is affected only by its own state and by the state of the coordinator but not by the dynamics of the other subsystems. For further reading about the concepts of this subsection the reader is referred to [26, 13, 14].

Below the concept of a Gaussian system is used. This is preceeded by the introduction of elementary notation. The integers are denoted by $\mathbb{Z}$ and the positive integers by $\mathbb{Z}_+$. The natural numbers are denoted by $\mathbb{N} = \{0, 1, \ldots\}$. For $n \in \mathbb{Z}_+$ denote the vector space of n tuples of the real numbers by $\mathbb{R}^n$. A probability space (Ω, F, P) is a triple consisting of a set Ω, a σ-algebra F, and a probability measure $P : F \to [0, 1]$. A Gaussian or normal distribution with parameters $m \in \mathbb{R}^n$ and a positive definite variance matrix $Q \in \mathbb{R}^{n \times n}$ is a probability distribution function which has a characteristic function described by the function

$$\exp(iu^T m - 1/2u^T Qu), \ \ \forall u \in \mathbb{R}^n.$$

A random variable $x : \Omega \to \mathbb{R}^n$ is said to be Gaussian if it has a Gaussian probability distribution and this will be denoted by $x \in G(m, Q)$ where the symbol $m \in \mathbb{R}^n$ denotes the mean and the symbol $Q \in \mathbb{R}^{n \times n}$ denotes the symmetric and positive-definite matrix of the variance of the distribution.

In this paper time is modeled as discrete and the notation for the time index set is $T = \{t_0, t_0+1, t_0+2, \ldots, t_1\} \subset \mathbb{Z}$. A *Gaussian white noise process* is a stochastic process $v : \Omega \times T \to \mathbb{R}^{m_v}$ such that the random variables $\{v(t),\ t \in T\}$ are independent and for every $t \in T$, $v(t) \in G(0, Q_v)$.

Definition 4.1 *A* Gaussian system *is a stochastic dynamic system with the state-space representation,*

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) + Mv(t),\ x(t_0) = x_0, && (1)\\
y(t) &= Cx(t) + Du(t) + Nv(t), && (2)\\
&\quad n,\ m,\ p \in \mathbb{Z}_+,\ \ v : \Omega \times T \to \mathbb{R}^m,\ \textit{a}\ \text{Gaussian white noise process},\ v(t) \in G(0, Q_v),\\
&\quad u : \Omega \times T \to \mathbb{R}^m,\ \textit{denotes the input process},\\
&\quad A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times m},\ C \in \mathbb{R}^{p \times n},\ D \in \mathbb{R}^{p \times m},\ \ M \in \mathbb{R}^{n \times m_v},\ N \in \mathbb{R}^{p \times m_v},\\
&\quad x : \Omega \times T \to \mathbb{R}^n,\ y : \Omega \times T \to \mathbb{R}^p,
\end{aligned}
$$

where x is called the state process *and y the* output process*. The dynamics proceeds for any time step by using the state $x(t)$, the current input $u(t)$, and the noise input $v(t)$ to determine the next state and the current output, $(x(t+1), y(t))$ according to the Equations (1,2). To save space in this paper, the initial conditions will mostly be omitted from the state recursions.*

Example 4.2 The concept of a coordinated Gaussian system is introduced by this example. Consider then a set of Gaussian systems and distinguish a coordinator, and two subsystems. Denote the state and the input of the coordinator by x_c and u_c respectively. The coordinator with the state x_c sends the reference signals r_{1c} and r_{2c} to the two subsystems with the states x_1 and x_2 respectively but the two subsystems do neither send signals to the coordinator nor to each other. The systems are described by the equations,

$$
\begin{aligned}
x_c(t+1) &= A_{cc}x_c(t) + B_{cc}u_c(t) + M_{cc}v_c(t),\\
r_{1c}(t) &= C_{1c}x_c(t) + D_{1c}u_c(t) + M_{1c}v_c(t),\\
r_{2c}(t) &= C_{2c}x_c(t) + D_{2c}u_c(t) + M_{2c}v_c(t),\\
x_1(t+1) &= A_{11}x_1(t) + B_{11}u_1(t) + B_{1c}r_{1c}(t) + M_{11}v_1(t),\\
x_2(t+1) &= A_{22}x_2(t) + B_{22}u_2(t) + B_{2c}r_{2c}(t) + M_{22}v_2(t).
\end{aligned}
$$

The interconnection of these systems then has the Gaussian coordinated system representation,

$$
\begin{aligned}
x(t) &= \begin{pmatrix} x_1(t) & x_2(y) & x_c(t) \end{pmatrix}^T,\\
u(t) &= \begin{pmatrix} u_1(t) & u_2(y) & u_c(t) \end{pmatrix}^T,\ v(t) = \begin{pmatrix} v_1(t) & v_2(y) & v_c(t) \end{pmatrix}^T,\\
x(t+1) &= \begin{pmatrix} A_{11} & 0 & B_{1c}C_{1c} \\ 0 & A_{22} & B_{2c}C_{2c} \\ 0 & 0 & A_{cc} \end{pmatrix} x(t) + \begin{pmatrix} B_{11} & 0 & B_{1c}D_{1c} \\ 0 & B_{22} & B_{2c}D_{2c} \\ 0 & 0 & B_{cc} \end{pmatrix} u(t) +\\
&\quad + \begin{pmatrix} M_{11} & 0 & B_{1c}M_{1c} \\ 0 & M_{22} & B_{2c}M_{2c} \\ 0 & 0 & M_{cc} \end{pmatrix} v(t).
\end{aligned}
$$

Definition 4.3 *The class of* Gaussian coordinated systems *is defined for* $k \in \mathbb{Z}_+$ *subsystems and a coordinator by the representation*

$$
\begin{aligned}
x(t+1) &= \begin{pmatrix} A_{11} & 0 & \dots & 0 & A_{1c} \\ 0 & A_{22} & 0 & \dots & A_{2c} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & A_{kk} & A_{kc} \\ 0 & 0 & \dots & 0 & A_{cc} \end{pmatrix} x(t) + \begin{pmatrix} B_{11} & 0 & \dots & 0 & B_{1c} \\ 0 & B_{22} & 0 & \dots & B_{2c} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & B_{kk} & B_{kc} \\ 0 & 0 & \dots & 0 & B_{cc} \end{pmatrix} u(t) + \\
&+ \begin{pmatrix} M_{11} & 0 & \dots & 0 & M_{1c} \\ 0 & M_{22} & 0 & \dots & M_{2c} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & M_{kk} & M_{kc} \\ 0 & 0 & \dots & 0 & M_{cc} \end{pmatrix} v(t),\ x(t_0) = x_0, \qquad (3) \\
y(t) &= \begin{pmatrix} C_{11} & 0 & \dots & 0 & C_{1c} \\ 0 & C_{22} & 0 & \dots & C_{2c} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & & C_{kk} & C_{kc} \\ 0 & 0 & \dots & 0 & C_{cc} \end{pmatrix} x(t) + \begin{pmatrix} D_{11} & 0 & \dots & 0 & D_{1c} \\ 0 & D_{22} & 0 & \dots & D_{2c} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & & D_{kk} & D_{kc} \\ 0 & 0 & \dots & 0 & D_{cc} \end{pmatrix} u(t) + \\
&+ \begin{pmatrix} N_{11} & 0 & \dots & 0 & N_{1c} \\ 0 & N_{22} & 0 & \dots & N_{2c} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & & N_{kk} & N_{kc} \\ 0 & 0 & \dots & 0 & N_{cc} \end{pmatrix} v(t). \qquad (4)
\end{aligned}
$$

Define the class of Gaussian M Systems *for* $k \in \mathbb{Z}_+$ *subsystems and a coordinator by the representation,*

$$
\begin{aligned}
x(t+1) &= \begin{pmatrix} A_{11} & 0 & \dots & 0 & A_{1c} \\ 0 & A_{22} & 0 & \dots & A_{2c} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & & A_{kk} & A_{kc} \\ A_{c1} & A_{c2} & \dots & A_{ck} & A_{cc} \end{pmatrix} x(t) + \begin{pmatrix} B_{11} & 0 & \dots & 0 & 0 \\ 0 & B_{22} & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & & B_{kk} & 0 \\ 0 & 0 & \dots & 0 & B_{cc} \end{pmatrix} u(t) + \\
&+ \begin{pmatrix} M_{11} & 0 & \dots & 0 & 0 \\ 0 & M_{22} & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & & M_{kk} & 0 \\ 0 & 0 & \dots & 0 & M_{cc} \end{pmatrix} v(t),\ x(t_0) = x_0. \qquad (5)
\end{aligned}
$$

The symbol M in the definition of an Gaussian M system denotes the term mammilary. Mammilary systems have been formulated in the research area of compartmental systems, see the book by J.A. Jacquez, [12]. Mammilary systems can be linear, nonlinear, or stochastic.

Observe that in a coordinated system there is no feedback from the subsystems to the coordinator. One may view this as a serious drawback of this concept. However, it will be argued that the class of coordinated systems is of interest for control theory. In fact, it can be shown that the interconnection of any two systems can be decomposed into a coordinated system. However, the dimensions of the coordinator and of the subsystems may differ with the particular system considered. The class of coordinated systems is also of interest as an approximation of the case where there is a relatively-weak influence from the subsystems to the coordinator. The class of M systems is therefore an alternative to those of the coordinated systems, in such a system there is feedback from the subsystems to the coordinator but there is no direct connection between the subsystems.

The concept of a coordinated system and of an M system can be easily generalized to nonlinear systems and to hybrid systems. This will not be described in this paper.

For later use the following classes of structured matrices are introduced.

Definition 4.4 *Define the following classes of structured real matrices:*

$$I = \{k, (m_1, m_2, \ldots, m_k, m_c), (n_1, n_2, \ldots, n_k, n_c)\} \subset \mathbb{Z} \times \mathbb{Z}^{(k+1)} \times \mathbb{Z}^{(k+1)}, \tag{6}$$

the index set of the indices used below;

$F_{ij} \in \mathbb{R}^{m_i \times n_j},\ i = 1, \ldots, k, c,\ j = 1, \ldots, k, c;$

define the set of arrow matrices *by,*

$$\mathbb{R}^{m \times m}_{\text{arrow}}(I) = \{F \in \mathbb{R}^{m \times n} | \text{ structured as indicated in (8),}\}, \tag{7}$$

$$F_{\text{arrow}} = \begin{pmatrix} F_{11} & 0 & \ldots & 0 & F_{1c} \\ 0 & F_{22} & 0 & \ldots & F_{2c} \\ \vdots & & \ddots & \ldots & \vdots \\ 0 & \ldots & & F_{kk} & F_{kc} \\ F_{c1} & F_{c2} & \ldots & F_{ck} & F_{cc} \end{pmatrix}, \tag{8}$$

define the set of coordination-structured matrices *by,*

$$\mathbb{R}^{m \times n}_{C}(I) = \{F \in \mathbb{R}^{m \times n} | \text{ structured as indicated in (10),}\}, \tag{9}$$

$$F_C = \begin{pmatrix} F_{11} & 0 & \ldots & 0 & F_{1c} \\ 0 & F_{22} & 0 & \ldots & F_{2c} \\ \vdots & & \ddots & \ldots & \vdots \\ 0 & \ldots & & F_{kk} & F_{kc} \\ 0 & 0 & \ldots & 0 & F_{cc} \end{pmatrix}, \tag{10}$$

define the set of block-diagonal matrices *by,*

$$\mathbb{R}^{m \times n}_{\text{Bdiag}}(I) = \{F \in \mathbb{R}^{m \times n} | \text{ structured as indicated in (12),}\}, \tag{11}$$

$$F_{\text{Bdiag}} = \begin{pmatrix} F_{11} & 0 & \ldots & 0 & 0 \\ 0 & F_{22} & 0 & \ldots & 0 \\ \vdots & & \ddots & \ldots & \vdots \\ 0 & \ldots & & F_{kk} & 0 \\ 0 & 0 & \ldots & 0 & F_{cc} \end{pmatrix}. \tag{12}$$

Note that the matrices of a Gaussian coordinated system are coordination-structured matrices. The system matrix of a Gaussian M system is a matrix in the class $\mathbb{R}^{m \times n}_{\text{arrow}}$ while the input and the noise matrices of an M system are block diagonal.

Proposition 4.5 *Consider the class of coordination-structured square matrices $\mathbb{R}^{n \times n}_{C}$, thus with the index set I such that $m_i = n_i$ for $i = 1, \ldots, k, c$. This class is a ring; equivalently, it is closed with respect to addition and with respect to multiplication. If the block-diagonal matrices are in addition invertible then it is an invertible ring meaning that the inverse exists and is also an element of the class.*

Note that if $A \in \mathbb{R}^{n \times n}_{C}$ then the exponential of that matrix, $\exp(A)$ is also a coordination-structured matrix.

4.3 The Concept of a Coordinated Discrete-Event System

The concepts formalized below were developed by several of the authors and the reader is referred to [20, 16, 18, 19, 17] for additional information.

The terminology of discrete-event systems (DES) is more or less according to the lecture notes of W.M. Wonham [44] and the book by C.S. Cassandras and S. Lafortune [7] but the notation of this paper differs slightly. A (deterministic) *generator*

$$G = (Q, E, f, q_0, Q_m), \tag{13}$$

is a mathematical structure with *state set* Q, an *event set* E, a *partial transition function* $f : Q \times E \to Q$, an *initial state* $q_0 \in Q$, and a *subset of marked states* $Q_m \subseteq Q$. A transition is also denoted as $q \overset{e}{\mapsto} q^+ = f(q, e)$. If a transition is defined then this is denoted by $f(q, e)!$ Extend the transition function f to $f : Q \times E^* \to Q$ by induction. Define respectively the *language* and the *marked language* of the generator as,

$$\begin{aligned} L(G) &= \{s \in E^* | f(q_0, s)!\}, && (14) \\ L_m(G) &= \{s \in L(G) | f(q_0, s) \in Q_m\}. && (15) \end{aligned}$$

A *controlled generator* is a structure (G, E_c, Γ_c), where G is a generator, $E_c \subseteq E$ is the subset of *controllable events*, $E_{uc} = E \backslash E_c$ is the subset of *uncontrollable events*, and $\Gamma_c = \{\gamma \subseteq E | E_{uc} \subseteq \gamma\}$, is called the *set of control patterns*. A *supervisory control* for the controlled generator G is a map $g : L(G) \to \Gamma_c$. The *supervisor* consists then of the supervisory control g which for historical reasons is also denoted by S. The *closed-loop system* associated with a controllable generator and a supervisory control as denoted above is defined as the language $L(S/G) \subseteq E^*$ and the marked language $L_m(S/G) \subseteq L(S/G)$ which satisfy respectively,

$$\begin{aligned} (1) \quad & \epsilon \in L(S/G), \\ (2) \quad & \text{if } s \in L(S/G),\ se \in L(G) \text{ and if } e \in g(s) \text{ then } se \in L(S/G); \\ L_m(S/G) &= L(S/G) \cap L_m(G). \end{aligned}$$

Recall that the *natural projection* $P : E^* \to E_o^*$ is a morphism of monoids such that $P(\epsilon) = \epsilon$ and P erases the events that are not in $E_o \subseteq E$. A supervisor with partial observations is a map $g : P(L(G)) \to \Gamma_c$.

It is important to distinguish for a generator between an event set and its associated reachable event set. Note that there may exist events of the event set which do not appear in any transition. Moreover, even if an event is used for a transition then this transition may not be reachable. This applies in particular to the case of the synchronous product of two generators.

Definition 4.6 *Consider a generator G denoted as above. Define the subset of* reachable events*, denoted by $E_r(G) \subseteq E$, if for any event $e \in E_r(G)$ there exists a string $s \in E^*$ containing the event e for which the function $f(q_0, s)$ is defined. Similarly, define for any language $L \subseteq E^*$ the subset of* reachable events $E_r(L) \subseteq E$ of the language *as the subset of events which occur in the strings of the language.*

Note the abuse of notation in $E_r(G)$ and $E_r(L)$. The complexity of computing the event set $E_r(G)$ is $O(n(G) \times m_E(G))$ where $n(G)$ denotes the number of states and $m_E(G)$ denotes the number of events of the generator G.

Definition 4.7 *Consider two event sets E_1 and E_2 and two languages $L_1 \subseteq E_1^*$ and $L_2 \subseteq E_2^*$. The* synchronous product *of the languages L_1 and L_2 is defined as*

$$L_1 \| L_2 = P_1^{-1}(L_1) \cap P_2^{-1}(L_2).$$

where $P_i : (E_1 \cup E_2)^* \to E_i^*$ for $i = 1, 2$. Their synchronous product is called the shuffle product *if*

$$\begin{aligned} \emptyset &= E_{r,sh} = E_r(L_1 \| L_2) \cap E_1 \cap E_2, && (16) \\ & \textit{and then one writes} \\ \text{shuffle}(L_1, L_2) &= L_1 \| L_2 = P_1^{-1}(L_1) \cap P_2^{-1}(L_2). && (17) \end{aligned}$$

The subset of reachable shared events $E_{r,sh}$ can be empty while $E_1 \cap E_2$ is not as a simple example shows. The corresponding synchronous product of generators [44] is known to satisfy

$$L(G_1 \| G_2) = L(G_1)||L(G_2), \quad (18)$$
$$L_m(G_1 \| G_2) = L_m(G_1)||L_m(G_2). \quad (19)$$

A distributed discrete-event system is a modular or a concurrent system with the global plant formed by the synchronous product of local subsystems.

Definition 4.8 *A* modular discrete-event system *with two modules is a structure* $(G_1, G_2, E_{1,c}, \Gamma_{1,c}, E_{2,c}, \Gamma_{2,c})$ *consisting of two or more modules in the form of controlled generators. The associated global system is their synchronous product* $G_1 \| G_2$. *Denote the natural projections by*

$$P_1 : (E_1 \cup E_2)^* \to E_1^*, \; P_2 : (E_1 \cup E_2)^* \to E_2^*.$$

Throughout the paper the special case of two modules is considered in order to simplify the exposition.

Concepts In this section the concept of conditionally independent generators and related notions are defined.

Conditional independence of σ-algebras is a concept of probability theory which has been used to put the concept of state of a stochastic system on a fundamental basis, see [33] and the references quoted there. A corresponding notion is useful in automata theory as well. This section presents the concepts, coordination control theory with these concepts is presented in the following sections.

Denote $E = E_1 \cup E_2$. The event set of the coordinator is denoted by E_k, where the symbol k is used rather than c to avoid confusion with the subset of controllable events. The following natural projections are needed: for $i = 1, 2$ let $P_{i \cup k} : E^* \to (E_i \cup E_k)^*$, $P_1^{i \cup k} : (E_i \cup E_k)^* \to E_i^*$, and its inverse projection $(P_1^{i \cup k})^{-1} : E_i^* \to \text{Pwrset}(E_i \cup E_k)^*$. Symmetrically, let $P_k^{i \cup k} : (E_i \cup E_k)^* \to E_k^*, \; i = 1, 2$. Also, let $P_{i \setminus k} : \; E^* \to (E_i \setminus E_k)^*, \; i = 1, 2$. The notation $P_{i \cap k}^i : E_i^* \to (E_i \cap E_k)^*, \; i = 1, 2$ is now self-explanatory.

Definition 4.9 *Consider event sets* $E_1, \; E_2, \; E_k$ *and languages* $L_1 \subseteq E_1^*$, $L_2 \subseteq E_2^*$, *and* $L_k \subseteq E_k^*$. *The languages* $L_1, \; L_2$ *are said to be* conditionally independent given L_k *if* $E_r(L_1 \| L_2) \cap E_1 \cap E_2 \subseteq E_k$.
Notation. $(L_1, L_2 | L_k) \in \text{CIL}$ *denotes that the languages* L_1, L_2 *are conditionally independent given* L_k.

Definition 4.10 *Consider three generators,* $G_k = (Q_k, E_k, f_k, q_{k,0}, Q_{k,m})$, $G_1 = (Q_1, E_1, f_1, q_{1,0}, Q_{1,m})$, $G_2 = (Q_2, E_2, f_2, q_{2,0}, Q_{2,m})$. *Call* G_1, G_2 conditionally independent generators *given* G_k *if* $(L(G_1), L(G_2) | L(G_k)) \in \text{CIL}$.

Note that conditional independence implies that if there exists an event which is reachable in G_1, G_2, and G_k, for which a transition simultaneously occurs in G_1 and G_2, then the coordinator automaton G_k also simultaneously transits with this event. The concept is easily extended to the case of three or more generators. Other related concepts are defined below.

Definition 4.11 *Consider the events sets* $E_1, \; E_2$ *and* E_k *and the languages* $L_1 \subseteq E_1^*$, $L_2 \subseteq E_2^*$, $L_k \subseteq E_k^*$, *and* $K \subseteq E^*$. *Assume that* $E_r(L_1 \| L_2) \cap E_1 \cap E_2 \subseteq E_c \subseteq E_1 \cup E_2 = E$. *Define the conditions: The language* K *is called* conditionally decomposable *with respect to the event sets* (E_1, E_2, E_k) *if*

$$K = P_{1 \cup k}(K) \| P_{2 \cup k}(K) \| P_k(K).$$

The term conditionally decomposable is used, because the projections $P_{i \cup k} : \; E^* \to E_{i \cup k}^*$ are involved. It should be clear that conditional decomposability is weaker than decomposability with respect to $P_1 : E^* \to E_1^*$, $P_2 : E^* \to E_2^*$, and $P_k : E^* \to E_k^*$ as defined in the literature on decentralized control. This is because $E_i \subseteq E_i \cup E_k$ implies that $P_i^{-1} P_i(K) \subseteq P_{i \cup k}^{-1} P_{i \cup k}(K)$. Also note that in the case $E_k = E = E_1 \cup E_2$ conditional decomposability is trivially satisfied. The concepts of conditional independence and conditional decomposability are closely related.

4.4 Algebraic-Geometric Characterization of Linear Coordinated Systems

An equivalent condition for a coordinated linear system is that of conditionally-independent linear subspaces of the state space given a coordinator subspace, combined with an invariance condition. The properties of conditionally-independent linear subspaces are investigated. These issues were first discussed in [26, 13].

The development in this essay builds on geometric control theory which has its origins in work of W.M. Wonham [42, 43], see also [30, 32]. We make use of concepts from linear algebra, in particular from the theory of invariant subspaces as described in [9].

Conditionally-Independent Linear Subspaces Consider a linear space X over a field $\mathbb{F}$. The set of linear subspaces of X is called the lattice of subspaces of X and denoted by $\mathrm{Lat}\ (X)$. We shall use the notation $X_1 \dot{+} X_2$ to denote the (not necessarily orthogonal) direct sum of two subspaces. Two subspaces $X_1, X_2 \subseteq X$ of a vector space X over a field $\mathbb{F}$ are called *linearly independent subspaces* if $X_1 \cap X_2 = \{0\}$.

Definition 4.12 *Consider a linear space X. Two subspaces $X_1, X_2 \in \mathrm{Lat}\ (X)$ are called* conditionally-independent *given a subspace $X_c \in \mathrm{Lat}\ (X)$ if there exist complements $X_{i\backslash c} \subseteq X_i$ of $X_i \cap X_c$, equivalently,*

$$X_i = (X_i \cap X_c) \dot{+} X_{i\backslash c},\ i = 1, 2, \tag{20}$$

such that $X_{1\backslash c}$ and $X_{2\backslash c}$ are linearly independent in X. The notation $(X_1, X_2 | X_c) \in \mathrm{CILS}$ denotes that the linear subspaces X_1, X_2 are conditionally-independent given X_c. In this case, the subspace X_c is called the coordinator subspace *for X_1 and X_2.*

The following observation is relatively straightforward.

Theorem 4.13 *Consider a linear space X and two subspaces $X_1, X_2 \in \mathrm{Lat}\ (X)$. Define $X_c = X_1 \cap X_2$, and let $X_{i\backslash c}$ be subspaces such that $X_i = (X_i \cap X_c) \dot{+} X_{i\backslash c}, i = 1, 2$. Then $X_{1\backslash c} \cap X_{2\backslash c} = \{0\}$. Consequently, $(X_1, X_2 | X_1 \cap X_2) \in \mathrm{CILS}$.*

Geometric Characterization of Linear Coordinated Systems

Theorem 4.14 *Let A be an $n \times n$ matrix, and consider the linear system $dx(t)/dt = Ax(t)$, $x(t_0) = x_0$. Consider linear subspaces $X_1, X_2 \in \mathrm{Lat}\ (X)$ with the property that $X_1 + X_2 = X$. Define $X_c = X_1 \cap X_2$, and let $X_{i\backslash c}$ be subspaces such that $X_i = X_c \dot{+} X_{i\backslash c}$. Then it follows from Theorem 4.13 that $(X_1, X_2 | X_c) \in$ CILS.*

There exists a basis of X such that with respect to this basis the linear system has the coordinated linear system representation as displayed in the following formula,

$$\frac{dx}{dt} = \begin{pmatrix} A_{11} & 0 & A_{1,c} \\ 0 & A_{22} & A_{2,c} \\ 0 & 0 & A_{c,c} \end{pmatrix} x(t),\ x(t_0) = x_0, \tag{21}$$

$$\text{with } n_1, n_2, n_c \in \mathbb{N},\ n_1 + n_2 + n_c = n, \qquad A_{i,j} \in \mathbb{R}^{n_i \times n_j},\ \forall i, j \in 1, 2, c,$$

if and only if there exist subspaces $X_{1\backslash c}$ and $X_{2\backslash c}$ as above such that they are A-invariant:

$$AX_{1\backslash c} \subseteq X_{1\backslash c},\ AX_{2\backslash c} \subseteq X_{2\backslash c}. \tag{22}$$

Consider a linear distributed system which consists of an interconnection of two subsystems. Denote the relevant state spaces of these subsystems by X_1 and X_2; suppose that $X = X_1 + X_2$. Denote $X_c = X_1 \cap X_2$. From Theorem 4.13 follows that $(X_1, X_2 | X_c) \in$ CILS. If the invariance condition of Theorem 4.14 holds then one can choose a basis of X such that the system has a representation as a coordinated linear system. In case the invariance condition (22) does not hold it is suggested to extend the coordinator subspace $X_c \subseteq X$ in such a way that the invariance condition holds, see [13]. The subspace $X_1 + X_2$ is a coordinator subspace but there may be smaller subspaces in the range $X_1 \cap X_2 \subseteq X_c \subseteq X_1 + X_2$.

Geometric Characterization of Linear Coordinated Control Systems Let A be an $n \times n$ real matrix, and let B be an $n \times m$ real matrix.

Definition 4.15 *A linear control system with representation*

$$dx(t)/dt = Ax(t) + Bu(t),\ x(t_0) = x_0, \tag{23}$$

is said to be a linear coordinated control system *if there exists a basis for the state space $X = \mathbb{R}^n$ and for the input space $U = \mathbb{R}^m$ such that with respect to those bases it has the representation*

$$\frac{dx}{dt} = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} x(t) + \begin{pmatrix} B_{11} & 0 & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{pmatrix} u(t), \qquad x(t_0) = x_0. \tag{24}$$

To avoid trivialities we shall assume from the start that $m \leq n$, and that B has full column rank, i.e., $\ker B = \{0\}$. The latter condition may be dropped at the expense of a slightly different procedure.

Problem 4.16 *Under which conditions on A and B are there invertible matrices $S \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{m\times m}$, and a matrix $F \in \mathbb{R}^{m\times n}$ such that*

$$S^{-1}(A + BF)S = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}, \qquad S^{-1}BT = \begin{pmatrix} B_{11} & 0 & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{pmatrix}. \tag{25}$$

In other words, in a basis-free formulation, we ask when there are decompositions

$$\mathbb{R}^n = X_1 \dot{+} X_2 \dot{+} X_3, \qquad \mathbb{R}^m = U_1 \dot{+} U_2 \dot{+} U_3,$$

such that with respect to these decompositions, the matrices $A + BF$ and B have the block forms of (25). Phrased yet differently, we ask when the pair (A, B) is feedback equivalent *to a pair of the form (25).*

To discuss the problem we need the notion of (A, B)-invariant subspace. Recall, see [43, 9] that a subspace $X_s \subset \mathbb{R}^n$ is called an (A, B)*-invariant subspace* if $AX_s \subset X_s + \operatorname{Im} B$.

Theorem 4.17 *Consider Problem 4.16. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ with $\ker B = \{0\}$. There exist invertible matrices $S \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{m\times m}$, and a feedback matrix $F \in \mathbb{R}^{m\times n}$ such that* (25) *holds, if and only if there are (A, B)-invariant subspaces X_1 and X_2 such that $X_1 \cap X_2 = \{0\}$.*

Note that it is possible that $U_i = \{0\}$, with $B_{ii} : U_i \to X_i$ being the zero operator.

4.5 Construction of a Coordinator

In this subsection the case is considered in which there is a system and one wants to transform it to a coordinated linear system, basically determining which parts of the system are the coordinator and the remaining parts of Subsystem 1 and of Subsystem 2. The procedure formulated below was first described in [13].

For the understanding of the reader we briefly describe the interaction of two systems. Consider then two linear systems of the form

$$\begin{aligned} dx_1(t)/dt &= A_1x_1(t) + B_1u_1(t),\ x_1(t_0) = x_0, \\ y_1(t) &= C_1x_1(t); \\ dx_2(t)/dt &= A_2x_2(t) + B_2u_2(t),\ x_2(t_0) = x_0, \\ y_2(t) &= C_2x_2(t). \end{aligned}$$

Consider the interconnection relations $u_1(t) = y_2(t)$ and $u_2(t) = y_1(t)$. The closed-loop system is then,

$$dx(t)/dt = \begin{pmatrix} A_1 & B_1C_2 \\ B_2C_1 & A_2 \end{pmatrix} x(t),\ x(t_0) = (x_{1,0}, x_{2,0})^T;\ x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

In case of three or more subsystems the interconnection relation then determines the closed-loop system.

Procedure 4.18 Construction of a coordinated linear system from an interconnected system. *Consider a linear system consisting of the interconnection of two or more subsystems, with representation*

$$dx(t)/dt = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x(t),\ x(t_0) = x_0.$$

1. *Take a state-space transformation such that the matrix pair* (A_{22}, A_{12}) *is transformed to the Kalman observable form,*

$$dx_a(t)/dt = \left(\begin{array}{c|cc} A_{11} & A_{121} & 0 \\ \hline A_{211} & A_{2211} & 0 \\ A_{212} & A_{2212} & A_{2222} \end{array}\right) x_a(t),\ x_a(t_0) = x_{a,0}.$$

2. *Take a state-space transformation such that the matrix pair* (A_{11}, A_{21}) *is transformed to the Kalman observable form,*

$$dx_b(t)/dt = \left(\begin{array}{cc|cc} A_{11,11} & 0 & A_{11,21} & 0 \\ A_{11,21} & A_{11,22} & A_{12,11} & 0 \\ \hline A_{21,11} & 0 & A_{22,11} & 0 \\ A_{21,21} & 0 & A_{22,21} & A_{22,22} \end{array}\right) x_b(t),\ x_b(t_0) = x_{b,0}.$$

3. *Next apply a permutation operation which permutes the blocks, symbolically written as* $(1, 2, 3, 4) \mapsto (2, 4, 1, 3)$ *and produces the coordinated linear system,*

$$dx_c(t)/dt = \left(\begin{array}{c|c|cc} A_{11,22} & 0 & A_{11,21} & A_{12,21} \\ \hline 0 & A_{22,22} & A_{21,21} & A_{22,21} \\ \hline 0 & 0 & A_{11,11} & A_{12,11} \\ 0 & 0 & A_{21,11} & A_{22,11} \end{array}\right) x_c(t),\ x_c(t_0) = x_{c,0}.$$

   Note that the unobservable parts of the two subsystems now form the local subsystems while the observable parts of the two original subsystems form the coordinator.

The coordinator constructed above may not be minimal. For a discussion of minimality see the paper [13].

4.6 Algebraic-Geometric Characterization of a Coordinated Discrete-Event System

Next the algebraic-geometric characterization of coordinated discrete-event systems is described. It has several analogies and differences with the case of linear systems.

Below the following notation is used. Consider an event set E and subsets $E_1,\ E_2,\ E_k \subseteq E$. Denote the set difference by $E_1 \backslash E_k = \{e_1 \in E_1 | e_1 \notin E_k\}$. Then $E_1 = (E_1 \cap E_k) \dot{\cup} (E_1 \backslash E_k)$ where $\dot{\cup}$ denotes a disjoint union.

Definition 4.19 *Consider an event set* E *and subsets* $E_1,\ E_2,\ E_k \subseteq E$.

(a) Call the subsets E_1 *and* E_2 independent subsets *if they are disjoint:* $E_1 \cap E_2 = \emptyset$.

(b) *Call the subsets E_1 and E_2* conditionally independent *given E_k if $E_1 \backslash E_k$ and $E_2 \backslash E_k$ are independent subsets; equivalently, if $(E_1 \backslash E_k) \cap (E_2 \backslash E_k) = \emptyset$. The notation of this property is $(E_1, E_2 | E_k) \in$* CISets.

Proposition 4.20 *Consider the set E and the subsets $E_1,\ E_2, E_k$. The following are equivalent:*

(a) $(E_1, E_2 | E_k) \in$ CISets.

(b) $(E_1 \cap E_2) \subseteq E_k$.

The elementary proof is omitted. The definition of a coordinated discrete-event system will next be related to the concepts introduced above. In automata theory, the predominant description is the behavior in terms of languages rather than the state and the transition description presented above for linear systems. Therefore the concept of conditional independent generators is defined in terms of languages.

Definition 4.21 *Consider three automata G_1, G_2, and G_k. The automata G_1 and G_2 are called* conditionally independent given *G_k if their corresponding event sets satisfy $(E_1, E_2 | E_k) \in$* CISets*; equivalently, by Proposition 4.20, if $(E_1 \cap E_2) \subseteq E_k$.*

The above definition is then identical to Def. 4.10. if there are no unreachable shared events.

4.7 System Theory of Coordinated Systems

The concept of a coordinated system leads to several system theoretic issues which are described below.

The main system theoretic issues of coordinated systems include: (1) The construction of a coordinator for the interconnection of two or more subsystems. (2) The minimality of the coordinator. (3) The controllability and the observability properties of such a system. These issues are discussed below.

The construction of a coordinated linear system is described in the paper [13] and is summarized in Subsection 4.5.

The minimality of the coordinator is discussed next. One can view a coordinated system as one in which on a set of subsystems a form of coordination is imposed by one of the subsystems, called the coordinator. The coordinator restricts the behavior of the collection of the subsystems. For this reason it seems useful that the degree of the restriction by the coordinator is as small as is necessary for the control objectives. Then the individual subsystems remain independent as much as possible. The restriction by the coordinator should be sufficient to meet the control objectives of the interconnected system which could not be met without coordination. For this reason, it is useful to have a coordinator of minimal dimension because the complexity of the coordinator of a linear system may be characterized in terms of its dimension.

Several concepts of minimality of the coordinator were discussed in the paper [13]. The concepts formulated in that paper need to be streamlined and therefore will not be stated in this paper.

A coordinated control system has several inputs. Therefore, controllability of such a system has to be distinguished in regard to those inputs. Concepts of controllability and of decompositions of a coordinated linear system have been defined, see [14].

The decomposition of the interconnection of three or more systems is much more complex than that of a tuple of systems. More research on this and its use in control synthesis remains to be explored.

5 Coordination Control - Control Synthesis

The synthesis of control laws for a coordinated system is treated in this subsection. Both control synthesis of Gaussian coordinated systems and of discrete-event coordinated systems are discussed.

The informal problem statement is to synthesize control laws based on an information structure defined below such that the closed-loop system meets prespecified control objectives. The primary control objectives are stability, minimal variance of the state process, and minimal use of the input process. For discrete-event systems the primary control objectives are safety and liveness. For Gaussian coordinated systems a quadratic cost function will be used. The above informal problem will be refined further below. References on optimal stochastic control include [4, 21].

5.1 Control Laws for Gaussian Coordinated Systems

The problem of control synthesis of distributed control systems is different from the centralized problem usually considered in which the control law is based on complete observations hence the input at any time depends on the full state. The input to the coordinator will be allowed to depend only on its own state while the input to any other subsystem is allowed to depend on its local state and on the coordinator state. For this specification, the concept of an information structure is useful which will be defined next.

For control theoretic purposes, information structures are defined for discrete-time nonlinear stochastic systems.

Definition 5.1 *Consider a discrete-time nonlinear stochastic system with the representation,*

$$
\begin{aligned}
x(t+1) &= f(t,x(t),u(t),v(t)),\ x(t_0)=x_0, \\
y(t) &= h(t,x(t),u(t),v(t)), \\
& \quad k \in \mathbb{Z}_+ \text{ number of stations},\ n, m_v \in \mathbb{Z}_+,\ \mathbb{Z}_k = \{1,2,\ldots,k\}, \\
& \quad \{m_1, m_2, \ldots, m_k \in \mathbb{Z}_+\},\ m_1+m_2+\ldots+m_k = m, \\
& \quad \{n_1, n_2, \ldots, n_k \in \mathbb{Z}_+\},\ n_1+n_2+\ldots+n_k = n, \\
& \quad U_i = \mathbb{R}^{m_i},\ U = \prod_{i=1}^{k} U_i,\ Y_i = \mathbb{R}^{m_i},\ Y = \prod_{i=1}^{k} Y_i,\ X = \mathbb{R}^n,\ V = \mathbb{R}^{m_v}, \\
T &= \{t_0, t_0+1, \ldots, t_1\},\ \text{the time index set}, \\
& \quad x_0 : \Omega \to \mathbb{R}^n,\ \text{a random variable, the initial state}, \\
& \quad v : \Omega \times T \to V,\ \text{a sequence of independent identically-distributed} \\
& \quad \text{random variables}, \\
& \quad f : T \times X \times U \times V \to X,\ \ h : T \times X \times U \times V \to Y,\ \text{measurable functions}, \\
& \quad u : \Omega \times T \to \mathbb{R}^m,\ u_i : \Omega \times T \to \mathbb{R}^{m_i},\ \forall\, i \in \mathbb{Z}_k, x : \Omega \times T \to \mathbb{R}^n,\ x_i : \Omega \times T \to \mathbb{R}^{n_i}, \\
& \quad y : \Omega \times T \to \mathbb{R}^m,\ y_i : \Omega \times T \to \mathbb{R}^{m_i},\ \forall\, i \in \mathbb{Z}_k, \\
y(t) &= \begin{pmatrix} y_1(t) & y_2(t) & \ldots & y_k(t) \end{pmatrix}^T,\ u(t) = \begin{pmatrix} u_1(t) & u_2(t) & \ldots & u_k(t) \end{pmatrix}^T, \\
y_i[t_0,t] &= \begin{pmatrix} y_i(t_0), & y_i(t_0+1), & \ldots, & y_i(t) \end{pmatrix} \in Y_i^{t-t_0+1}, \\
y[t_0,t] &= \begin{pmatrix} y_1[t_0,t] & y_2[t_0,t] & \ldots & y_k[t_0,t] \end{pmatrix}^T \in Y^{t-t_0+1}, \\
u_i[t_0,t] &= \begin{pmatrix} u_i(t_0), & u_i(t_0+1), & \ldots, & u_i(t) \end{pmatrix} \in U_i^{t-t_0+1}, \\
u[t_0,t] &= \begin{pmatrix} u_1[t_0,t], & u_2[t_0,t], & \ldots, & u_k[t_0,t] \end{pmatrix}^T \in U^{t-t_0+1}.
\end{aligned}
$$

Station $i \in \mathbb{Z}_k$ receives directly from the system the observation process y_i and provides to the system the input process u_i.

Definition 5.2 *Consider the distributed control system 5.1. Define the* information structure *of this system as the sets,*

$$
\begin{aligned}
& \forall\, i \in \{1,2,\ldots,k\}, \\
IS_i(t) \;\; & \subseteq \;\; \prod_{j=1}^{k} (y_j[t_0,t], u_j[t_0,t]) \in \prod_{j=1}^{k} (Y_j^{t-t_0+1} \times U_j^{t-t_0+1}), \\
& \textit{the } \text{information structure of Station } i \text{ at time } t \in T, \\
IS(t) \;\; & = \;\; \{IS_1(t), \ldots, IS_k(t)\}, \;\; \textit{the } \text{information structure at time } t \in T, \\
IS \;\; & = \;\; \{IS(t_0), IS(t_0+1), \ldots, IS(t_1)\}, \;\; \textit{the } \text{information structure of the system.}
\end{aligned}
$$

H. Witsenhausen in [40, 39] uses for the information structure not the actual observations but index sets for the labels of the stations and for the time in the index set. In this paper the formulation provided above is preferred.

Definition 5.3 *Specific information structures of the system of Definition 5.1 follow.*

(a) *The* classical information structure *is that defined by*

$$
IS_i(t) \;\; = \;\; (y_i[t_0,t], u_i[t_0,t]) \in Y_i^{t-t_0+1} \times U_i^{t-t_0+1}, \; \forall\, i \in \{1,2,\ldots,k\}, \; \forall\, t \in T.
$$

Note that Station i only receives the observations y_i from the plant and it recalls its own outputs and inputs. This information structure corresponds to distributed control or distributed filtering/state estimation both without communication. An information structure is called strictly classical *if it is classical and there is only one station ($k = 1$).*

(b) *The* coordination information structure *is defined by the expressions*

$$
\begin{aligned}
IS_c(t) \;\; & = \;\; (y_c[t_0,t], u_c[t_0,t]) \;\in Y_c^{t-t_0+1} \times U_c^{t-t_0+1}, \\
IS_i(t) \;\; & = \;\; (y_i[t_0,t], u_i[t_0,t], y_c[t_0,t], u_c[t_0,t]) \in Y_i^{t-t_0+1} \times U_i^{t-t_0+1} \times Y_c^{t-t_0+1} \times U_c^{t-t_0+1}, \\
& \quad\;\; \forall\, i \in \{1,2,\ldots,k\}, \; \forall\, t \in T.
\end{aligned}
$$

(c) *The* one-step delayed sharing information structure *is defined by*

$$
\begin{aligned}
IS1D_i(t) \;\; & = \;\; \left((y_i[t_0,t], u_i[t_0,t]), \prod_{j=1, j \neq i}^{k} (y_j[t_0,t-1], u_j[t_0,t-1]) \right), \\
& \quad\;\; \forall\, i \in \{1,2,\ldots,k\}, \; \forall\, t \in T.
\end{aligned}
$$

A multi-step delayed sharing information structure *can then be defined similarly. The information structure may be distinguished into the* common information structure *of all stations and the* private information structure *of each station.*

The one or multi-step delayed sharing information structures are appropriate models for control of communication networks where the communication of the information often takes a small delay.

Definition 5.4 *Define the following properties of information structures.*

(a) *An information structure is said to possess* perfect recall *if*

$$
IS_i(t) \;\; \subseteq \;\; IS_i(t+1), \; \forall\, t \in \{t_0, t_0+1, \ldots, t_1-1\}, \; \forall\, i \in \{1,2,\ldots,k\}.
$$

Note that a classical information structure has perfect recall.

In this subsection for control of Gaussian coordinated systems attention is restricted to time-invariant control laws and to infinite-horizon average-cost control problems. Below use is made of the structured matrices, see Definition 4.4.

Definition 5.5 *Consider a Gaussian coordinated system. Define the classes of time-invariant control laws,*

$$\begin{aligned} G &= \{g : X \to U | g \text{ measurable function}\}, && (26)\\ G_L &= \{g : X \to U | g(x) = Fx, \;\; F \in \mathbb{R}^{m\times n}\}, && (27)\\ G_M &= \{g_{\text{arrow}} : X \to U | g_{\text{arrow}}(x) = Fx, \;\; F \in \mathbb{R}^{m\times n}_{\text{arrow}}\}, && (28)\\ G_C &= \{g_C : X \to U | g_C(x) = Fx, \;\; F \in \mathbb{R}^{m\times n}_{C}\}, && (29)\\ G_{\text{Bdiag}} &= \{g_{\text{Bdiag}} : X \to U | g_{\text{Bdiag}}(x) = Fx, \;\; F \in \mathbb{R}^{m\times n}_{\text{Bdiag}}\}. && (30) \end{aligned}$$

An interpretation of the control laws follows. Each control law depends on a particular information structure. Below the reader first finds notation and then in text the description of the class of control laws.

$$\begin{aligned} &G_{\text{Bdiag}} && u_i(t) = F_i x_i(t), \;\; i = 1, 2, \ldots, k, c; && (31)\\ &G_C && u_i(t) = F_i x_i(t) + F_{ic} x_c(t), \;\; i = 1, 2, \ldots, k, && (32)\\ & && u_c(t) = F_{cc} x_c(t); && (33)\\ &G_M && u_i(t) = F_i x_i(t) + F_{ic} x_c(t), \;\; i = 1, 2, \ldots, k, && (34)\\ & && u_c(t) = F_{cc} x_c(t) + \sum_{i=1}^{k} F_{ci} x_i(t). && (35) \end{aligned}$$

The class of control laws G_{Bdiag} is such that the information structure of any subsystem is the past of its local state, $IS_i(t) = \{x_i(t_0), x_i(t_0 + 1), \ldots, x_i(t)\}$ for $i \in \mathbb{Z}_k$, while the input to any subsystem is restrained to be a linear function of the elements of the information structure $u_i(t) = F_{ii} x_i(t)$ for all $i \in \mathbb{Z}_k$ and $i = c$. The class could be characterized by stating that there is local control exclusively.

The class of control laws G_C is such that the information structure of a subsystem is $IS_i(t) = \{x_i(t_0), x_i(t_0+1), \ldots, x_i(t), x_c(t_0), x_c(t_0+1), \ldots, x_c(t)\}$, $i \in \mathbb{Z}_k$, and the information structure of the coordinator is $IS_c(t) = \{x_c(t_0), x_c(t_0 + 1), \ldots, x_c(t)\}$, while the control laws of the subsystems and of the coordinator are restrained to be linear functions of the current state, $u_i(t) = F_{ii} x_i(t) + F_{ic} x_c(t)$ for all $i \in \mathbb{Z}_k$ and $u_c(t) = F_{cc} x_c(t)$. The information structure in this case is compatible with the structure of the coordinated system, the coordinator does not receive the states of the other subsystems while the other subsystems receive the state of the coordinator.

The class of control laws G_M is such that the information structure of a subsystem is $IS_i(t) = \{x_i(t_0), x_i(t_0+1), \ldots, x_i(t), x_c(t_0), x_c(t_0+1), \ldots, x_c(t)\}$, $i \in \mathbb{Z}_k$, and the information structure of the coordinator is $IS_c(t) = \{x_c(t_0), x_c(t_0 + 1), \ldots, x_c(t)\} \cup_{i=1}^{k} IS_i(t)$, while the control laws are restrained to be linear functions of the current states, $u_i(t) = F_{ii} x_i(t) + F_{ic} x_c(t)$ for all $i \in \mathbb{Z}_k$ and $u_c(t) = F_{cc} x_c(t) + \sum_{i=1}^{k} F_{ci} x_i(t)$. The information structure is again compatible with the structure of the Gaussian M system, any subsystem observes the local state and receives from the coordinator its state, while the coordinator observes its local state x_c and receives from the subsystems their states $x_i(t)$, $i \in \mathbb{Z}_n$. Note that a subsystem does not receive the states of the other subsystems, not counting the coordinator.

The classes of control laws G_L and G are then clear from the formulas.

5.2 Optimal Control of Gaussian Coordinated Systems

Problem 5.6 *Consider a coordinated Gaussian system. Consider the classes of control laws G, G_M, G_C and G_{Bdiag}. Consider the cost functions*

$$
\begin{aligned}
& J : G \to \mathbb{R}_+, \\
J(g) &= \lim_{t_1 \to \infty} \frac{1}{t_1} E\left[\sum_{s=0}^{t_1-1} \begin{pmatrix} x(s) \\ u(s) \end{pmatrix}^T \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} x(s) \\ u(s) \end{pmatrix} \right], \quad (36) \\
& \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}^T \geq 0, \ R > 0. \quad (37)
\end{aligned}
$$

Solve the optimal stochastic control problems:

$$
\begin{aligned}
J^* &= \inf_{g \in G} J(g); \ \ J_C^* = \inf_{g_C \in G_C} J(g_C); \ \ J_L^* = \inf_{g_L \in G_L} J(g_L); \quad (38) \\
J_M^* &= \inf_{g_M \in G_M} J(g_M); \ \ J_{\text{Bdiag}}^* = \inf_{g_{\text{Bdiag}} \in G_{\text{Bdiag}}} J(g_{\text{Bdiag}}). \quad (39)
\end{aligned}
$$

5.3 Control Synthesis - Eigenvalue Assignment

In this section control synthesis is studied for eigenvalue placement of linear coordinated systems. The result below was first described in [26].

Proposition 5.7 *Consider a coordinated linear control system with representation (24). For any complex-conjugate subset of the complex numbers, $S_{specification} \subset \mathbb{C}$ there exist control laws $g_1(x) = F_{11}x_1 + F_{13}x_3$, $g_2(x) = F_{22}x_2 + F_{23}x_3$, $g_3(x) = F_{33}x_3$, such that the inputs $u_1(t) = g_1(x(t))$, $u_2(t) = g_2(x(t))$, $u_3(t) = g_3(x(t))$, yield a closed-loop linear system with eigenvalues of the system matrix in $S_{specification}$ if and only if the matrix pairs $(A_{11}, B_{11}), (A_{22}, B_{22})$, and$(A_{33}, B_{33})$ are controllable.*

The resulting closed-loop system is then $dx(t)/dt = A_{cl}x(t)$, $x(t_0) = x_0$, where A_{cl} is given by

$$
\begin{pmatrix} A_{11} + B_{11}F_{11} & 0 & A_{13} + B_{11}F_{13} + B_{13}F_{33} \\ 0 & A_{22} + B_{22}F_{22} & A_{23} + B_{22}F_{23} + B_{23}F_{33} \\ 0 & 0 & A_{33} + B_{33}F_{33} \end{pmatrix},
$$

which has the required structure of a coordination-structured matrix.

The reader can now easily formulate the result corresponding to the above proposition for which only exponential stability of the closed-loop linear system is required, in terms of stabilizability.

5.4 Control Synthesis - Optimal Control of Gaussian Coordinated Systems

The optimal control of a Gaussian coordinated system is presented below.

Recall the system equations (3,4), the coordination information structure of Def. 5.3, the class of coordinated control laws G_C of Def. 5.5, and the cost function of Problem 5.6. Note that the information structures of each tuple of subsystems differ and the information structure of the coordinator is again different from that of all of the subsystems. Therefore control synthesis of centralized systems is not applicable. However, the information structure of the coordinator is contained in the information structure of each of the subsystems. This relation between the information structures are also described as nested. Decision making with nested information structures were studied by Y.C. Ho and K.C. Chu, see [11]; thus in this problem each team takes a decision once. Nested information structures in distributed control have not been investigated extensively. There is a relation with the concept of Stackelberg games. A Stackelberg game involves only one decision. The concept of a Stackelberg dynamic game is much more involved and several concepts are possible, see [3]. A research issue is to investigate coordination control with nested information structures in more depth.

Problem 5.8 *Consider Problem 5.6. Determine the minimal cost J_C^* and the optimal control law g_C^* for this problem with respect to the class of coordinated control laws:*

$$J_C^* = \inf_{g_C \in G_C} J(g_C) = J(g_C^*). \tag{40}$$

As stated before, the above problem cannot be solved by the standard LQG theory because the information structures of the coordinator and of the subsystems differ. However, the optimal control problem can be solved by a search procedure for the feedback matrix of the coordinator as described below. The theorem and the procedure which follow are due to P.L. Kempker (personal communication) and to N. Pambakian [25].

Theorem 5.9 *Consider Problem 5.6 and Problem 5.8. Assume that there exists a coordinated control law such that the closed-loop system is asymptotically stable. Am appropriate stabilizability condition will imply that the previous assumption holds.*

(a) The cost function decomposes additively as

$$\begin{aligned}
& && g_C \in G_C,\ u = Fx,\ F \in \mathbb{R}_C^{m \times n}, \\
J(g_C) &=& & \text{trace}([Q_x[Q + SF + F^T S^T + F^T RF]) && (41)\\
&=& & J_{11}(F_{11}, F_{1c}, F_{cc}) + J_{1c}(F_{11}, F_{1c}, F_{cc}) + \\
& && +J_{22}(F_{22}, F_{2c}, F_{cc}) + J_{2c}(F_{22}, F_{2c}, F_{cc}) + \\
& && +J_{cc}(F_{cc});\ \ \textit{where}, && (42)\\
J_{11}(F_{11}, F_{1c}, F_{cc}) &=& & \text{trace}(Q_{x,11}[Q_{11} + S_{11}F_{11} + F_{11}^T S_{11}^T + F_{11}^T R_{11} F_{11}]), && (43)\\
J_{1c}(F_{11}, F_{1c}, F_{cc}) &=& & 2\text{trace}(Q_{x,1c}[Q_{1c} + S_{11}F_{11} + F_{1c}^T S_{11}^T + F_{11}^T R_{11} F_{1c}]^T) + \\
& && +\text{trace}(Q_{cc} F_{1c}^T R_{11} F_{1c}), && (44)\\
J_{cc}(F_{1c}, F_{2c}, F_{cc}) &=& & \text{trace}(Q_{x,cc}[Q_{cc} + S_{cc}F_{cc} + F_{cc}^T S_{cc}^T + F_{cc}^T R_{cc} F_{cc}]), && (45)
\end{aligned}$$

and further formulas provided below.

(b) The optimization problem decomposes into several nested problems according to the formula (46),

$$\begin{aligned}
& \inf_{F_{11}, F_{22}, F_{1c}, F_{2c}, F_{cc}} J(F) && (46)\\
= \ & \inf_{F_{cc} \in \mathbb{R}^{m_c \times n_c}} \left[\sum_{i=1}^{2} \inf_{F_{ic}} \left(J_{cc,i}(F_{ic}, F_{cc}) + \inf_{F_{ii}} [J_{ii}(F_{ii}, F_{ic}, F_{cc}) + J_{ic}(F_{ii}, F_{ic}, F_{cc})] \right) + \right. \\
& \left. + J_{cc}(F_{cc}) \right].
\end{aligned}$$

For the computation of the optimal control law a search procedure is defined over the matrix $F_{cc} \in \mathbb{R}^{m_c \times n_c}$. A procedure to compute the optimal cost for a fixed F_{cc} matrix follows which may be used for the search procedure stated above.

Procedure 5.10 *Consider Problem 5.6. Data $F_{cc} \in \mathbb{R}^{m_c \times n_c}$.*

1. *Compute the variance matrix $Q_{x,cc}(F_{cc}) \in \mathbb{R}^{n_c \times n_c}$ as the solution of the following discrete-time Lyapunov equation (Stein equation) and note that $Q_{x,cc}$ is a rational function of F_{cc}.*

$$\begin{aligned}
Q_{x,cc} &= A_{cc,cl} Q_{x,cc} A_{cc,cl}^T + M_{cc} Q_{v,cc} M_{cc}^T,\ \textit{where}, && (47)\\
A_{cc,cl} &= A_{cc} + B_{cc} F_{cc}. && (48)
\end{aligned}$$

2. *For $i = 1, 2, \ldots, k$ do:*

(a) *Compute the matrix $Q_{c,ii} \in \mathbb{R}^{n_i \times n_i}$ as the solution of the algebraic Riccati equation:*

$$\begin{aligned} Q_{c,ii} &= A_{ii}^T Q_{c,ii} A_{ii} + Q_{ii} + \\ &\quad -[A_{ii}^T Q_{c,ii} B_{ii} + S_{ii}][B_{ii}^T Q_{c,ii} B_{ii} + R_{ii}]^{-1}[A_{ii}^T Q_{c,ii} B_{ii} + S_{ii}]^T \end{aligned} \tag{49}$$

The matrix $Q_{c,ii}$ does not depend on the matrix F_{cc}.

(b) *Compute the feedback matrix $F_{ii} \in \mathbb{R}^{m_i \times n_i}$ by the following formula. Note that the matrix F_{ii} does not depend on the matrix F_{cc}.*

$$F_{ii} = -[B_{ii}^T Q_{c,ii} B_{ii} + R_{ii}]^{-1}[A_{ii}^T Q_{c,ii} B_{ii} + S_{ii}]^T. \tag{50}$$

(c) *Compute the matrix $Q_{c,ic} \in \mathbb{R}^{n_i \times n_c}$ by solving the following asymmetric Lyapunov equation. Note that $Q_{c,ic}$ is a rational function of the elements of F_{cc}.*

$$\begin{aligned} Q_{c,ic} &= \left[A_{ii}^T - \left(A_{ii}^T Q_{c,ii} B_{ii} + S_{ii}\right)[B_{ii}^T Q_{c,ii} B_{ii} + R_{ii}]^{-1} B_{ii}^T\right] Q_{c,ic}\left(A_{cc} + B_{cc} F_{cc}\right) + \\ &\quad + Q_{ic} + \left[A_{ii}^T Q_{c,ii}\left(A_{ic} + B_{ic} F_{cc}\right)\right] + \\ &\quad - \left[A_{ii}^T Q_{c,ii} B_{ii} + S_{ii}\right]\left[B_{ii}^T Q_{c,ii} B_{ii} + R_{ii}\right]^{-1}\left[B_{ii}^T Q_{c,ii}(A_{ic}^T + B_{ic} + F_{cc})\right] \end{aligned} \tag{51}$$

(d) *Compute the feedback matrix $F_{ic} \in \mathbb{R}^{m_i \times n_i}$ by the following formula. Note that the matrix F_{ic} is a rational function of the elements of the matrix F_{cc}.*

$$\begin{aligned} F_{ic} &= -[B_{ii} Q_{c,ii} B_{ii}^T + R_{ii}]^{-1} \times \\ &\quad \times [B_{ii}^T Q_{c,ii}(A_{ic} + B_{ic} F_{cc}) + B_{ii} Q_{c,ic}(A_{cc} + B_{cc} F_{cc})]. \end{aligned} \tag{52}$$

(e) *Compute the matrix $Q_{x,ic} \in \mathbb{R}^{n_i \times n_c}$ as the solution of the Lyapunov equation,*

$$Q_{x,ic} = A_{ii,cl} Q_{x,ic} A_{cc,cl}^T + A_{ic,cl} Q_{x,cc} A_{cc,cl}^T + + M_{ii} Q_{v,ic} M_{cc}^T + M_{ic} Q_{v,cc} M_{cc}^T. \tag{53}$$

Note that it is a rational function of the elements of the matrix F_{cc} because of its dependence via

$$A_{ic,cl}(F_{cc}) = A_{ic} + B_{ii} F_{ic}(F_{cc}) + B_{ic} F_{cc}, \tag{54}$$

$$A_{cc,cl}(F_{cc}) = A_{cc} + B_{cc} F_{cc}. \tag{55}$$

(f) *Compute the matrix $Q_{x,ii} \in \mathbb{R}^{n_i \times n_i}$ by solving the Lyapunov equation*

$$\begin{aligned} Q_{x,ii} &= A_{ii,cl} Q_{x,ii} A_{ii,cl}^T + A_{ic,cl} Q_{x,ic}^T A_{ii,cl}^T + A_{ic,cl} Q_{x,cc} A_{ic,cl}^T + \\ &\quad + M_{ii} Q_{v,ii} M_{ii}^T + M_{ic} Q_{v,ic}^T M_{ii}^T + M_{ii} Q_{v,ic} M_{ic}^T + M_{ic} Q_{v,cc} M_{ic}^T. \end{aligned} \tag{56}$$

Note that $Q_{x,ii}$ is a rational function of the elements of F_{cc} because of the dependence on $A_{ic,cl}(F_{cc})$, $Q_{x,ic}(F_{cc})$, and on $Q_{x,cc}(F_{cc})$.

3. *Compute the full feedback matrix from the previous steps as*

$$F = \begin{pmatrix} F_{11} & 0 & \dots & 0 & F_{1c} \\ 0 & F_{22} & 0 & \dots & F_{2c} \\ \vdots & & \ddots & \dots & \vdots \\ 0 & \dots & & F_{kk} & F_{kc} \\ 0 & 0 & \dots & 0 & F_{cc} \end{pmatrix}. \tag{57}$$

and compute the overall state variance matrix

$$Q_x = \begin{pmatrix} Q_{x,11} & 0 & \dots & 0 & Q_{x,1c} \\ 0 & Q_{x,22} & 0 & \dots & Q_{x,2c} \\ \vdots & & \ddots & \dots & \vdots \\ 0 & \dots & & Q_{x,kk} & Q_{x,kc} \\ Q_{x,1c}^T & Q_{x,2c}^T & \dots & Q_{x,kc}^T & Q_{x,cc} \end{pmatrix}. \tag{58}$$

Both matrices are rational functions of the elements of F_{cc}.

4. *Compute the cost function*

$$J(F_{cc}) = \text{trace}\left(Q_x(F_{cc})[Q + F(F_{cc})^T RF(F_{cc})]\right). \tag{59}$$

which is then a rational function of the elements of F_{cc}.

Proposition 5.11 *The above procedure to compute the cost as a function of the feedback matrix F_{cc} is correct. The cost is a rational function of the elements of the feedback matrix F_{cc}.*

The rational dependence follows because the solution of a Lyapunov function, which is a linear matrix equation, is a rational function of the elements of its matrices by Cramér's rule. In general a rational function is not convex. Only if special conditions hold will it be convex on a domain. It is not clear whether $J(F_{cc})$ is a convex function of the elements of the matrix F_{cc}. More research for this issue is required. For many examples the procedure has produced satisfactory control laws.

An application of control synthesis of Gaussian coordinated systems is the formation flying of autonomous underwater vehicles (AUVs), see [15, 25].

5.5 The Cost of Coordination

Of interest to control theory is the comparison of the costs of the various forms of coordination.

Definition 5.12 *Consider the optimal stochastic control problem of Problem 5.6.*

(a) If $J_L^* = J_C^*$ then one says that there is no loss in cost *for this problem by using coordinated control laws compared with an arbitrary linear control law. If $J_L^* < J_C^*$ then call $(J_C^* - J_L^*)$ the* price of coordination control *in comparison with a linear control law in terms of full state information.*

(b) If $J_C^* = J_{\text{Bdiag}}^*$ then one says that there is no loss in cost *for this problem by using only local state information compared with coordinated state information. If $J_C^* < J_{\text{Bdiag}}^*$ then call $(J_{\text{Bdiag}}^* - J_C^*)$ the* gain *of coordinated control compared to distributed control in terms of the costs.*

Only for numerical examples is there so far information on the price of coordination. A research issue is to determine which properties of the system and of the cost function characterize the differences $(J_C^* - J_L^*)$ and $(J_{\text{Bdiag}}^* - J_C^*)$.

5.6 Control Synthesis of Coordinated M Systems

M systems were introduced in Section 5 as an alternative to coordinated systems. Recall that in a M system each subsystem receives from the coordinator its state and sends to the coordinator its own state. However, there is no direct communication between the individual subsystems. The structure of an M system appears many times in a hierarchical system.

Control synthesis of Gaussian M systems is briefly discussed. The conclusion is summarized at the start of the story: the optimal control law of a Gaussian M system within the class of linear control laws is likely not to have the M structure. Yet, for several control engineering systems the M-structured control law may be of interest and the performance of the closed-loop system has to be investigated.

Recall the definition of a Gaussian M system for the case of two subsystems.

$$\begin{aligned} x(t+1) &= \begin{pmatrix} A_{11} & 0 & A_{1c} \\ 0 & A_{22} & A_{2c} \\ A_{c1} & A_{c2} & A_{cc} \end{pmatrix} x(t) + \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & B_{cc} \end{pmatrix} u(t) + \\ &+ \begin{pmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & 0 \\ 0 & 0 & M_{cc} \end{pmatrix} v(t),\ x(t_0) = x_0. \end{aligned} \tag{60}$$

Definition 5.13 *Consider a Gaussian M system. Define the information structures,*

$$\begin{aligned} IS_i(t) &= \{x_i[t_0,t], x_c[t_0,t]\}, &(61)\\ IS_c(t) &= \{x_c[t_0,t], \{x_j[t_0,t],\ \forall\, j \in \mathbb{Z}_k\}\}. &(62)\end{aligned}$$

Define the set of M control laws as,

$$\begin{aligned} G_M &= \{g : X \to U | g(x) = Fx,\ F \in \mathbb{R}^{m\times n}_{\text{arrow}}\}, &(63)\\ F &= \begin{pmatrix} F_{11} & 0 & F_{1c} \\ 0 & F_{22} & F_{2c} \\ F_{c1} & F_{c2} & F_{cc} \end{pmatrix} \in \mathbb{R}^{m\times n}_{\text{arrow}}. \end{aligned}$$

An input determined by a control law in the class G_M is represented as,

$$\begin{aligned} u_i &= g_i(x) = F_{ii}x_i + F_{ic}x_c,\\ u_c &= g_c(x) = F_{cc}x_c + \sum_{j=1}^{k} F_{ci}x_i. \end{aligned}$$

Problem 5.14 *Consider a Gaussian M system, the class G_M of control laws, and the quadratic cost function of Problem 5.6. Solve the optimal control problem for the optimal control law $g_M^* \in G_M$ and the value J_M^* satisfying*

$$J_M^* = \inf_{g_M \in G_M} J(g_M) = J(g_M^*). \qquad (64)$$

The main research issues are to determine an algorithm for the computation of $g_M^* \in G_M$ and the value J_M^*; and to determine a characterization of the difference $(J_L^* - J_M^*)$, where J_L^* is defined in Def. 5.6.

In general the difference will satisfy $J_L^* - J_M^* < 0$ or, equivalently, a strictly lower cost can be achieved by using an optimal control law based on the full state than by using a control law with the structure of an arrow matrix. Note that the optimal control problem with the class of control laws G_L is the standard LQG optimal control problem with complete observations whose solution is well known.

The argument for the above strict inequality is that all subsystems are related via the coordinator. Therefore, control of a subsystem will benefit from using the state of the other subsystems by which it may be influenced. Stated another way, the optimal control law is likely to be of the form $g \in G_L$, $g(x) = Fx$ with $F \in \mathbb{R}^{m\times n}$ but $F \notin \mathbb{R}^{m\times n}_{\text{arrow}}$. As to whether the full linear control is useful in practice is primarily dependent on the strengths of the interactions of the subsystems with the coordinator and on the linear algebraic structure of this interaction. A decomposition of the relations between the subsystems and the coordinator may clarify this. A research issue is to develop concepts and theorems for the interaction of subsystems and control.

An approximation is to ignore the interaction of the subsystems and to compute a control law in the class G_M by optimization techniques. If the performance is satisfactory then the control law can be useful. It limits the amount of communication in the coordinated system.'

Control of coordinated M systems is motivated by its frequent occurence in hierarchical systems. Further research on this seems useful.

5.7 Control Synthesis of Coordinated Discrete-Event Systems

The purpose of this subsection is to present a condition for safety of concurrent discrete-event systems. We are interested in safety of modular systems composed with the coordinator. It will be assumed that a control objective is specified in terms of the closed-loop system both for the coordinator and for the remaining parts of the subsystems. The results of this subsection are described in more detail in [18, 19, 16, 17].

An equivalent condition for verification of safety is conditional safety defined below as safety of the coordinator and safety of each local subsystem when combined with the coordinator.

Problem 5.15 *Consider two generators G_1, G_2, and a coordinator G_k, which makes G_1 and G_2 conditionally independent. Consider a specification language $K \subseteq E^*$. Assume that the language K is conditionally decomposable with respect to the reachable event sets (E_1, E_2, E_k). Determine sufficient or equivalent conditions such that $L_m(G_1) \| L_m(G_2) \| L_m(G_k) \subseteq K$.*

We first have the following result.

Proposition 5.16 *Let $K \subseteq E^*$ be conditionally decomposable with respect to the reachable event sets (E_k, E_1, E_2) and let there exist a coordinator G_k over E_k^* such that*

(i) $L_m(G_k) \subseteq P_k(K)$,

(ii) $L_m(G_1) \| L_m(G_k) \subseteq P_{1\cup k}(K)$,

(iii) $L_m(G_2) \| L_m(G_k) \subseteq P_{2\cup k}(K)$.

Then $L_m(G_1) \| L_m(G_2) \| L_m(G_k) \subseteq K$.

Proof Since synchronous product is associative, commutative, and idempotent we obtain:

$$\begin{aligned} & L_m(G_1) \| L_m(G_2) \| L_m(G_k) \\ = \; & L_m(G_k) \| (L_m(G_1) \| L_m(G_k)) \| (L_m(G_2) \| L_m(G_k)) \\ \subseteq \; & P_k(K) \| P_{1\cup k}(K) \| P_{2\cup k}(K) = K, \end{aligned}$$

because of Definition 4.11. □

Now we present a weaker (string based) necessary and sufficient conditions for safety similar to conditions used for nonblockingness.

Definition 5.17 *Consider the setting of Problem 5.15. The system and the specification language $K \subset E^*$ are said to be* conditionally safe *with respect to the reachable event sets (E_k, E_1, E_2) if*

$$(1) \quad L_m(G_k) \subseteq P_k(K); \tag{65}$$

$$(2) \quad (2.i) \quad L_m(G_1) \| L_m(G_k) \cap (P_k^{1\cup k})^{-1} P_k^{2\cup k}(L_m(G_2) \| L_m(G_k)) \subseteq P_{1\cup k}(K), \tag{66}$$

$$(2.ii) \quad L_m(G_2) \| L_m(G_k) \cap (P_k^{2\cup k})^{-1} P_k^{1\cup k}(L_m(G_1) \| L_m(G_k)) \subseteq P_{2\cup k}(K). \tag{67}$$

The main result of this section now follows.

Theorem 5.18 *Consider Problem 5.15.*
(a) If the system and the specification language are conditionally safe with respect to the reachable event sets (E_1, E_2, E_k) then $L_m(G_1) \| L_m(G_2) \| L_m(G_k) \subseteq K$.
(b) Conversely, if $L_m(G_1) \| L_m(G_2) \| L_m(G_k) \subseteq K$ and $L_m(G_k) \subseteq P_k(L_m(G_1) \| L_m(G_2))$ then the system and the specification language are conditionally safe with respect to the reachable event sets (E_1, E_2, E_k).

The theorem above provides a necessary and a sufficient condition for a coordinated discrete-event system to be conditionally safe. Let us remark that $L_m(G_k) \subseteq P_k(L_m(G_1) \| L_m(G_2))$ is often satisfied in coordination control, because coordinators for safety as well as coordinators for nonblockingness typically do not add additional behavior to the composed systems, i.e. $L_m(G_c)$ is included in the projected behavior.

In the remainder of this subsection the overall control synthesis is presented. Using the coordination scheme, first a supervisor for the coordinator is synthesized that takes care of the part $P_k(K)$ of the specification K. Then $S_i, \ i = 1, 2$, are synthesized such that the corresponding part of the specification, i.e. $P_{i\cup k}(K)$, is taken care of with respect to the corresponding plant languages $G_i \| (S_c/G_k)$. Let $E_u \subseteq E$ be the set of uncontrollable events and $E_{i,u} = E_u \cap E_i, \ i = 1, 2, k$ the corresponding sets of local uncontrollable events.

Problem 5.19 *Consider generators $G_1,\ G_2,\ G_k$ and a specification language $K \subseteq (E_1 \cup E_2 \cup E_k)^*$. Assume that the coordinator G_k makes the two generators $G_1,\ G_2$ conditionally independent and that the language K is conditionally decomposable.*

Determine supervisors $S_1,\ S_2,\ S_k$ for the respective generators such that the closed-loop system with S_k/G_k as coordinator for S_1/G_1 and S_2/G_2 is such that

$$L(S_1/[G_1\|(S_k/G_k)])\|L(S_2/[G_2\|(S_k/G_k)])\|L(S_k/G_k) = K. \tag{68}$$

Definition 5.20 *Consider the setting of Problem 5.19. Call the specification language $K \subset E^*$* conditionally controllable *for generators (G_1, G_2, G_k) and for the event subsets $(E_{1,u}, E_{2,u}, E_{k,u})$ if*

1. *The language $P_k(K) \subseteq E_k^*$ is controllable with respect to G_k and $E_{k,u}$; equivalently,*

   $$P_k(K)E_{k,u} \cap L(G_k) \subseteq P_k(K). \tag{69}$$

   Then there exists a supervisor S_k for G_k such that $L(S_k/G_k) = P_k(K)$. The supervisor S_k is used in the remaining part of the definition.

2. *The language $P_{1\cup k}(K) \subseteq (E_1 \cup E_k)^*$ is controllable with respect to $L(G_1\|(S_k/G_k))$ and $E_{1+k,u} = E_u \cap (E_1 \cup E_k)$; equivalently,*

   $$P_{1\cup k}(K)E_{1+k,u} \cap L(G_1\|(S_k/G_k)) \subseteq P_{1\cup k}(K)$$

3. *The language $P_{2\cup k}(K) \subseteq (E_2 \cup E_k)^*$ is controllable with respect to $L(G_2\|(S_k/G_k))$ and $E_{2+k,u} = E_u \cap (E_2 \cup E_k)$; equivalently,*

   $$P_{2\cup k}(K)E_{2+k,u} \cap L(G_2\|(S_k/G_k)) \subseteq P_{2\cup k}(K)$$

The conditions of Definition 5.20 can be checked by efficient algorithms as is directly clear from the computational complexity of controllability in the case of only one subsystem. The computational complexity of checking conditional controllability is much less than that of checking controllability of the global system, $L(G_1\|G_2\|G_k)$.

Theorem 5.21 *Consider Problem 5.19 of control for safety. There exists a set of supervisors $(S_k,\ S_1,\ S_2)$ such that*

$$L(S_1/[G_1\|(S_k/G_k)])\|L(S_2/[G_2\|(S_k/G_k)])\|L(S_k/G_k)) = K, \tag{70}$$

if and only if the specification language K is conditionally controllable with respect to (G_1, G_2, G_k) and $(E_{1,u}, E_{2,u}, E_{k,u})$.

The theorem above provides a necessary and sufficient condition for the existence of a supervisory control which achieves the specification language for the considered closed-loop system. The interest in Theorem 5.21 is in the saving of the computation of the supervisor, the distributed way of computing successively the supervisors S_k, S_1, and S_2 is much less complex than that of the supervisor constructed for the system $G_1\|G_2\|G_k$.

An application of coordination control of discrete-event systems has been investigated. The application concerns a laboratory example of supervisory control of a scale model called the *paint factory*, located at the Faculty of Mechanical Engineering of the Eindhoven University of Technology. The machine produces cups of colored fluids. It has vessels to store and mix fluids, a switched network of pipes and pumps to drive the fluids, and a turntable where the cups are eventually filled. The operations include the pumping of the fluids between vessels, from a vessel to the turn table, and the cleaning operations of the mixing vessel and of the pipes. The model is in terms of automata as described before, and a hierarchical system has been formulated. Control synthesis based on coordination control of the common pipes has been carried out using the software packages libFAUDES [24] and DESTool [23]. The computational saving of coordination control compared to centralized control is significant. The details are planned to be published in a report.

5.8 Further Research Issues

Research results are available on several other issues of coordination control of Gaussian coordinated systems. Those are not stated in this paper for lack of space. Filtering of Gaussian coordinated systems can be carried out in a coordinated way. The Kalman gain has a coordinated structure and the matrices of the structured Kalman gain can be computed in a way analogous to that of the coordinated feedback matrix described above in this section. The optimal control problem with partial observations does not have the separation property because of the information structure imposed. However, after decomposition of the control problem into several subproblems, each of the subproblems has the separation property. The control law computation then separates with the help of that assumption. These results are due to P.L. Kempker (personal communication) and to N. Pambakian, see [25].

It should be clear that the investigation of coordination control is incomplete. Research issues to be investigated further include:

- Experience with the procedure for the computation of the optimal control laws and its convergence.
- Information on the costs of coordination, in particular the gain in the cost by coordination compared with local control, $J^*_{\mathrm{Bdiag}} - J^*_C$, and the loss in cost due to coordination compared with full state control, $J^*_L - J^*_C$.
- The communication in a coordinated system. For example, the complexity of communication from the coordinator to the subsystems. In terms of the control law of a Gaussian coordinated system the complexity of the communication from the coordinator to Subsystem i can be defined as $\mathrm{rank}(F_{ic})$ if the control law of the i-th subsystem is $g_i(x) = F_{ii}x_i + F_{ic}x_c$. What does $F_{ic}x_c$ represent? The study of this and related questions depends on the decomposition of a coordinated system in terms of controllability and observability mentioned in Subsection 4.7. How to quantize the information to be communicated? etc.
- The effect of noise suppression in Gaussian coordinated systems. From examples it appears that the variance of the noise is amplified if one compares the coordinator with a subsystems. This effect can be counteracted by the cost function but this relates to the other control objectives. The decomposition of a Gaussian coordinated system in regard to effect of the different noise processes on the different parts of the systems will have to be investigated for this issue.

6 Distributed Control with Communication

The purpose of this section is to introduce to the reader the topic of distributed control with communication.

It is a fact that many distributed engineering systems are controlled in such a way that the local controllers, often one controller per subsystem, communicate with each other. Examples are mentioned below. The form of the communication depends on the problem considered. This form of communication requires the use of a communication channel and processing of the received information. A problem issue is therefore whether the sum of the communication and the of the associated computational costs is lower than the decrease in the performance costs due to communication? This economic aspect has to be combined with control and communication synthesis.

Control theory has not spent much attention on the overall theory for such a form of control. This section aims at formulation of the problem and a discussion of the problem issues.

Examples of systems with the control architecture of distributed control with communication were mentioned in Section 3.

Problem The problem of distributed control with communication for a distributed control system is to synthesize and to design communication laws and control laws so as to achieve the global control objectives.

The general control issues are: With whom should a local controller communicate? What should be communicated? When should a controller communicate? How is the communication process to be organized? How should the received communication be used? The following forms of distributed control with communication have been considered: (a) A controller sends a subset of its observations. (b) (Informing) A controller sends its latest observation when it presumes this is useful to other controllers. (c) A controller requests information of another controller if it presumes that controller has information which is useful to it.

More concretely, the following control issues are addressed:

1. (What?) Which information is useful to the control tasks of a controller which that controller does not observe but other controllers observe? This requires an analysis of the distributed system, of the observations, and of the control objectives. A principle could be that the stronger the interaction of two or more subsystems of the distributed system, the more information the respective controllers should exchange. In the case of almost no interaction this should then lead to no communication. An approach for this research issue is to develop the concepts of common and private information originally proposed by H.S. Witsenhausen.

2. (When?) When should a controller request information of other controllers or when should it sent information of its own observations to others?

3. (Whom?) To which other controllers should the information be sent? The answer depends much on the structure of the interaction of the distributed system.

4. (Communication) How is the encoding and the decoding to be formulated in a protocol for the communication from a controller to another? The operation of the communication network is a factor in this.

5. (Processing) How to formulate a framework for processing the information received by a controller not only directly from the plant but also from other controllers? H.S. Witsenhausen [39] has formulated the concept of an information structure for this problem issue and this concept deserves to be better known. The use of this concept for state estimation and control was also described by Witsenhausen. A related question is how to integrate the two streams of observed events received by a supervisor in supervisory control. The difficulty here is the fact that the events received from two or more streams have to be ordered by the distributed system.

6. (Control) How to carry out control synthesis for distributed control with communication of a distributed system? It seems that even after the information of other controllers has been received, one still is faced

with the distributed control problem discussed in Section 3. Determination of a person-by-person equilibrium is the most likely solution.

Overview of Theoretical Contributions Distributed control with communication for a discrete-event system has been investigated for more than 15 years in the area of supervisory control. The problem was formulated in the paper [41] where a sufficient condition for its effectiveness is stated. The result is that the combined observations should allow the controller that receives the extra observations to control the discrete-event system such that the control objective of safety is met. Major research contributions were provided by G. Barrett and S. Lafortune, [2], and by L. Ricker and K. Rudie, [28]. Both provide algorithms on how to request communication and how to integrate the received communications into the control for the plant. The two approaches differ in that the first paper requests the information at the latest moment while the second paper communicates information at the earliest possible moment. For other publications see [27]. Further research is required into the framework for this problem.

Various forms of nearest neighbor communication have been explored in various parts of engineering. The authors are not aware of a comprehensive paper of this approach.

Distributed control with communication for stochastic systems was formulated in the paper [34]. The paper presents several forms of communication. Work is in progress on the formulated problems.

The reader is referred to the following references for more details on the topic of this section, [39, 41, 5, 34].

Distributed Control with Communication of Gaussian Systems Consider a Gaussian distributed system. The interconnection of the systems may be described with a directed graph. Examples of such graphs are a graph consisting of a line, a grid, or an arbitrary graph. Denote the directed graph by $G = (N, E)$ in which N is the set of nodes and E is the set of edges. The concept of nearest neighbors in a graph is needed. Define for node $i \in N$ of a directed graph, the set of downstream nearest neighbors, the set of upstream nearest neighbors, and the set of nearest neighbors as

$$NN_i^+ = \{j \in N | \exists \, (i,j) \in E\}, \; NN_i^- = \{j \in N | \exists \, (j,i) \in E\}, \; NN_i = NN_i^+ \cup NN_i^-.$$

Consider for the remainder of this subsection a Gaussian distributed system with a line graph structure. The theory of nearest neighbor systems with other graphs is similar.

Definition 6.1 *Define a* Gaussian NNL distributed system *with line structure, and with five subsystems, and nearest neighbor interaction by the representation,*

$$\begin{aligned} x(t+1) \;=\; & \begin{pmatrix} A_{11} & A_{12} & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 \\ 0 & A_{32} & A_{33} & A_{34} & 0 \\ 0 & 0 & A_{43} & A_{44} & A_{45} \\ 0 & 0 & 0 & A_{54} & A_{55} \end{pmatrix} x(t) + \begin{pmatrix} B_{11} & 0 & 0 & 0 & 0 \\ 0 & B_{22} & 0 & 0 & 0 \\ 0 & 0 & B_{33} & 0 & 0 \\ 0 & 0 & 0 & B_{44} & 0 \\ 0 & 0 & 0 & 0 & B_{55} \end{pmatrix} u(t) + \\ & + \begin{pmatrix} M_{11} & 0 & 0 & 0 & 0 \\ 0 & M_{22} & 0 & 0 & 0 \\ 0 & 0 & M_{33} & 0 & 0 \\ 0 & 0 & 0 & M_{44} & 0 \\ 0 & 0 & 0 & 0 & M_{55} \end{pmatrix} v(t). \end{aligned} \tag{71}$$

Examples of distributed system with nearest neighbor interaction are described in the literature. C.A. Desoer studied a string of cars with directed nearest neighbor structure on an automated highway in the early 1990's. Coordinated load balancing was studied in [6].

Control of Gaussian NNL distributed systems is discussed next.

Definition 6.2 *Consider a Gaussian NNL distributed system with the represention presented above. Define the information structures of the local controllers as*

$$\begin{aligned} IS_i(t) &= \{x_j[t_0,t], u_j[t_0,t],\ j = i-1, i, i+1\},\ \ i = 2, \ldots, k, \\ IS_1(t) &= \{x_j[t_0,t], u_j[t_0,t],\ j = 1,2\},\ \ IS_k(t) = \{x_j[t_0,t], u_j[t_0,t],\ j = k-1, k\}. \end{aligned}$$

Define the corresponding class of control laws by

$$G_{NN} = \{g : X \to U | g(x) = Fx,\ F \in \mathbb{R}^{m \times n}_{\mathrm{B3D}}\}, \tag{72}$$

$$F_{\mathrm{B3D}} = \begin{pmatrix} F_{11} & F_{12} & 0 & 0 & 0 \\ F_{21} & F_{22} & F_{23} & 0 & 0 \\ 0 & F_{32} & F_{33} & F_{34} & 0 \\ 0 & 0 & F_{43} & F_{44} & F_{45} \\ 0 & 0 & 0 & F_{54} & F_{55} \end{pmatrix} \in \mathbb{R}^{m \times n}_{\mathrm{B3D}}. \tag{73}$$

An example of a control law of the type defined in the previous definition is the backpressure algorithm of communication networks, see [31]. Another example is control of an urban road network where the control law of an intersection is based also on the state of the nearest-neighbor intersections.

An input determined by a control law in the class G_M is represented as,

$$u_i = g_i(x) = F_{i,i-1}x_{i-1} + F_{ii}x_i + F_{i,i+1}x_{i+1},\ i = 2, \ldots, k-1.$$

Problem 6.3 *Consider a Gaussian NNL distributed system, the class G_{NN} of control laws, and the quadratic cost function defined before. Solve the optimal control problem for the optimal control law $g^*_{NN} \in G_{NN}$ and the value J^*_{NN} satisfying*

$$J^*_{NN} = \inf_{g_{NN} \in G_{NN}} J(g_{NN}) = J(g^*_{NN}), \tag{74}$$

The main research issues are to determine an algorithm for the computation of $g^*_{NN} \in G_{NN}$ and the value J^*_{NN}; and to determine a characterization of the difference $(J^*_{NN} - J^*_L)$, where J^*_L is defined in Def. 5.6.

It is expected that in general the above inequality is strict, it is optimal to use control with complete observations on the full state and not a control law in the class G_{NN}.. The argument is that the systems interact via their interconnections. The usefulness of the extra state information from non-nearest neighbors depends on the strength of the interactions of the neighbors in the network.

For particular systems one may approximate and limit attention to a control law based on state information of nearest neighbors only. There then appears a problem of evaluating the performance of an NNL distributed system controlled by a control law in the class of G_{NN}. Stability is a concern, as it was for the string network of vehicles on an automated highway. Of interest is also which components of the state of the nearest neighbors are of interest to be communnicated. Here again the backpressure algorithm may help to develop useful conjectures.

7 Concluding Remarks

This tutorial paper describes distributed systems, both Gaussian distributed systems and distributed discrete-event systems. Four control architectures for distributed systems are formulated and distinguished. The concepts of a Gaussian coordinated system and of a coordinated discrete-event system were defined. Control synthesis is summarized for a Gaussian coordinated system and for coordinated discrete-event system. Distributed control with communication between controllers is discussed.

The plan of the authors is to continue research on control of distributed systems with more emphasis on distributed control with communication, coordination control, and hierarchical control.

Acknowledgements

The European Commission is thanked for the financial support in part via the Project Control for Coordination of Distributed Systems (C4C Project; Grant Agreement number INFSO-ICT-223844). The authors are very gratefull to all researchers of the C4C Research Teams who were active for the C4C Project. Yet, a few researchers are cited for their valuable scientific discussions on control of distributed systems: René K. Boel, Charalambos D. Charalambous, Pieter J. Collins, Frank Ottenhof, and Fernando Lobo Pereira. They also thank Alain Bensoussan, P.R. Kumar, Demostenis Teneketzis, and Pravin Varaiya for comments on the research issues.

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