Numerical solution of Atmospheric Boundary Layer flows problems

T.Bodnár, K. Kozel

Department of Technical Mathematics, Faculty of Mechanical Engineering
Czech Technical University of Prague
Karlovo náměstí 13, 121 35 Prague 2, Czech Republic

I. - Viscosity role in CFD models

II. - Application of CFD models in ABL simulations

Contents

Part I

Viscosity role in CFD models

Chapter 1

Physical viscosity

1.1. Compressible Navier-Stokes equations

Momentum equations

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v}) = -\nabla p + \rho \boldsymbol{f} + \nabla \cdot \mathcal{T} \qquad (1.1)$$

Stokes law for viscous fluid

$$\mathcal{T} = \mu[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T] - \lambda \mathbb{I} \nabla \cdot \boldsymbol{v} \qquad (1.2)$$

$$\nabla \cdot \mathcal{T} = \mathrm{div}\mathcal{T} = \Big(\sum_{j=1}^{3} \frac{\partial \tau_{j1}}{\partial x_j}, \sum_{j=1}^{3} \frac{\partial \tau_{j2}}{\partial x_j}, \sum_{j=1}^{3} \frac{\partial \tau_{j3}}{\partial x_j}\Big) \qquad (1.3)$$

1.2. Simplified models

Compressible Navier-Stokes system

$$\begin{aligned} \nabla \cdot \mathcal{T} &= \nabla \cdot \{\mu[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T] - \lambda \mathbb{I} \nabla \cdot \boldsymbol{v}\} \\ &= \nabla \cdot (\mu \nabla \boldsymbol{v}) + (\lambda + \mu) \nabla (\nabla \cdot \boldsymbol{v}) \end{aligned} \tag{1.4}$$

Incompressible Navier-Stokes system $\operatorname{div} \boldsymbol{v} = 0$

$$\nabla \cdot \mathcal{T} = \nabla \cdot (\mu \nabla \boldsymbol{v}) \tag{1.5}$$

Euler system $\mu = \lambda = 0$

$$\nabla \cdot \mathcal{T} = 0 \tag{1.6}$$

1.3. Transport equation

$$\frac{\partial(\rho c)}{\partial t} + \nabla \cdot (\rho\, c\, \boldsymbol{v}) = \rho f_{pc} - \nabla \cdot \boldsymbol{q}_c \tag{1.7}$$

Fourier's law for concentration flux

$$\boldsymbol{q}_c = -k_c \nabla c \tag{1.8}$$

$$\frac{\partial(\rho c)}{\partial t} + \nabla \cdot (\rho\, c\, \boldsymbol{v}) = f_{pc} + \nabla \cdot (k_c \nabla c) \tag{1.9}$$

1.4. Turbulence effects

Boussinesq hypothesis

$$\mu \longrightarrow K = \mu_L + \mu_T \tag{1.10}$$

Turbulent diffusion coefficient =

=Laminar (molecular) viscosity +

Turbulent (eddy) viscosity

Example

$$\mu_L = const$$

$$\mu_T = \ell^2 \left\| \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \right\|$$

1.5. Summary

Physical diffusion coefficients

- Dynamical viscosity μ
- Bulk viscosity λ
- Diffusion coefficient k
- Turbulent diffusion $K = \mu_L + \mu_T$

Chapter 2

One-dimensional model problems

2.1. Model equations

Advection $u_t + au_x = 0$

$$\hat{u}(\xi,t) = e^{-i\xi at}\hat{\eta}(\xi) \Longrightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i\xi at}\hat{\eta}(\xi)e^{i\xi x}d\xi$$

Diffusion $u_t + bu_{xx} = 0$

$$\hat{u}(\xi,t) = e^{-i^2\xi^2 bt}\hat{\eta}(\xi) \Longrightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\xi^2 bt}\hat{\eta}(\xi)e^{i\xi x}d\xi$$

Dispersion $u_t + cu_{xxx} = 0$

$$\hat{u}(\xi,t) = e^{-i^3\xi^3 ct}\hat{\eta}(\xi) \Longrightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\xi^3 ct}\hat{\eta}(\xi)e^{i\xi x}d\xi$$

Initial data

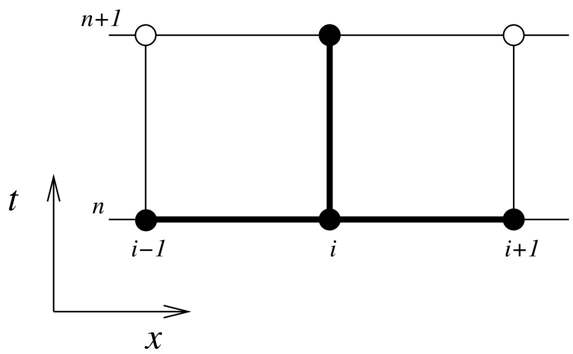
$$u(x,t=0) = \eta(x)$$

$$\hat{u}(\xi,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} u(x,t)e^{-i\xi x}dx \qquad \& \qquad u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{u}(\xi,t)e^{i\xi x}d\xi$$

2.2. Numerical solution of advection equation

Model equation $\quad u_t + au_x = 0 \qquad a > 0$

Initial data $\quad u(x, t = 0) = \eta(x)$



2.2.1. Classical explicit schemes

Up-wind scheme [U]

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{\Delta x}(u_i^n - u_{i-1}^n)$$

Down-wind scheme [D]

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{\Delta x}(u_{i+1}^n - u_i^n)$$

Central scheme [C]

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$$

Lax-Friedrichs scheme [LF]

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$$

Lax-Wendroff scheme [LW]

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{a^2\Delta t^2}{2\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

2.3. Behaviour of numerical schemes - Summary

Scheme	Accuracy	Stability	Behaviour
U	$(\Delta t)^1$ / $(\Delta x)^1$	Stable	Diffusive
D	$(\Delta t)^1$ / $(\Delta x)^1$	UnStable	Dispersive
C	$(\Delta t)^1$ / $(\Delta x)^2$	UnStable	Dispersive
LF	$(\Delta t)^1$ / $(\Delta x)^1$	Stable	Diffusive
LW	$(\Delta t)^2$ / $(\Delta x)^2$	Stable	Dispersive

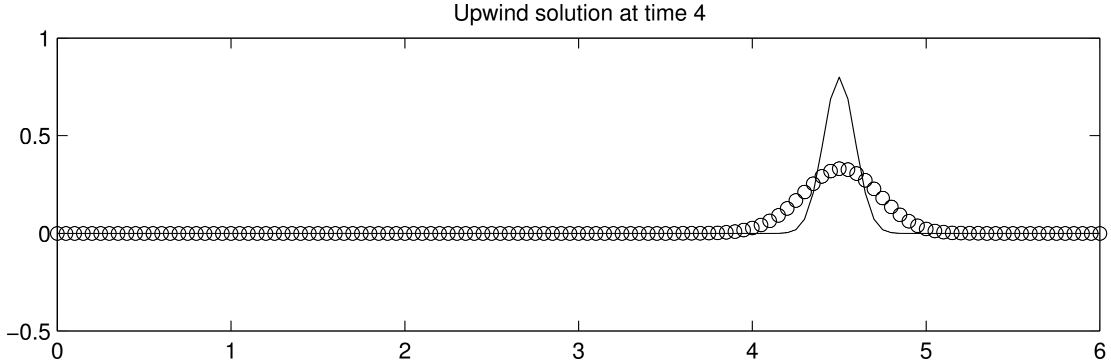
Accuracy

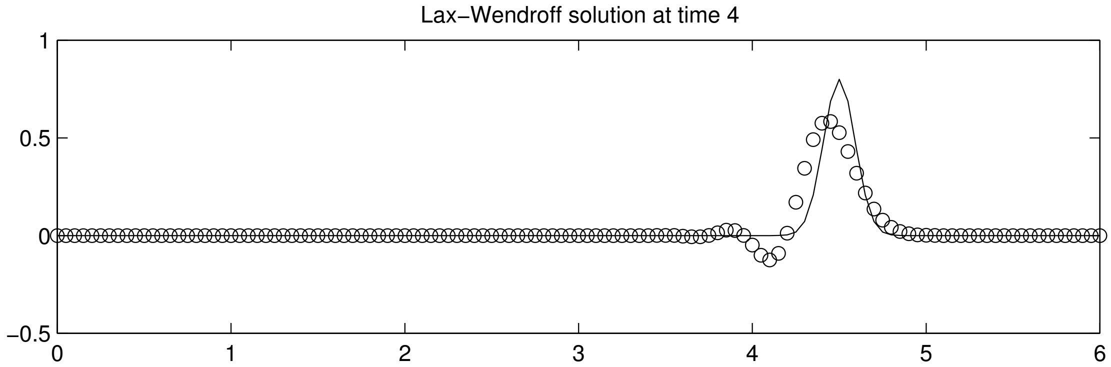
$$u_i^n = u(x_i, t_n) + \mathcal{O}\left((\Delta t)^p + (\Delta x)^q\right)$$

Stability

Von Neumann stabilty analysis

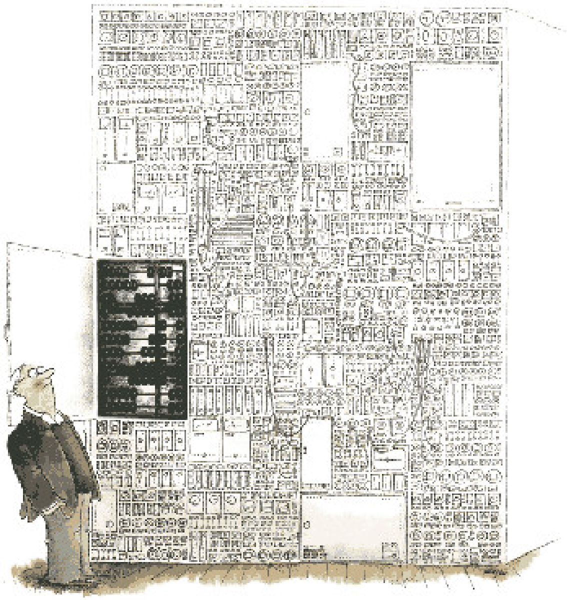
2.3.1. Diffusive & Dispersive behaviour





Chapter 3

Analysis of Numerical Schemes



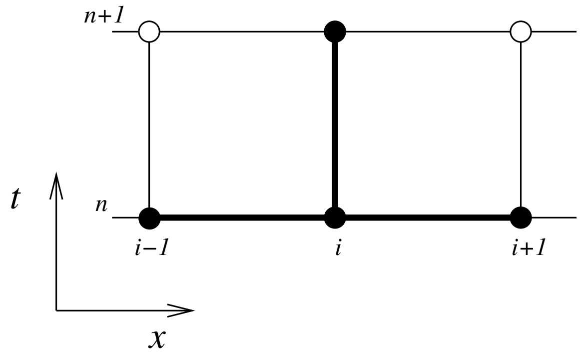
3.1. "Discrete analysis"

3.1.1. Up-wind & Down-wind decomposition

$$u_t + a u_x = 0 \qquad a > 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\left[\alpha\left(\frac{u_{i+1}^n - u_i^n}{\Delta x}\right) + (1-\alpha)\left(\frac{u_i^n - u_{i-1}^n}{\Delta x}\right)\right] = 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\left[\alpha\{\mathsf{D}\} + (1-\alpha)\{\mathsf{U}\}\right] = 0$$



$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\left[\alpha\{\mathsf{D}\} + (1-\alpha)\{\mathsf{U}\}\right] = 0$$

Scheme	Coefficient α
U	$\mathbf{0}$
D	$\mathbf{1}$
C	$\frac{1}{2}$
LF	$\frac{1}{2} - \frac{\Delta x}{2a\Delta t}$
LW	$\frac{1}{2} - \frac{a\Delta t}{2\Delta x}$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\left[\alpha\{\mathsf{D}\} + (1-\alpha)\{\mathsf{U}\}\right] = 0$$

3.1.2. Central & Upwind decomposition

$$\{\mathsf{C}\} = \frac{\{\mathsf{D}\} + \{\mathsf{U}\}}{2} \qquad \Longrightarrow \qquad \{\mathsf{D}\} = 2\{\mathsf{C}\} - \{\mathsf{U}\}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\left[2\alpha\{\mathsf{C}\} + (1-2\alpha)\{\mathsf{U}\}\right] = 0$$

3.1.3. Central & Viscous decomposition

$$\{\mathsf{V}\} = \frac{\{\mathsf{D}\} - \{\mathsf{U}\}}{\Delta x} \qquad \Longrightarrow \qquad \{\mathsf{U}\} = \{\mathsf{C}\} - \frac{\Delta x}{2}\{\mathsf{V}\}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\left[\{\mathsf{C}\} - (1-2\alpha)\frac{a\Delta x}{2}\{\mathsf{V}\}\right] = 0$$

3.2. Numerical viscosity

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\{\mathsf{C}\} = (1 - 2\alpha)\frac{a\Delta x}{2}\{\mathsf{V}\}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\{\mathsf{C}\} = \mu\{\mathsf{V}\}$$

$$u_t + au_x = 0 \qquad \longrightarrow \qquad u_t + au_x = \mu u_{xx}$$

Advection $\longrightarrow$ Advection-diffusion

$$\mu = \underbrace{(1 - 2\alpha)}_{\epsilon}\frac{a\Delta x}{2}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\{\mathsf{C}\} = \mu\{\mathsf{V}\}$$

$$\mu = \underbrace{(1 - 2\alpha)}_{\epsilon} \frac{a\Delta x}{2}$$

Scheme	Coefficient α	Coefficient ϵ	Coefficient μ
U	0	1	$\frac{a\Delta x}{2}$
D	1	-1	$-\frac{a\Delta x}{2}$
C	$\frac{1}{2}$	0	0
LF	$\frac{1}{2} - \frac{\Delta x}{2a\Delta t}$	$\frac{1}{\gamma}$	$\frac{\Delta x^2}{2\Delta t}$
LW	$\frac{1}{2} - \frac{a\Delta t}{2\Delta x}$	γ	$\frac{a^2\Delta t}{2}$

Parameter $\gamma = \frac{a\Delta t}{\Delta x}$ is bounded by stabilty condition.

CFL condition $\implies$ $\gamma < 1$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\{\mathsf{C}\} = \mu\{\mathsf{V}\}$$

$$\mu = \underbrace{(1 - 2\alpha)}_{\epsilon} \frac{a\Delta x}{2}$$

Scheme	ϵ	μ	Accuracy	Stability	Behaviour
D	-1	$-\frac{a\Delta x}{2} < 0$	$(\Delta t)^1/(\Delta x)^1$	UnStable	Dispersive
C	0	0	$(\Delta t)^1/(\Delta x)^2$	UnStable	Dispersive
LW	$\gamma < 1$	$\frac{a^2\Delta t}{2}$	$(\Delta t)^2/(\Delta x)^2$	Stable	Dispersive
U	1	$\frac{a\Delta x}{2}$	$(\Delta t)^1/(\Delta x)^1$	Stable	Diffusive
LF	$\frac{1}{\gamma} > 1$	$\frac{\Delta x^2}{2\Delta t}$	$(\Delta t)^1/(\Delta x)^1$	Stable	Diffusive

CFL condition $\Longrightarrow$ $\gamma < 1$

3.2.1. “Discrete analysis” - Summary

- It is possible to rewrite the schemes as a sum of “inviscid, unstable” part and “viscous, stabilising” part
- Each scheme contains some amount of “imbeded, implicit” numerical viscosity
- The amount of viscosity influences the stability, accuracy and dispersive-diffusive behaviour of the scheme

3.3. "Continuous analysis"

3.3.1. Motivation example

$$\frac{dy}{dt} = f(y) \qquad y(t=0) = y_0$$

Euler method

$$y(t+\Delta t) = y(t) + \Delta t \frac{dy}{dt} + \mathcal{O}(\Delta t^2)$$

$$\Longrightarrow \frac{dy}{dt} = \underbrace{\frac{y(t+\Delta t) - y(t)}{\Delta t}}_{f(y)} + \mathcal{O}(\Delta t)$$

Modified equation

$$y(t+\Delta t) = y(t) + \Delta t \frac{dy}{dt} + \frac{\Delta t^2}{2!}\frac{d^2y}{dt^2} + \mathcal{O}(\Delta t^3)$$

$$\Longrightarrow \frac{dy}{dt} = \underbrace{\frac{y(t+\Delta t) - y(t)}{\Delta t}}_{f(y)} + \frac{\Delta t}{2!}\frac{d^2y}{dt^2} + \mathcal{O}(\Delta t^2)$$

$$\frac{dy}{dt} = f(y) \qquad \& \qquad \frac{d^2y}{dt^2} = \frac{df}{dy}\frac{dy}{dt} = \frac{df}{dy} f(y)$$

$$\frac{dy}{dt} = f(y) + \frac{\Delta t}{2!}\frac{df}{dy} f(y)$$

“Original equation”

$$\frac{dy}{dt} = f(y)$$

“Modified equation”

$$\frac{dy}{dt} = f(y) + \frac{\Delta t}{2!}\frac{df}{dy}f(y)$$

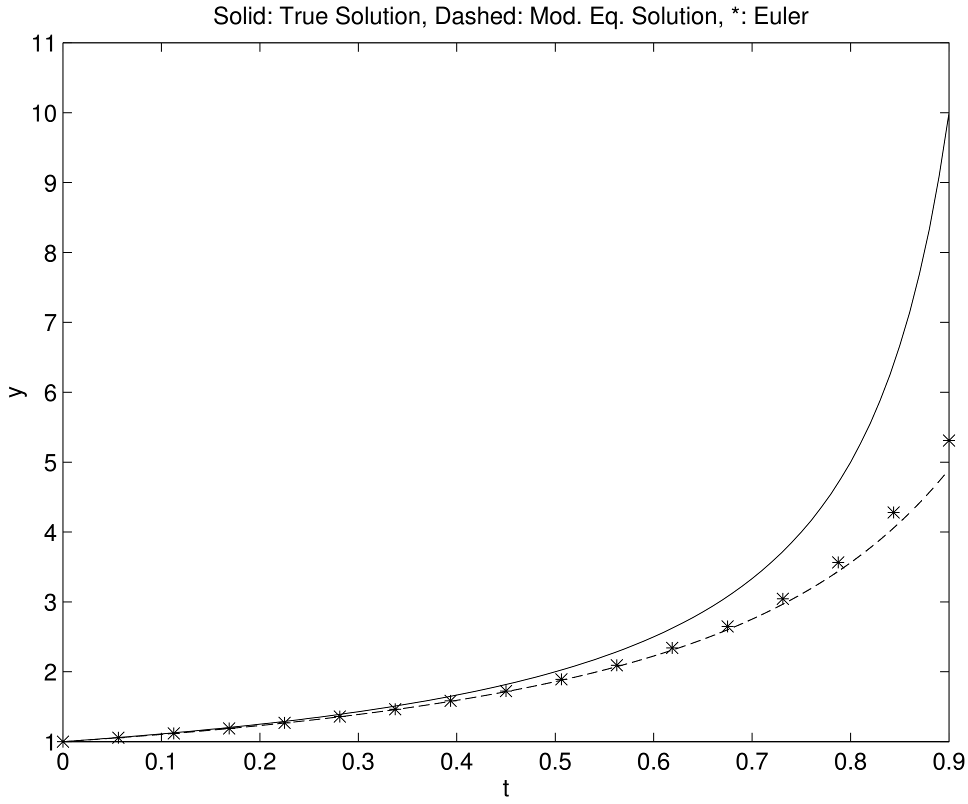
Example

$$\frac{dy}{dt} = y^2 \qquad y(t = 0) = 1$$

“Modified equation”

$$\frac{dy}{dt} = y^2 + \Delta t y^3$$

$$\frac{dy}{dt} = y^2 \qquad y(t = 0) = 1 \qquad t \in < 0; 0.9 > \qquad \Delta t = 0.9/16$$


Solid: True Solution, Dashed: Mod. Eq. Solution, *: Euler
y
t
1
2
3
4
5
6
7
8
9
10
11
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

3.3.2. Derivation of “Modified equation”

The advection equation

$$u_t + au_x = 0 \qquad a > 0$$

Up-wind scheme

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\left[\frac{u_i^n - u_{i-1}^n}{\Delta x}\right] = 0$$

Continuous interpolant $u(x_i, t_n) = u_i^n$

$$\frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{\Delta t} + a\left[\frac{u(x_i, t_n) - u(x_{i-1}, t_n)}{\Delta x}\right] = 0$$

Taylor expansions

$$
\begin{aligned}
u(x_i, t_{n+1}) &= u(x_i, t_n) + \Delta t u_t(x_i, t_n) + \frac{\Delta t^2}{2} u_{tt}(x_i, t_n) \\
&+ \frac{\Delta t^3}{6} u_{ttt}(x_i, t_n) + \mathcal{O}(\Delta t^4)
\end{aligned}
$$

$$
\begin{aligned}
u(x_{i-1}, t_n) &= u(x_i, t_n) - \Delta x u_x(x_i, t_n) + \frac{\Delta x^2}{2} u_{xx}(x_i, t_n) \\
&- \frac{\Delta x^3}{6} u_{xxx}(x_i, t_n) + \mathcal{O}(\Delta x^4)
\end{aligned}
$$

$$\frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{\Delta t} = u_t(x_i, t_n) + \frac{\Delta t}{2} u_{tt}(x_i, t_n) + \frac{\Delta t^2}{6} u_{ttt}(x_i, t_n) + \mathcal{O}(\Delta t^3)$$

$$\frac{u(x_i, t_n) - u(x_{i-1}, t_n)}{\Delta x} = u_x(x_i, t_n) - \frac{\Delta x}{2} u_{xx}(x_i, t_n) + \frac{\Delta x^2}{6} u_{xxx}(x_i, t_n) + \mathcal{O}(\Delta x^3)$$

$$\frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{\Delta t} + a \left[\frac{u(x_i, t_n) - u(x_{i-1}, t_n)}{\Delta x} \right] =$$

$$= \quad u_t(x_i, t_n) + a u_x(x_i, t_n) + \ldots\ldots\ldots\ldots = 0$$

$$u_t(x_i,t_n) + a u_x(x_i,t_n) = -\frac{\Delta t}{2} u_{tt}(x_i,t_n) - \frac{\Delta t^2}{6} u_{ttt}(x_i,t_n) + \mathcal{O}(\Delta t^3)$$

$$+ \frac{a\Delta x}{2} u_{xx}(x_i,t_n) - \frac{a\Delta x^2}{6} u_{xxx}(x_i,t_n) + \mathcal{O}(\Delta x^3)$$

$$u_t + a u_x = 0 \implies \frac{\partial\cdot}{\partial t} = -a\frac{\partial\cdot}{\partial x}$$

$$u_{tt} = a^2 u_{xx} \qquad \& \qquad u_{ttt} = -a^3 u_{xxx}$$

$$u_t(x_i,t_n) + a u_x(x_i,t_n) = -\frac{a^2\Delta t}{2} u_{xx}(x_i,t_n) + \frac{a^3\Delta t^2}{6} u_{xxx}(x_i,t_n) + \mathcal{O}(\Delta t^3)$$

$$+ \frac{a\Delta x}{2} u_{xx}(x_i,t_n) - \frac{a\Delta x^2}{6} u_{xxx}(x_i,t_n) + \mathcal{O}(\Delta x^3)$$

$$u_t(x_i,t_n)+au_x(x_i,t_n) = \left(\frac{a\Delta x}{2} - \frac{a^2\Delta t}{2}\right) u_{xx}(x_i,t_n)+\mathcal{O}(\Delta t^2;\Delta x^2)$$

$$u_t(x_i,t_n) + au_x(x_i,t_n) \doteq \frac{a\Delta x}{2}(1-\gamma)u_{xx}(x_i,t_n)$$

$$\boldsymbol{u_t + au_x = 0 \quad \xrightarrow{Up-wind} \quad u_t + au_x = \frac{a\Delta x}{2}(1-\gamma)u_{xx}}$$

1^{st} order approximation of Original equation

$$u_t + au_x = 0 + \mathcal{O}(\Delta t;\Delta x)$$

2^{nd} order approximation of Modified equation

$$u_t + au_x = \frac{a\Delta x}{2}(1-\gamma)u_{xx} + \mathcal{O}(\Delta t^2;\Delta x^2)$$

3.3.3. Examples of modified equations

Scheme	Modified equation
Up-wind	$u_t + au_x = (1-\gamma)\frac{a\Delta x}{2}u_{xx}$
Down-wind	$u_t + au_x = -(1+\gamma)\frac{a\Delta x}{2}u_{xx}$
Central	$u_t + au_x = -\gamma\frac{a\Delta x}{2}u_{xx}$
Lax-Friedrichs	$u_t + au_x = (\frac{1}{\gamma}-\gamma)\frac{a\Delta x}{2}u_{xx}$
Lax-Wendroff	$u_t + au_x = -\frac{a\Delta x^2}{6}(1-\gamma^2)u_{xxx}$
Beam-Warming	$u_t + au_x = \frac{a\Delta x^2}{6}(2-3\gamma+\gamma^2)u_{xxx}$
Wendroff	$u_t + au_x = -\frac{a\Delta x^2}{12}(2+3\gamma+\gamma^2)u_{xxx}$
Crank-Nicolson	$u_t + au_x = -\frac{a\Delta x^2}{12}(2+\gamma^2)u_{xxx}$

3.3.4. Generalisation of modified equation concept

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} + \; a \; \left[\alpha_1\left(\frac{u_{i+1}^n-u_i^n}{\Delta x}\right)+\alpha_2\left(\frac{u_i^n-u_{i-1}^n}{\Delta x}\right)\right]$$

$$+ \; a \; \left[\alpha_3\left(\frac{u_{i+1}^{n+1}-u_i^{n+1}}{\Delta x}\right)+\alpha_4\left(\frac{u_i^{n+1}-u_{i-1}^{n+1}}{\Delta x}\right)\right] = 0$$

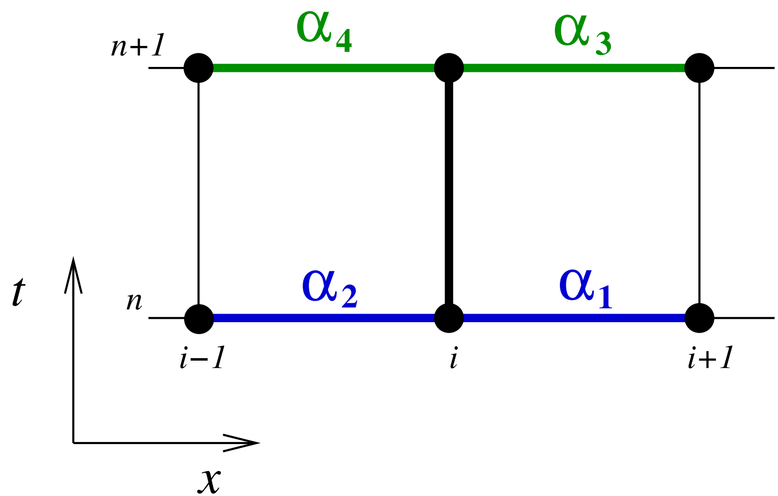
Consistency $\Longrightarrow$ $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 1$

Explicit schemes

$|\alpha_3| + |\alpha_4| = 0$

Implicit schemes

$|\alpha_3| + |\alpha_4| \neq 0$



Scheme	α_1	α_2	α_3	α_4
U	0	1	0	0
D	1	0	0	0
C	$\frac{1}{2}$	$\frac{1}{2}$	0	0
LF	$\frac{1}{2} - \frac{\Delta x}{2a\Delta t}$	$\frac{1}{2} + \frac{\Delta x}{2a\Delta t}$	0	0
LW	$\frac{1}{2} - \frac{a\Delta t}{2\Delta x}$	$\frac{1}{2} + \frac{a\Delta t}{2\Delta x}$	0	0
W	$\frac{1}{2}$	0	0	$\frac{1}{2}$
CN	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Modified equation - up to 3^{rd} order terms

$$u_t + au_x = \epsilon_2 u_{xx} + \epsilon_3 u_{xxx}$$

$$\epsilon_2 = -\frac{a\Delta x}{2}\Big\{(\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_4) + \gamma\left[(\alpha_1 + \alpha_2)^2 - (\alpha_3 + \alpha_4)^2\right]\Big\}$$

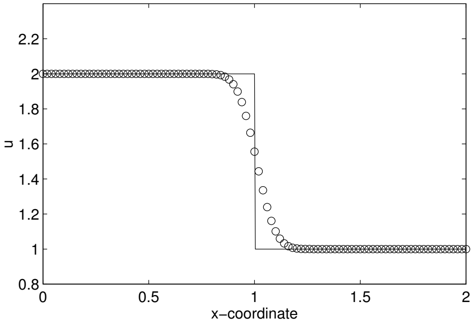
$$\epsilon_3 = -\frac{a\Delta x^2}{6}\Big\{1 + 3\gamma\left[(\alpha_1^2 - \alpha_2^2) - (\alpha_3^2 - \alpha_4^2)\right] + 2\gamma^2\left[(\alpha_1 + \alpha_2)^3 + (\alpha_3 + \alpha_4)^3\right]\Big\}$$

$$u_t + a u_x = \epsilon_2 u_{xx} + \epsilon_3 u_{xxx}$$

$$= \tilde{\epsilon}_2 \frac{a\Delta x}{2} u_{xx} + \tilde{\epsilon}_3 \frac{a\Delta x^2}{6} u_{xxx}$$

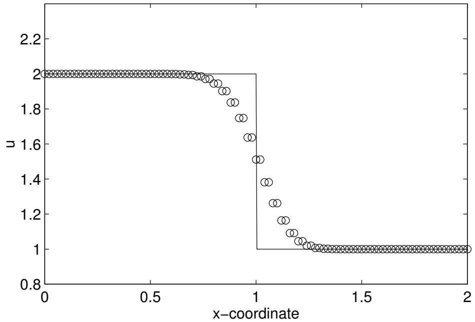
Scheme	$\tilde{\epsilon}_2$	$\tilde{\epsilon}_3$	—	—
U	$1-\gamma$	$\cdots$	$\epsilon_2 > 0$	diffusive
D	$-1-\gamma$	$\cdots$	$\epsilon_2 < 0$	unstable
C	$-\gamma$	$\cdots$	$\epsilon_2 < 0$	unstable
LF	$\frac{1}{\gamma}-\gamma$	$\cdots$	$\epsilon_2 > 0$	diffusive
LW	0	$-(1-\gamma^2)$	$\epsilon_3 \neq 0$	dispersive
W	0	$-\frac{1}{2}(2+3\gamma+\gamma^2)$	$\epsilon_3 \neq 0$	dispersive
CN	0	$-\frac{1}{2}(2+\gamma^2)$	$\epsilon_3 \neq 0$	dispersive

Up-wind



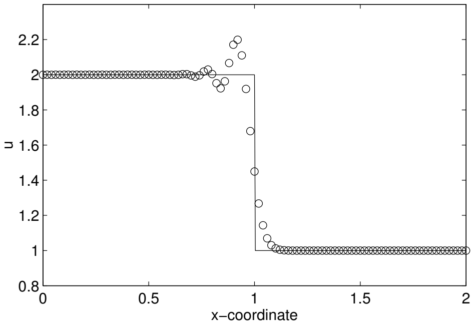
$$u_t + au_x = (1-\gamma)\frac{a\Delta x}{2}u_{xx}$$

Lax-Friedrichs



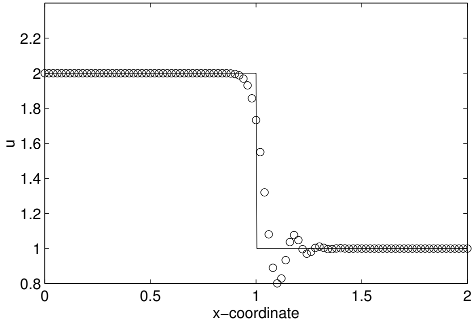
$$u_t + au_x = (\frac{1}{\gamma}-\gamma)\frac{a\Delta x}{2}u_{xx}$$

Lax-Wendroff



$$u_t + au_x = -(1-\gamma^2)\frac{a\Delta x^2}{6}u_{xxx}$$

Beam-Warming



$$u_t + au_x = (2-3\gamma+\gamma^2)\frac{a\Delta x^2}{6}u_{xxx}$$

3.4. Numerical schemes analysis - Summary

- Numerical solution of the advection equation is “much closer” to the solution of advection-diffusion or advection-dispersion equation
- The behaviour and quality of numerical solution is essentially dependent on the coefficients of modified equation
- The detailed knowledge of the structure of the leading order terms in the modified equation can be used to construct the “high resolution” numerical methods

Chapter 4

Introduction to High Resolution Methods

1. Improved classical schemes
2. "Blended schemes"
3. "Composite schemes"
4. Artificial viscosity methods

4.1. Improved classical schemes

Lax-Friedrichs

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$$

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{1}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

Modified Lax-Friedrichs

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{\epsilon}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \qquad \epsilon \in (0;1)$$

Modification of the internal numerical viscosity of the scheme

4.2. Blended schemes

Central & Upwind

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\left[\mathbf{2\alpha}\{\mathsf{C}\} + \mathbf{(1 - 2\alpha)}\{\mathsf{U}\}\right] = 0 \qquad \boldsymbol{\alpha \in (0;1)}$$

Central - low diffusion & Upwind - high diffusion
$\Longrightarrow$ Blended - "optimal" diffusion

- constant blending coefficient
- variable blending coefficient

  Dependent on the local solution behaviour (e.g. on the solution gradient)

4.3. Composite schemes

1. Advance m steps by the higher order, dispersive method (e.g. Lax-Wendroff)
2. Advance n steps by the low order, diffusive method (e.g. Lax-Friedrichs)

Lax-Wendroff - solving & Lax-Friedrichs - smoothing

Example

$m = 3$, $n = 1$

Step	Scheme
1.	LW
2.	LW
3.	LW
4.	LF
5.	LW
6.	LW
7.	LW
8.	LF

4.4. Artificial viscosity methods

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\{\mathbf{C}\} = (1 - 2\alpha)\frac{a\Delta x}{2}\{\mathbf{V}\}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right] = (1 - 2\alpha)\frac{a\Delta x}{2}\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}\right]$$

Generalisation - Artificial viscosity

$$u^{n+1} = \mathbb{L}u^n + \mathbf{D}u^n$$

$\mathbb{L}$ ... discrete evolution operator

$\mathbf{D}$ ... artificial diffusion operator

Example - Central scheme

$$u_i^{n+1} = \underbrace{u_i^n - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)}_{\mathbb{L}u^n} + \mathsf{D}u_i^n$$

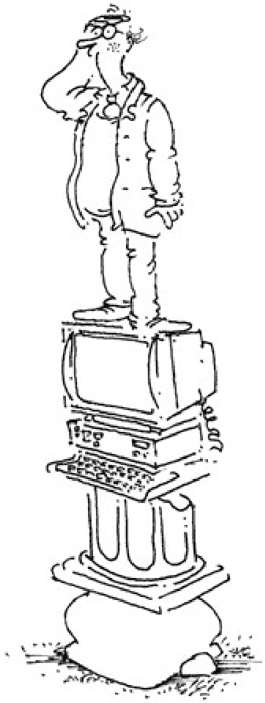
How to design the numerical viscosity?

$$\mathsf{D}u^n = \mathsf{D}_2 u^n + \mathsf{D}_4 u^n$$

$$\begin{aligned} \mathsf{D}_2 u^n &= \epsilon_2 \Delta x^2 u_{xx} \approx \epsilon_2 (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \\ \mathsf{D}_4 u^n &= \epsilon_4 \Delta x^4 u_{xxxx} \approx \epsilon_4 (u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n) \end{aligned}$$

Chapter 5

Viscosity - Overview



Viscosity role in CFD models

Viscosity:

1. **Physical**

   - coming from the mathematical description of the fluid flow

   (a) **Laminar** - basic property of the fluid. Usualy constant or temperature dependent

   (b) **Turbulent** - depends on the local flow field and is given by turbulence model

2. **Numerical**

   - introduced as a consequence of numerical method used for the solution

   (a) **Internal** - implicitly involved in the numerical discretisation

   (b) **External** - explicitly added into the numerical solution to improve the stability and accuracy of the numerical method

Part II

Application of CFD Models in ABL Simulations

Chapter 6

Definition of the Problem

6.1. Geometry of the domain

6.1.1. Wind-tunnel scale tests

3D case

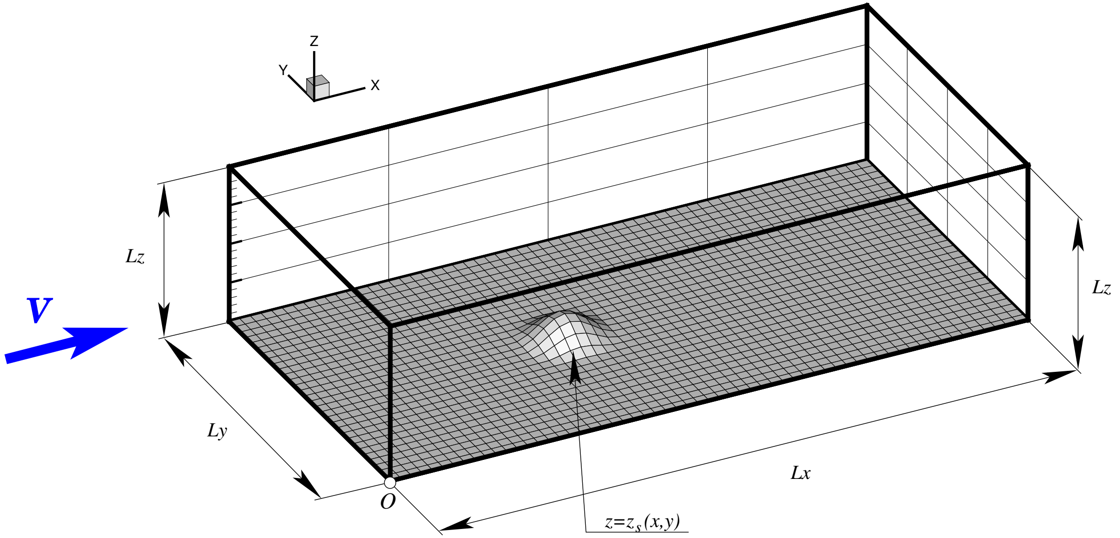


Figure 6.1: Three-dimensional domain with sinusoidal hill

2D case

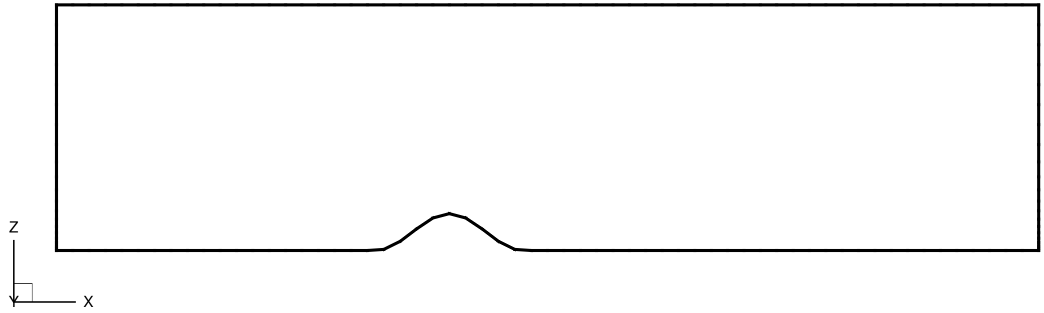


Figure 6.2: Two-dimensional domain with sinusoidal hill

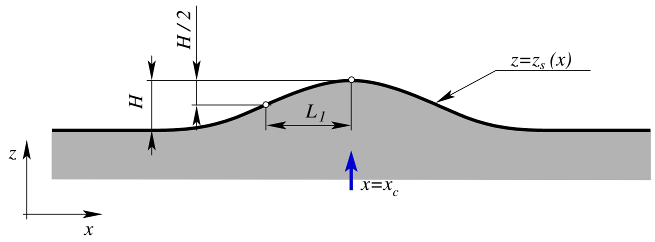


Figure 6.3: Hill geometry for 2D problems

Hill	slope	height H	length L_1
S3H4	0.3	4 cm	6.67 cm
S3H7	0.3	7 cm	11.67 cm
S5H4	0.5	4 cm	4.0 cm
S5H7	0.5	7 cm	7.0 cm

Table 6.1: 2D hill setup

6.1.2. Real scale tests

Real terrain orography

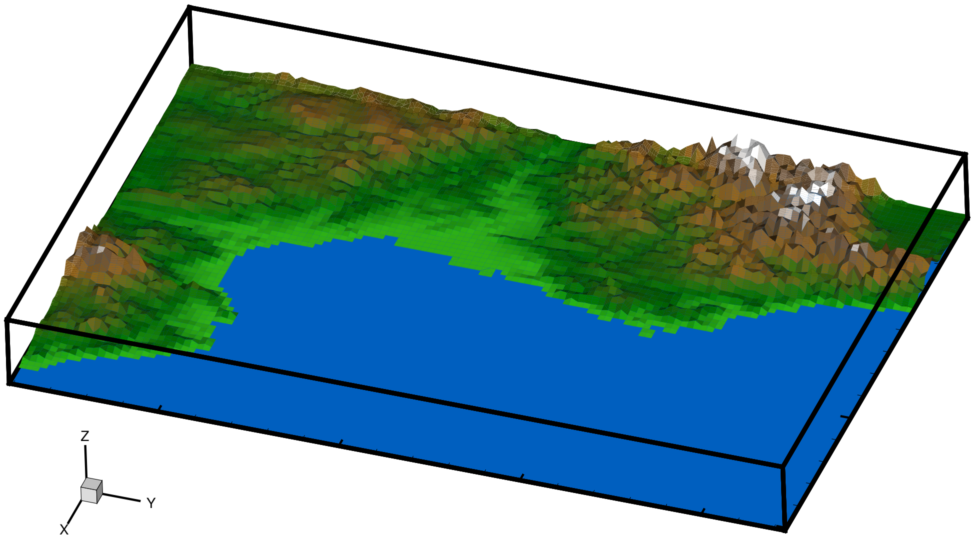
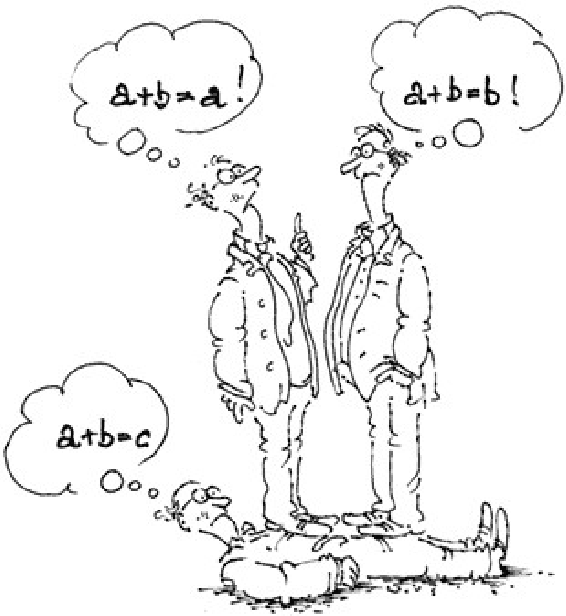


Figure 6.4: 3D domain with real terrain topography (Mediterranean coast)

Chapter 7

Governing equations



7.1. Reynolds averaged Navier-Stokes equations

$$\begin{aligned} u_x + v_y + w_z &= 0 \\ u_t + (u^2+p)_x + (uv)_y + (uw)_z &= [Ku_x]_x + [Ku_y]_y + [Ku_z]_z + f_c v \\ v_t + (uv)_x + (v^2+p)_y + (vw)_z &= [Kv_x]_x + [Kv_y]_y + [Kv_z]_z - f_c u \\ w_t + (uw)_x + (vw)_y + (w^2+p)_z &= [Kw_x]_x + [Kw_y]_y + [Kw_z]_z \end{aligned}$$

In vector form:

$$\tilde{R}\mathbf{W}_t + \mathbf{F}_x + \mathbf{G}_y + \mathbf{H}_z = \mathbf{R}_x + \mathbf{S}_y + \mathbf{T}_z + \mathbf{f}_{\mathbf{W}}$$

Where $\mathsf{W} = (0, u, v, w)^T$, $\mathsf{f}_{\mathsf{W}} = (0, f_c v, -f_c u, 0)^T$ and $\tilde{R} = diag(0, 1, 1, 1)$

$$\begin{array}{ll} \mathsf{F} = (u, u^2+p, uv, uw)^T & \mathsf{R} = (0, Ku_x, Kv_x, Kw_x)^T \\ \mathsf{G} = (v, uv, v^2+p, vw)^T & \mathsf{S} = (0, Ku_y, Kv_y, Kw_y)^T \\ \mathsf{H} = (w, uw, vw, w^2+p)^T & \mathsf{T} = (0, Ku_z, Kv_z, Kw_z)^T \end{array}$$

7.2. RANS - nonconservative form

$$u_x + v_y + w_z = 0$$

$$
\begin{aligned}
u_t + uu_x + vu_y + wu_z &= -\frac{p_x}{\rho} + \{[Ku_x]_x + [Ku_y]_y + [Ku_z]_z\} + f_c v \\
v_t + uv_x + vv_y + wv_z &= -\frac{p_y}{\rho} + \{[Kv_x]_x + [Kv_y]_y + [Kv_z]_z\} - f_c u \\
w_t + uw_x + vw_y + ww_z &= -\frac{p_z}{\rho} + \{[Kw_x]_x + [Kw_y]_y + [Kw_z]_z\}
\end{aligned}
$$

In vector form:

$$\boldsymbol{u_x + v_y + w_z = 0}$$

$$\mathbf{V}_t + u\mathbf{V}_x + v\mathbf{V}_y + w\mathbf{V}_z = -\frac{\nabla p}{\rho} + \{[K\mathbf{V}_x]_x + [K\mathbf{V}_y]_y + [K\mathbf{V}_z]_z\} + \mathbf{f}_{\mathbf{V}}$$

Here $\mathrm{V} = (u, v, w)^T$, $\mathrm{f}_{\mathrm{V}} = (f_c v, -f_c u, 0)^T$.

7.3. Boussinesq approximation

$$(\rho_0 u)_x + (\rho_0 v)_y + (\rho_0 w)_z = 0$$

$$u_t + uu_x + vu_y + wu_z = -\frac{p''_x}{\rho_0} + \frac{1}{\rho_0}\{[\rho_0 K u_x]_x + [\rho_0 K u_y]_y + [\rho_0 K u_z]_z\} + f_c v$$

$$v_t + uv_x + vv_y + wv_z = -\frac{p''_y}{\rho_0} + \frac{1}{\rho_0}\{[\rho_0 K v_x]_x + [\rho_0 K v_y]_y + [\rho_0 K v_z]_z\} - f_c u$$

$$w_t + uw_x + vw_y + ww_z = -\frac{p''_z}{\rho_0} + \frac{1}{\rho_0}\{[\rho_0 K w_x]_x + [\rho_0 K w_y]_y + [\rho_0 K w_z]_z\} - \frac{\Theta''}{\Theta_0} g$$

In vector form:

$$(\rho_0 u)_x + (\rho_0 v)_y + (\rho_0 w)_z = 0$$

$$\mathbf{V}_t + u\mathbf{V}_x + v\mathbf{V}_y + w\mathbf{V}_z = -\frac{\nabla p''}{\rho_0} + \frac{1}{\rho_0}\{[\rho_0 K \mathbf{V}_x]_x + [\rho_0 K \mathbf{V}_y]_y + [\rho_0 K \mathbf{V}_z]_z\} + \mathbf{f}_\mathrm{v}$$

Here $\mathrm{V} = (u, v, w)^T$, $\mathrm{f}_\mathrm{v} = (f_c v, -f_c u, \frac{\Theta''}{\Theta_0} g)^T$.

7.4. Transport equations

$$
\begin{aligned}
C^i_t + uC^i_x + vC^i_y + wC^i_z &= \left[K\frac{C^i_x}{\sigma_{C^i}}\right]_x + \left[K\frac{C^i_y}{\sigma_{C^i}}\right]_y + \left[K\frac{C^i_z}{\sigma_{C^i}}\right]_z \\
\Theta_t + u\Theta_x + v\Theta_y + w\Theta_z &= \left[K\frac{\Theta_x}{\sigma_\Theta}\right]_x + \left[K\frac{\Theta_y}{\sigma_\Theta}\right]_y + \left[K\frac{\Theta_z}{\sigma_\Theta}\right]_z
\end{aligned}
$$

7.5. Turbulence modelling

Algebraic turbulent closure $K = \nu_L + \nu_T$

Turbulent viscosity . . . $\nu_T = \ell^2 \left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right]^{1/2} \mathcal{G}$

Stability function . . .

$$\mathcal{G} = (1 + \beta\, Ri)^{-2} \qquad \text{for} \quad Ri > 0$$

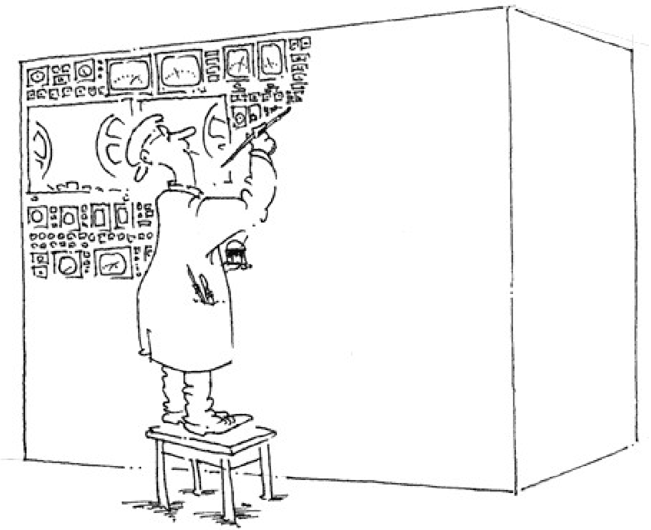
$$\mathcal{G} = (1 - \beta\, Ri)^{2} \qquad \text{for} \quad Ri \le 0$$

Mixing length . . . $\ell = \frac{\kappa(z+z_0)}{1+\kappa\frac{(z+z_0)}{\ell_\infty}}$ $\qquad \ell_\infty = \frac{27\, \|\boldsymbol{v}_g\|\, 10^{-5}}{\lambda}$

Type of surface	z_0 $[m]$
snow, ice	10^{-5}
flat hayfield, grass of height 1 cm	10^{-3}
grass of height 10 cm	10^{-2}
grass of height 50 cm	$7 \cdot 10^{-2}$
city estate	$1/10 \div 1/20$ average height of buildings

Chapter 8

Numerical Solution



8.1. Artificial compressibility formulation

$$\frac{p_t}{\beta^2} + u_x + v_y + w_z = 0$$

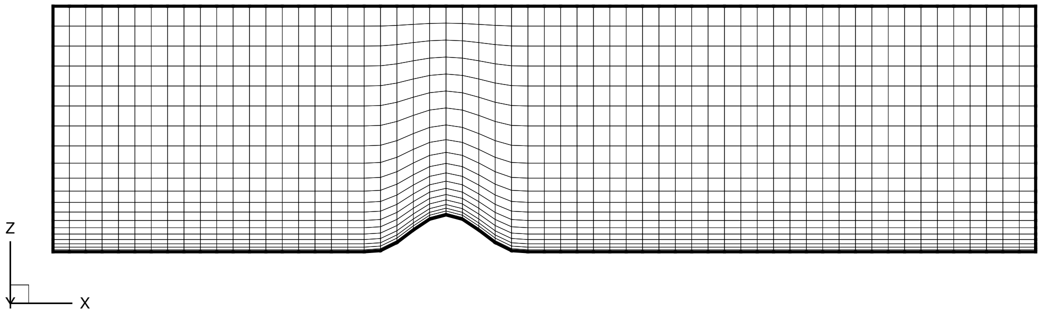
$$\mathbf{V}_t + u\mathbf{V}_x + v\mathbf{V}_y + w\mathbf{V}_z = -\frac{\nabla p'}{\rho_0} + [K\mathbf{V}_x]_x + [K\mathbf{V}_y]_y + [K\mathbf{V}_z]_z + \vec{f}$$

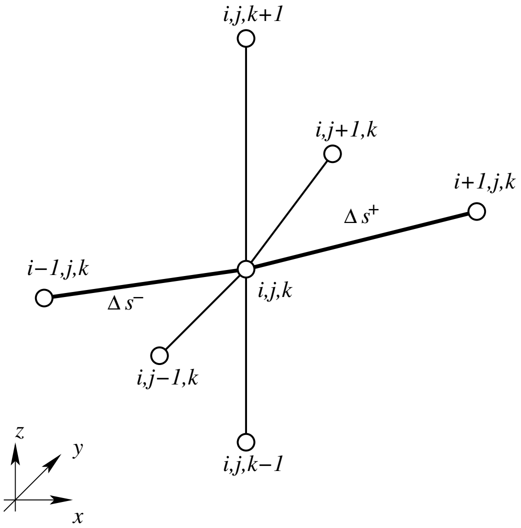
$$C^i_t + uC^i_x + vC^i_y + wC^i_z = \left[K\frac{C^i_x}{\sigma_{C^i}}\right]_x + \left[K\frac{C^i_y}{\sigma_{C^i}}\right]_y + \left[K\frac{C^i_z}{\sigma_{C^i}}\right]_z$$

$$\Theta_t + u\Theta_x + v\Theta_y + w\Theta_z = \left[K\frac{\Theta_x}{\sigma_\Theta}\right]_x + \left[K\frac{\Theta_y}{\sigma_\Theta}\right]_y + \left[K\frac{\Theta_z}{\sigma_\Theta}\right]_z$$

where V $= col(u, v, w)$

8.2. Finite difference discretisation





$$\overleftarrow{\delta_s} = \frac{\mathsf{V}_i - \mathsf{V}_{i-1}}{\Delta s^-}$$

$$\overrightarrow{\delta_s} = \frac{\mathsf{V}_{i+1} - \mathsf{V}_i}{\Delta s^+}$$

$$\delta_s = \frac{1}{2}(\overleftarrow{\delta_s} + \overrightarrow{\delta_s})$$

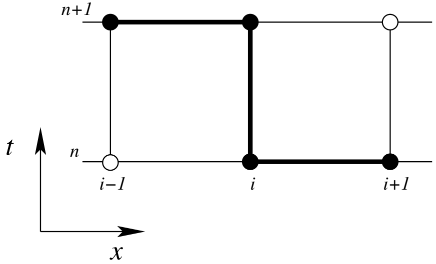
8.3. One-dimensional model schemes

Advection equation:

$$u_t + au_x = 0$$

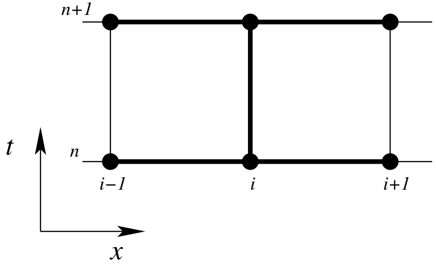
X-scheme - Wendroff

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{a}{2}\left[\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} + \frac{u_{i+1}^n - u_i^n}{\Delta x}\right] = 0$$



Y-scheme - Crank-Nicolson

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{a}{2}\left[\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} + \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right] = 0$$



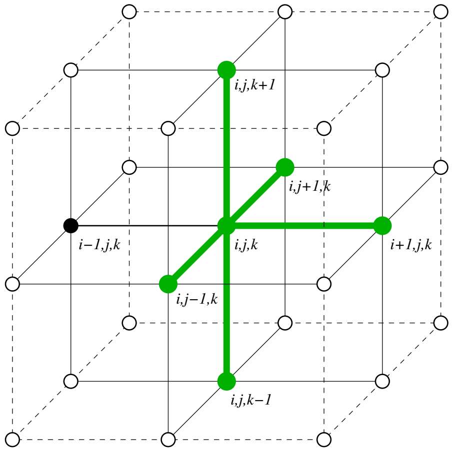
8.4. Semi-implicit scheme

$$
\begin{aligned}
\mathsf{V}_t &\sim \overrightarrow{\delta_t}\mathsf{V}^n_{i,j,k} \\
u\mathsf{V}_x &\sim \frac{1}{2}\left(u^n_{i+1/2}\overrightarrow{\delta_x}\mathsf{V}^n_{i,j,k} + u^n_{i-1/2}\overleftarrow{\delta_x}\mathsf{V}^{n+1}_{i,j,k}\right) \\
v\mathsf{V}_y &\sim \frac{1}{2}\left\{\frac{1}{2}\left(v^n_{j+1/2}\overrightarrow{\delta_y}\mathsf{V}_{i,j,k} + v^n_{j-1/2}\overleftarrow{\delta_y}\mathsf{V}_{i,j,k}\right)^n + \right. \\
&\quad \left. +\frac{1}{2}\left(v^n_{j+1/2}\overrightarrow{\delta_y}\mathsf{V}_{i,j,k} + v^n_{j-1/2}\overleftarrow{\delta_y}\mathsf{V}_{i,j,k}\right)^{n+1}\right\} \\
w\mathsf{V}_z &\sim \frac{1}{2}\left\{\frac{1}{2}\left(w^n_{k+1/2}\overrightarrow{\delta_z}\mathsf{V}_{i,j,k} + w^n_{k-1/2}\overleftarrow{\delta_z}\mathsf{V}_{i,j,k}\right)^n + \right. \\
&\quad \left. +\frac{1}{2}\left(w^n_{k+1/2}\overrightarrow{\delta_z}\mathsf{V}_{i,j,k} + w^n_{k-1/2}\overleftarrow{\delta_z}\mathsf{V}_{i,j,k}\right)^{n+1}\right\}
\end{aligned}
$$

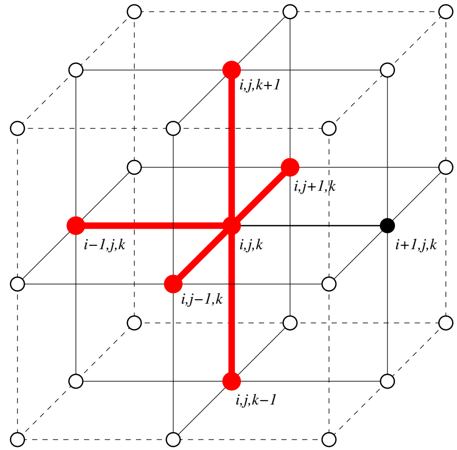
Computational stencil

Semi-implicit scheme

Time level n

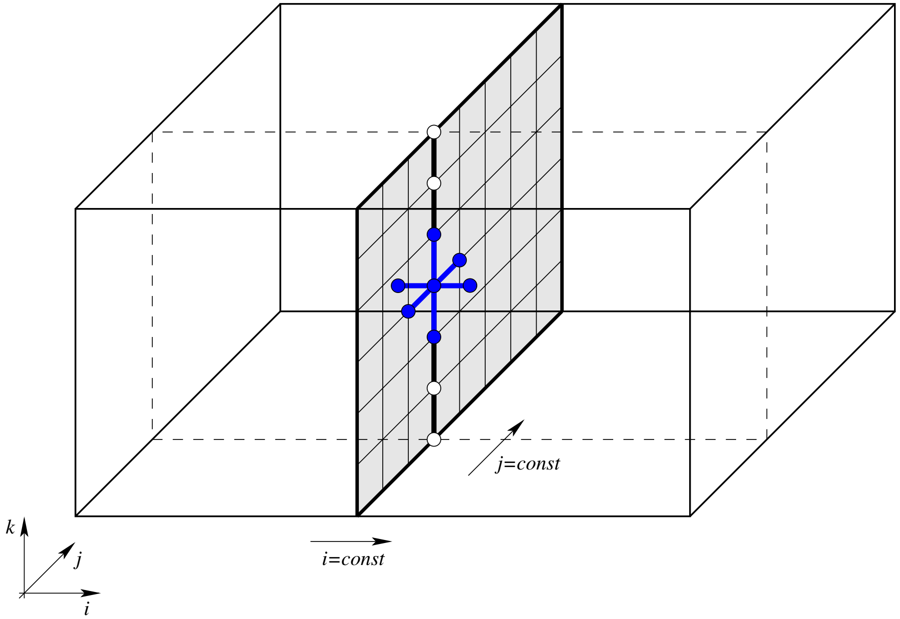


Time level $n+1$



Numerical Solver

Semi-implicit scheme

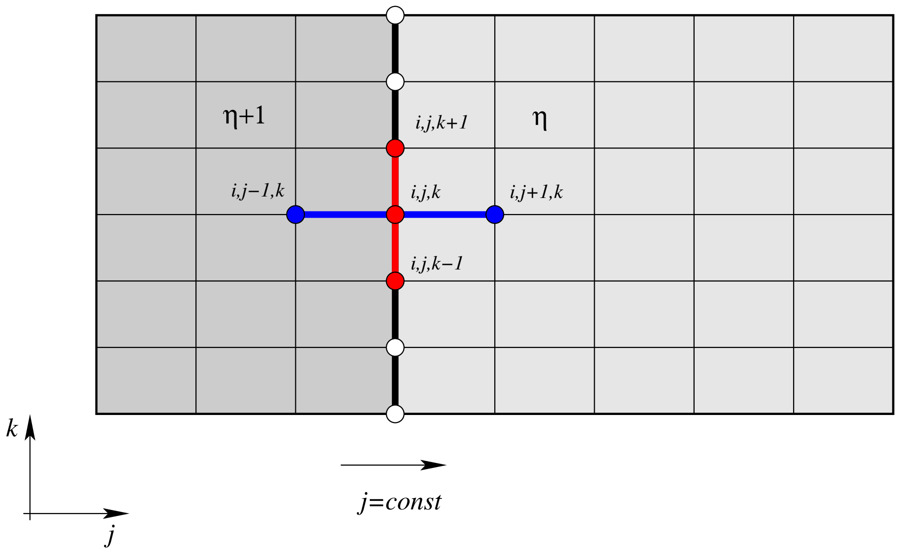


Five-diagonal system of linear equations

$$\color{blue}{a_1 \mathsf{V}^{n+1}_{i,j+1,k}} + a_2 \mathsf{V}^{n+1}_{i,j,k} \color{blue}{+ a_3 \mathsf{V}^{n+1}_{i,j-1,k}} + a_4 \mathsf{V}^{n+1}_{i,j,k+1} + a_5 \mathsf{V}^{n+1}_{i,j,k-1} = RHS$$

Numerical Solver

Semi-implicit scheme



Three-diagonal iterative procedure

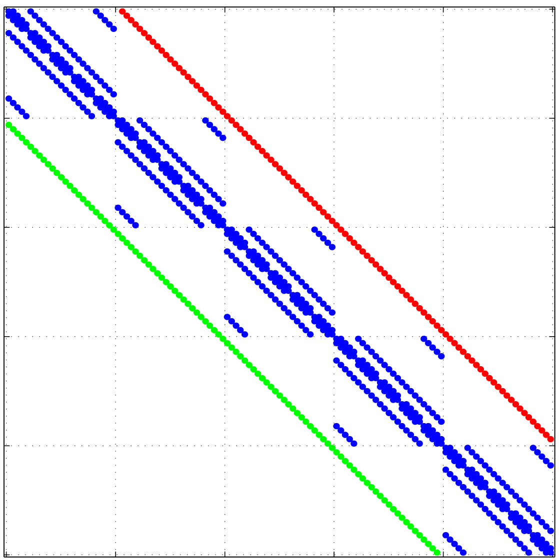
$$a_5 V^{\eta+1}_{i,j,k-1} + a_2 V^{\eta+1}_{i,j,k} + a_4 V^{\eta+1}_{i,j,k+1} = RHS - a_1 V^{\eta}_{i,j+1,k} - a_3 V^{\eta+1}_{i,j-1,k}$$

$$\eta = 1 \ldots m-1 \qquad V^{m}_{i,j,k} \longrightarrow V^{n+1}_{i,j,k}$$

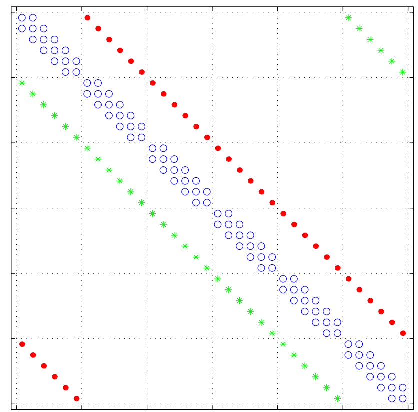
Sparse structure of system matrix

Semi-implicit scheme

Fully implicit matrix



Diagonal block



8.5. Pressure resolution

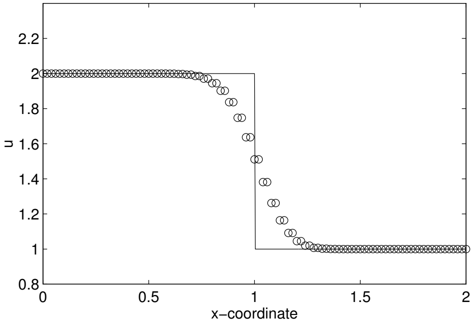
Pressure is updated from the modified continuity equation

$$p'_t = -(u_x + v_y + w_z)$$

The derivatives are discretized by central differences at time level $(n + 1/2)$

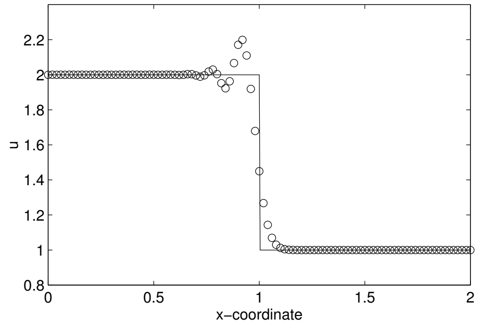
$$\begin{aligned} p_t &\sim \delta_t\, p_{i,j,k}^{n+1/2} \\ u_x &\sim \frac{1}{2}\{\delta_x u_{i,j,k}^{n} + \delta_x u_{i,j,k}^{n+1}\} \\ v_y &\sim \frac{1}{2}\{\delta_y v_{i,j,k}^{n} + \delta_y v_{i,j,k}^{n+1}\} \\ w_z &\sim \frac{1}{2}\{\delta_z w_{i,j,k}^{n} + \delta_z w_{i,j,k}^{n+1}\} \end{aligned}$$

Lax-Friedrichs



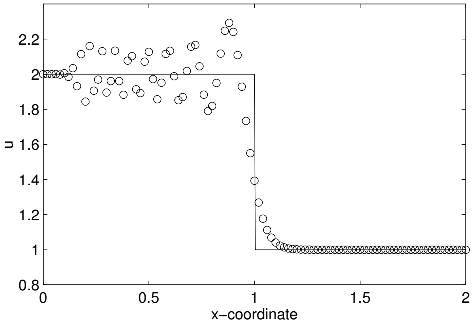
$$u_t + a u_x = \left(\frac{1}{\gamma} - \gamma\right)\frac{a\Delta x}{2} u_{xx}$$

Lax-Wendroff



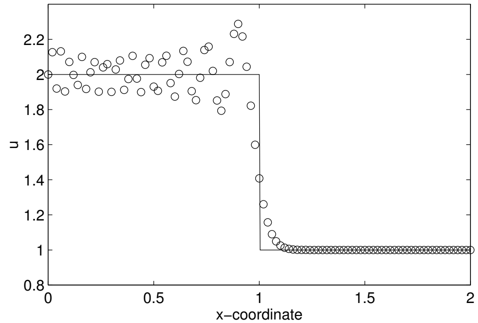
$$u_t + a u_x = -(1-\gamma^2)\frac{a\Delta x^2}{6} u_{xxx}$$

Wendroff



$$u_t + a u_x = -(2+3\gamma+\gamma^2)\frac{a\Delta x^2}{12} u_{xxx}$$

Crank-Nicolson



$$u_t + a u_x = -(2+\gamma^2)\frac{a\Delta x^2}{12} u_{xxx}$$

8.6. Artificial viscosity terms

The non-physical oscillations are dumped by combination of artificial viscosity of second and fourth order.

$$DV_i^n = D^2V_i^n + D^4V_i^n$$

$$\begin{aligned} D^2V_i^n &= \tilde{\epsilon}_2 \Delta x^3 \frac{\partial}{\partial x} |V_x| V_x \\ &= \tilde{\epsilon}_2 \Delta x^2 (\epsilon_{i+1/2} V_x - \epsilon_{i-1/2} V_x) \end{aligned}$$

$$\epsilon_{i+1/2} = \begin{cases} |V_{i+1} - V_i| & \text{for} \quad |V_{i+1} - V_i| < \frac{K}{10} \\ \frac{K}{10} & \text{for} \quad |V_{i+1} - V_k| \geq \frac{K}{10} \end{cases}$$

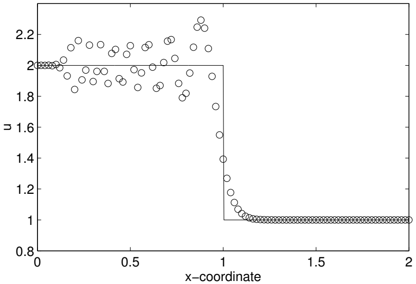
$$\begin{aligned} D^4V_i^n &= \tilde{\epsilon}_4 \Delta x^4 V_{xxxx} \\ &= \tilde{\epsilon}_4 \left(V_{i-2}^n - 4V_{i-1}^n + 6V_i^n - 4V_{i+1}^n + V_{i+2}^n \right) \end{aligned}$$

8.7. One-dimensional test

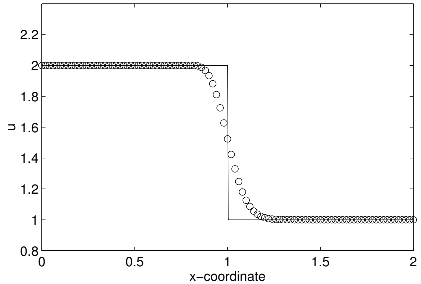
Linear advection equation test

$$u_t + au_x = 0$$

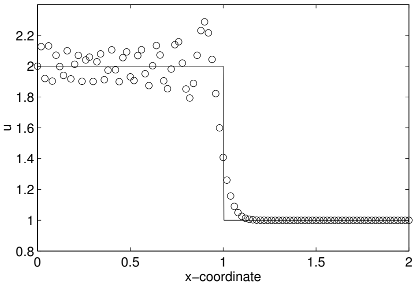
X-scheme without art. viscosity



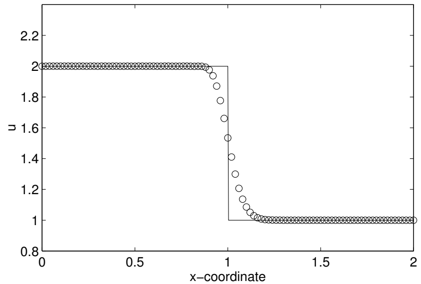
X-scheme with art. viscosity



Y-scheme without art. viscosity



Y-scheme with art. viscosity

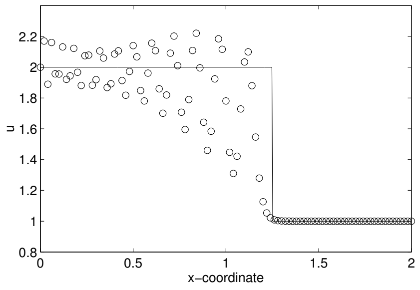


Home Page
Title Page
Contents
◀◀ ▶▶
◀ ▶
Page 75 of 104
Go Back
Full Screen
Close
Quit

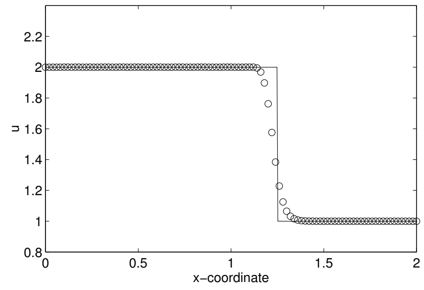
Non-linear Burgers equation test

$$u_t + uu_x = 0 \qquad u_t + \left(\frac{u^2}{2}\right)_x = 0$$

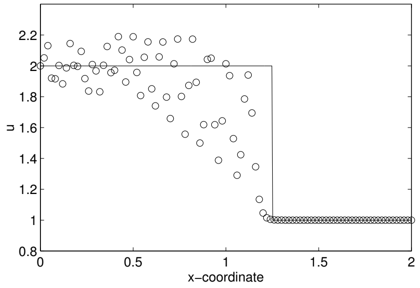
X-scheme without art. viscosity



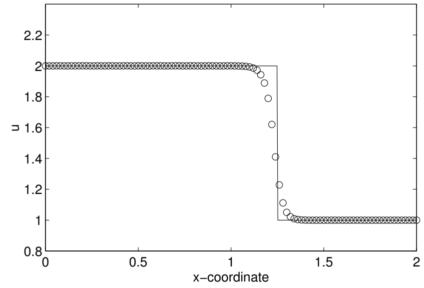
X-scheme with art. viscosity



Y-scheme without art. viscosity



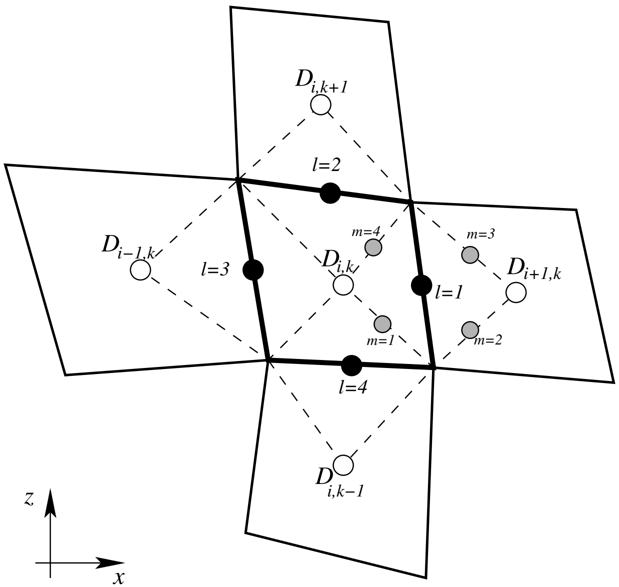
Y-scheme with art. viscosity



Home Page
Title Page
Contents
◀◀ ▶▶
◀ ▶
Page 76 of 104
Go Back
Full Screen
Close
Quit

8.8. Finite-volume scheme for 2D RANS

Structured finite-volume grid



Mac Cormack finite-volume scheme

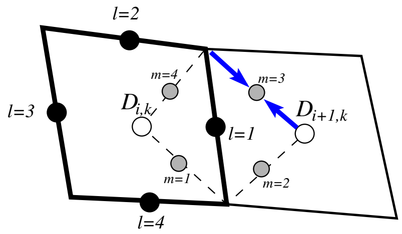
$$\begin{aligned}
\mathsf{W}_{i,k}^{n+\frac{1}{2}} &= \mathsf{W}_{i,k}^{n} - \frac{\Delta t}{|D_{i,k}|}\sum_{l=1}^{4}\{(\mathsf{F}_l^n - \mathsf{R}_l^n)\Delta z_l - (\mathsf{H}_l^n - \mathsf{T}_l^n)\Delta x_l\} \\
\overline{\mathsf{W}}_{i,k}^{n+1} &= \frac{1}{2}\Big[\mathsf{W}_{i,k}^{n} + \mathsf{W}_{i,k}^{n+\frac{1}{2}} \\
&- \frac{\Delta t}{|D_{i,k}|}\sum_{l=1}^{4}\{(\mathsf{F}_l^{n+\frac{1}{2}} - \mathsf{R}_l^{n+\frac{1}{2}})\Delta z_k - (\mathsf{H}_l^{n+\frac{1}{2}} - \mathsf{T}_l^{n+\frac{1}{2}})\Delta x_l\}\Big] \\
\mathsf{W}_{i,k}^{n+1} &= \overline{\mathsf{W}}_{i,k}^{n+1} + \mathsf{DW}_{i,k}^{n}
\end{aligned}$$

Mac Cormack numerical flux

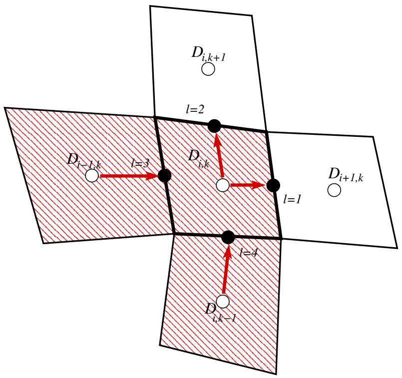
Mac Cormack inviscid fluxes

Predictor	Corrector
$F_1^n = F(W_{i,k}^n)$	$F_1^{n+\frac{1}{2}} = F(W_{i+1,k}^{n+\frac{1}{2}})$
$F_2^n = F(W_{i,k}^n)$	$F_2^{n+\frac{1}{2}} = F(W_{i,k+1}^{n+\frac{1}{2}})$
$F_3^n = F(W_{i-1,k}^n)$	$F_3^{n+\frac{1}{2}} = F(W_{i,k}^{n+\frac{1}{2}})$
$F_4^n = F(W_{i,k-1}^n)$	$F_4^{n+\frac{1}{2}} = F(W_{i,k}^{n+\frac{1}{2}})$

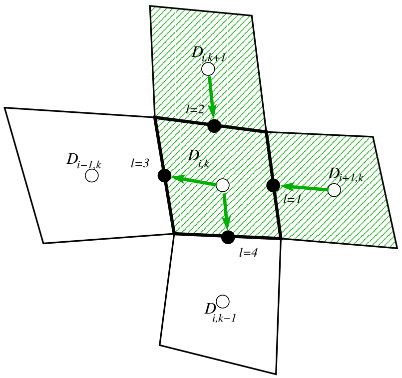
Viscous flux
dual control volume



Mac Cormack numerical flux



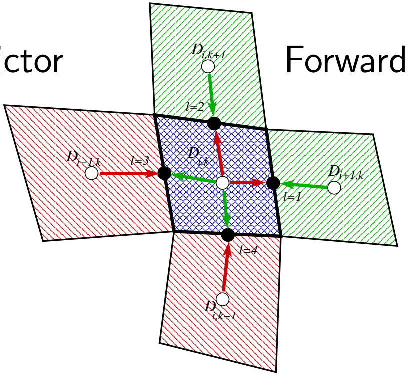
$+$



Backward - Predictor

Forward - Corrector

$=$



Central

Home Page
Title Page
Contents
◀◀ ▶▶
◀ ▶
Page 80 of 104
Go Back
Full Screen
Close
Quit

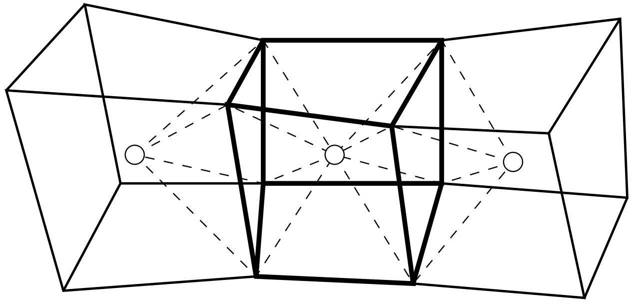
Artificial viscosity

$$\begin{aligned} \mathsf{DW}_k^n &= \tilde{\epsilon}_2 \left[\left| \mathsf{W}_{k+1}^n - \mathsf{W}_k^n \right| \left(\mathsf{W}_{k+1}^n - \mathsf{W}_k^n \right) - \left| \mathsf{W}_k^n - \mathsf{W}_{k-1}^n \right| \left(\mathsf{W}_k^n - \mathsf{W}_{k-1}^n \right) \right] \\ &+ \tilde{\epsilon}_4 \left(\mathsf{W}_{k-2}^n - 4\mathsf{W}_{k-1}^n + 6\mathsf{W}_k^n - 4\mathsf{W}_{k+1}^n + \mathsf{W}_{k+2}^n \right); \tilde{\epsilon}_2, \tilde{\epsilon}_4 \in \mathbb{R} \end{aligned}$$

$$\begin{aligned} \mathsf{DW}_k^n &= \tilde{\epsilon}_2 \left[\left| \frac{p_{k+1}^n - 2p_k^n + p_{k-1}^n}{p_{k+1}^n + 2p_k^n + p_{k-1}^n} \right| \left(\mathsf{W}_{k+1}^n - \mathsf{W}_k^n \right) - \left| \frac{p_k^n - 2p_{k-1}^n + p_{k-2}^n}{p_k^n + 2p_{k-1}^n + p_{k-2}^n} \right| \left(\mathsf{W}_k^n - \mathsf{W}_{k-1}^n \right) \right] \\ &+ \tilde{\epsilon}_4 \left(\mathsf{W}_{k-2}^n - 4\mathsf{W}_{k-1}^n + 6\mathsf{W}_k^n - 4\mathsf{W}_{k+1}^n + \mathsf{W}_{k+2}^n \right); \tilde{\epsilon}_2, \tilde{\epsilon}_4 \in \mathbb{R} \end{aligned}$$

8.9. 3D Finite-volume scheme for RANS

Finite-volume semi-discretization



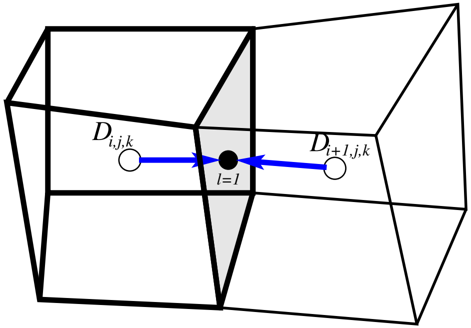
$$\frac{\partial \mathsf{W}_{ijk}}{\partial t} = -\mathcal{L}\,\mathsf{W}_{i,j,k}$$

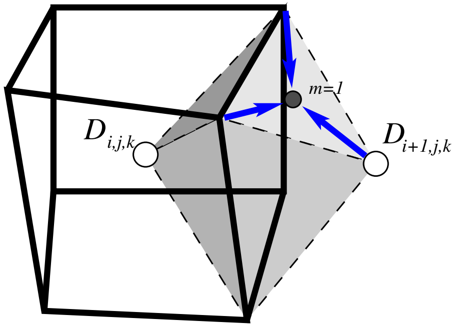
$$\mathcal{L}\mathsf{W}_{i,j,k} = \frac{1}{|D|}\left\{ \underbrace{\oint_{\partial D} [\mathsf{F},\mathsf{G},\mathsf{H}] \cdot \boldsymbol{\nu}\, dS}_{\text{inviscid flux}} - \underbrace{\oint_{\partial D} [\mathsf{R},\mathsf{S},\mathsf{T}] \cdot \boldsymbol{\nu}\, dS}_{\text{viscous flux}} + \underbrace{\int_{D} \tilde{\mathsf{f}}_{\mathsf{W}}\, dV}_{\text{external force}} \right\}$$

Finite-volume semi-discretization

Inviscid flux

Viscous flux





$$\mathsf{F}_1^n = \frac{1}{2}[\mathsf{F}(\mathsf{W}_{i,j,k}^n) + \mathsf{F}(\mathsf{W}_{i+1,j,k}^n)]$$

$$u_x \approx \oint_{\partial \tilde{D}} u\, \nu^x \, dy\, dz \approx \sum_{m=1}^{8} u_m \nu_m^x S_m$$

Runge-Kutta time integration

$$\frac{d\mathsf{W}_{ijk}}{dt} = -\tilde{\mathcal{L}}\,\mathsf{W}_{i,j,k}$$

$$\begin{aligned} \mathsf{W}^{(0)}_{i,j,k} &= \mathsf{W}^{n}_{i,j,k} \\ \mathsf{W}^{(r+1)}_{i,j,k} &= \mathsf{W}^{(0)}_{i,j,k} - \alpha_{(r)}\Delta t\tilde{\mathcal{L}}\mathsf{W}^{(r)}_{i,j,k} \qquad r = 1, \ldots, m \\ \mathsf{W}^{n+1}_{i,j,k} &= \mathsf{W}^{(m)}_{i,j,k} \end{aligned}$$

The three-stage explicit RK scheme has coefficients:

$$\alpha_{(1)} = 1/2,\ \ \alpha_{(2)} = 1/2,\ \ \alpha_{(3)} = 1$$

Artificial viscosity

$$\bar{\tilde{\mathcal{L}}}\mathsf{W}^{(r)}_{i,j,k} = \tilde{\mathcal{L}}\mathsf{W}^{(r)}_{i,j,k} + \mathsf{DW}^{(r)}_{i,j,k}$$

$$\mathsf{DW}^{n}_{i} = \epsilon_2 \Delta x^2\, \mathsf{W}_{xx}\big|^n_i + \epsilon_4 \Delta x^4 \mathsf{W}_{xxxx}\big|^n_i$$

$$\begin{aligned} \mathsf{DW}^n_i &= \tilde{\epsilon}_2(\mathsf{W}^n_{i-1} - 2\mathsf{W}^n_i + \mathsf{W}^n_{i+1}) \\ &+ \tilde{\epsilon}_4\left(\mathsf{W}^n_{i-2} - 4\mathsf{W}^n_{i-1} + 6\mathsf{W}^n_i - 4\mathsf{W}^n_{i+1} + \mathsf{W}^n_{i+2}\right); \qquad \tilde{\epsilon}_2, \tilde{\epsilon}_4 \in \mathbb{R} \end{aligned}$$

Chapter 9

Selected Numerical Results

9.1. Wind-tunnel scale tests

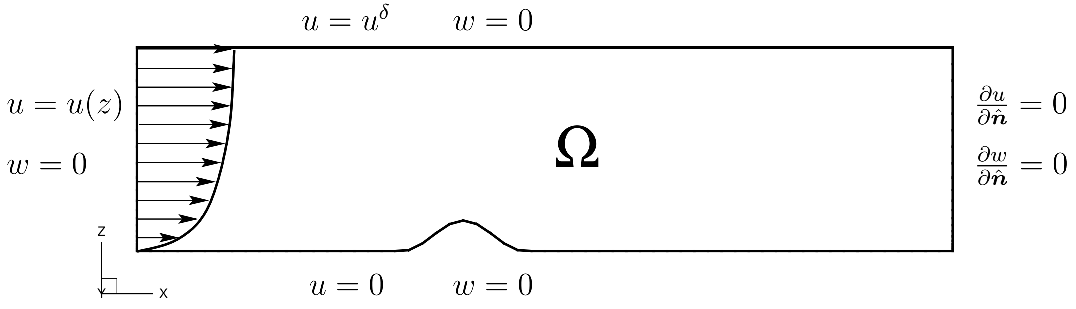


Figure 9.1: Boundary conditions for 2D incompressible Navier-Stokes model

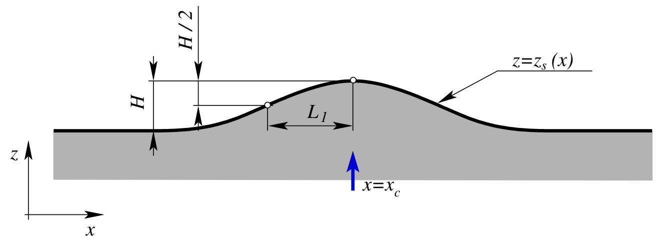
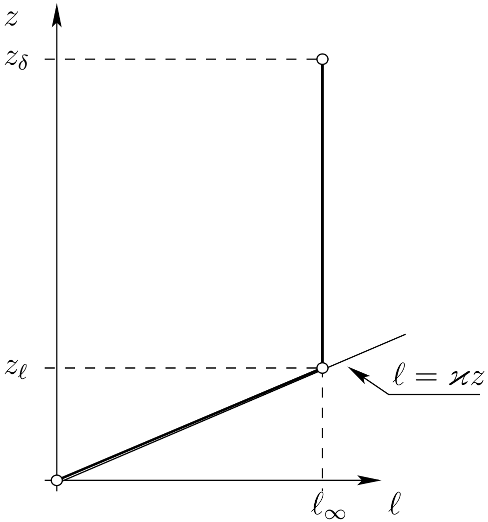


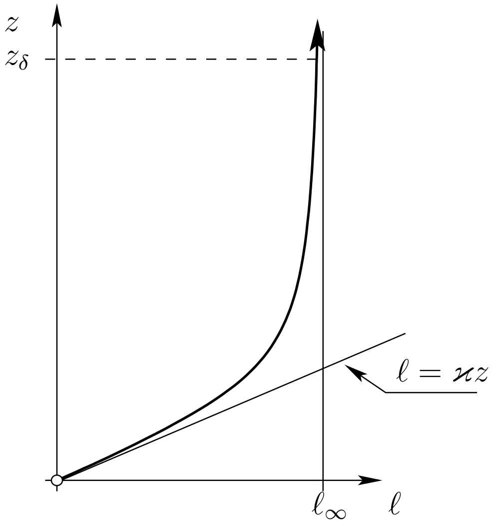
Figure 9.2: Hill geometry for 2D problems

Hill	slope	height H	length L_1
S3H4	0.3	4 cm	6.67 cm
S3H7	0.3	7 cm	11.67 cm
S5H4	0.5	4 cm	4.0 cm
S5H7	0.5	7 cm	7.0 cm

Table 9.1: 2D hill setup

9.2. Turbulence model adjustement





$$\nu_T = \ell^2 \, [(\frac{\partial u}{\partial z})^2 + (\frac{\partial v}{\partial z})^2]^{1/2} \quad \& \quad \frac{1}{\ell} = \frac{1}{\varkappa z} + \frac{1}{\ell_\infty} \qquad \Longrightarrow \qquad \ell = \frac{\varkappa z}{1 + \frac{\varkappa z}{\ell_\infty}}$$

Atmosphere

$$\ell_\infty = \frac{27 \, \|\boldsymbol{v}_{\boldsymbol{G}}\| \, 10^{-5}}{f_c}$$

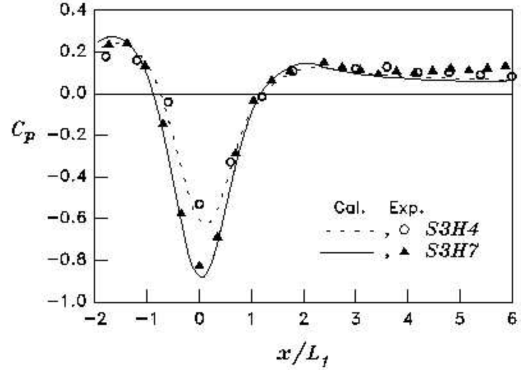
Wind-tunnel

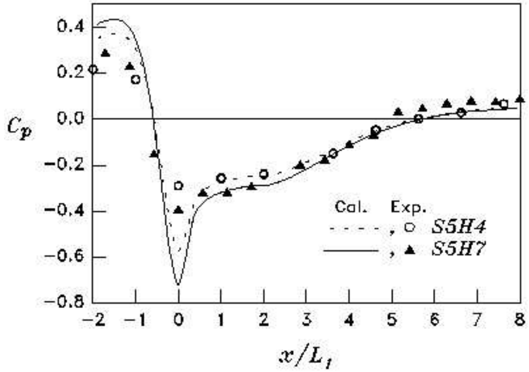
$$\ell_\infty = C_\ell \, l_{in}$$

$$l_{in} \ln(l_{in}/z_0) = 2\varkappa^2 \, L_1$$

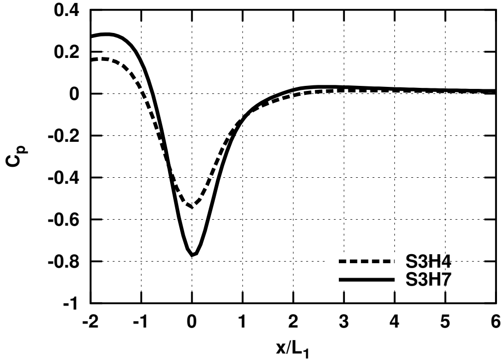
9.3. Validation in 2D - Pressure distribution

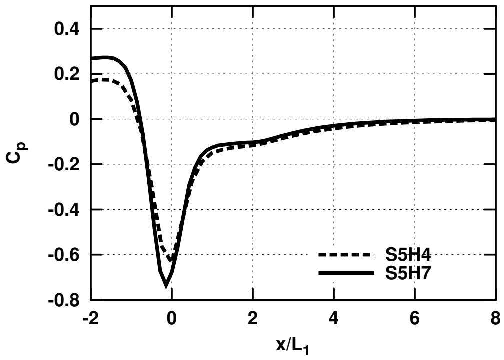
Experiment
H.G.Kim & all. 1997





Computation

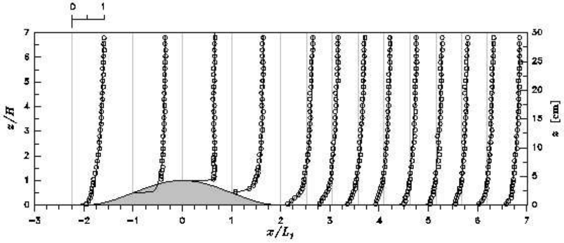


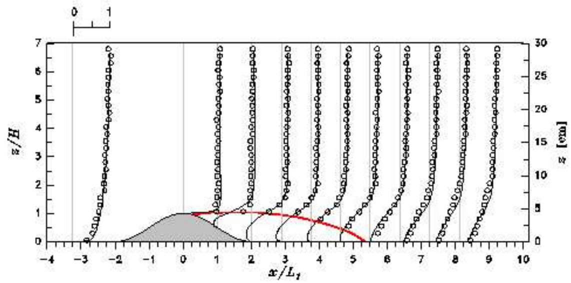


9.4. Validation in 2D - Separated flow

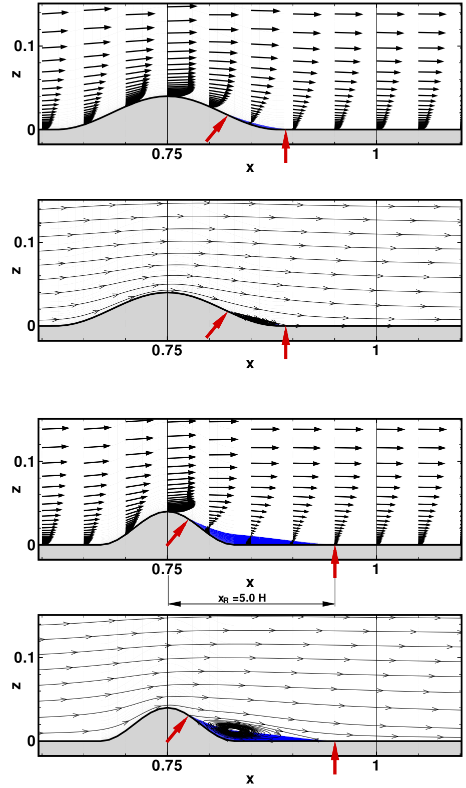
Experiment

H.G.Kim & all. 1997

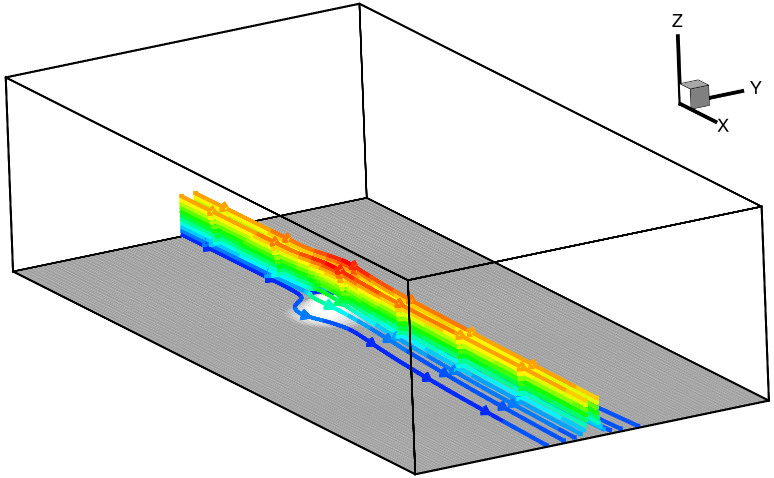


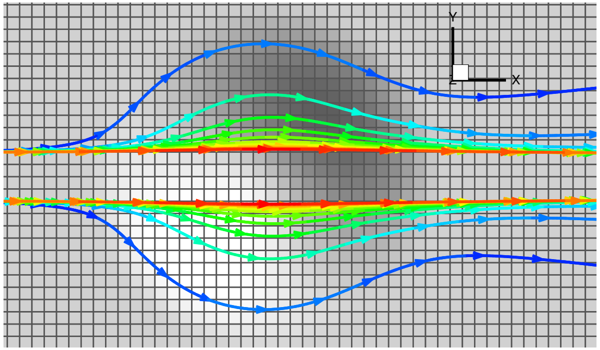


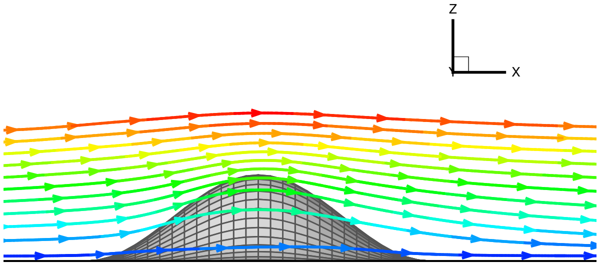
Computation



Home Page
Title Page
Contents
◀◀ ▶▶
◀ ▶
Page 90 of 104
Go Back
Full Screen
Close
Quit

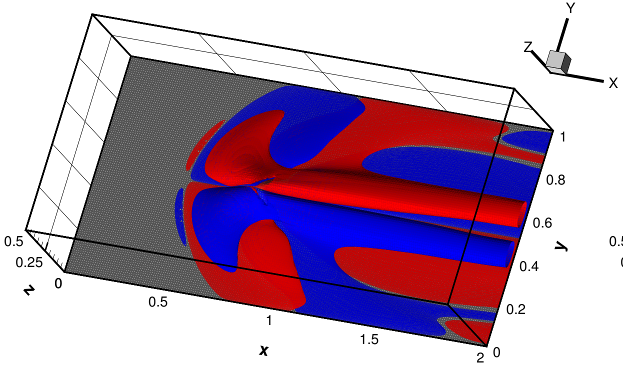


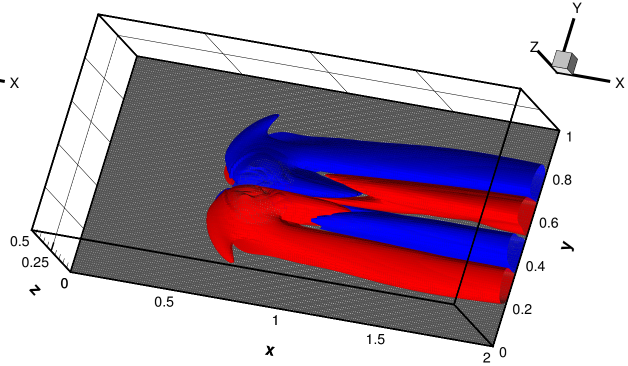


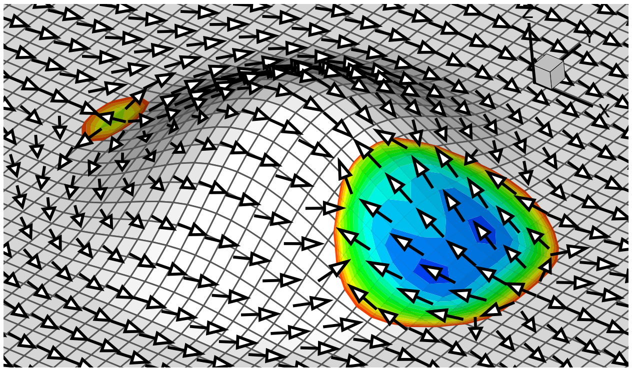


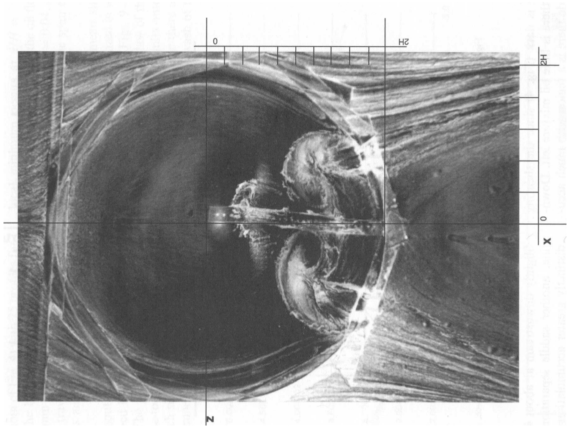
Home Page
Title Page
Contents
◀◀ ▶▶
◀ ▶

Go Back
Full Screen
Close
Quit



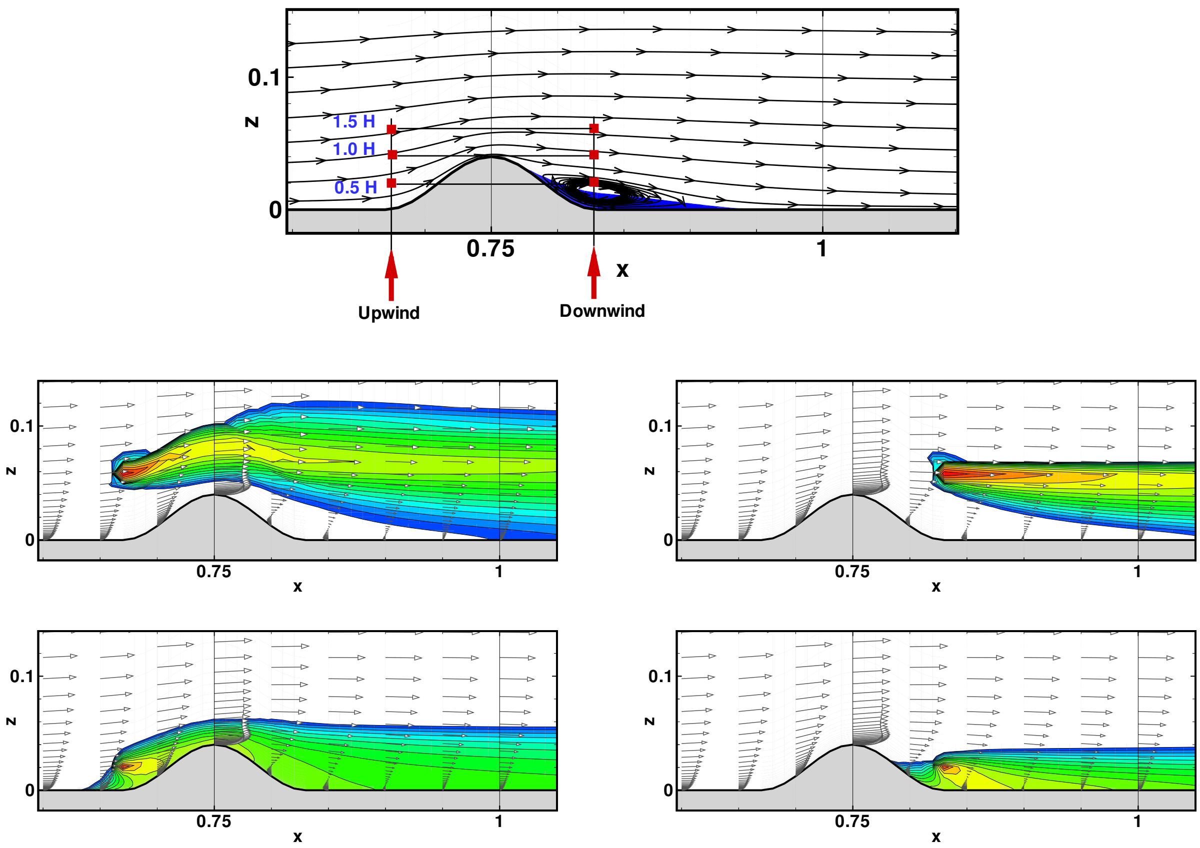


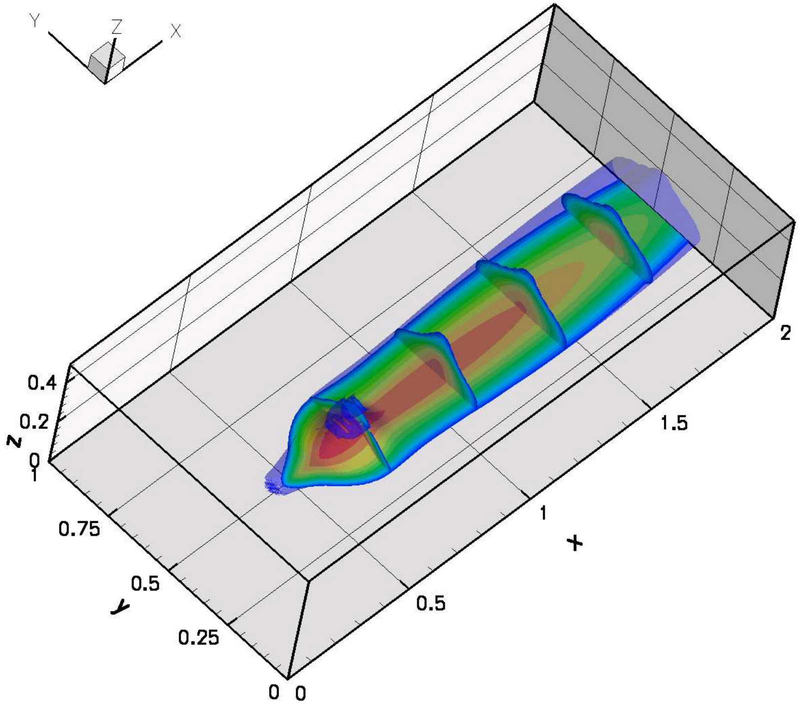




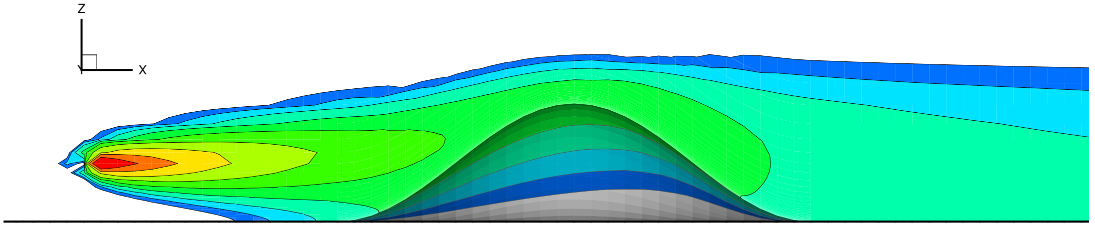
Home Page
Title Page
Contents
Page 92 of 104
Go Back
Full Screen
Close
Quit

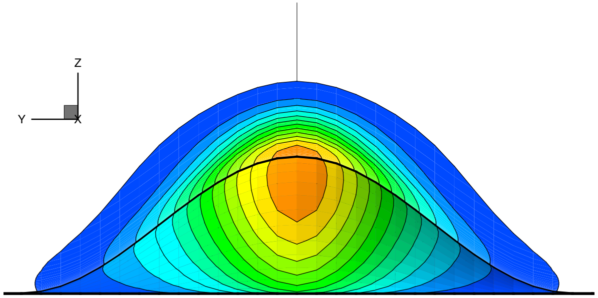
9.5. Pollution dispersion





Y
Z
X
0.4
0.2
Z
0
1
0.75
0.5
Y
0.25
0
0
0.5
1
X
1.5
2





Home Page
Title Page
Contents
Page 95 of 104
Go Back
Full Screen
Close
Quit

9.6. Real scale tests

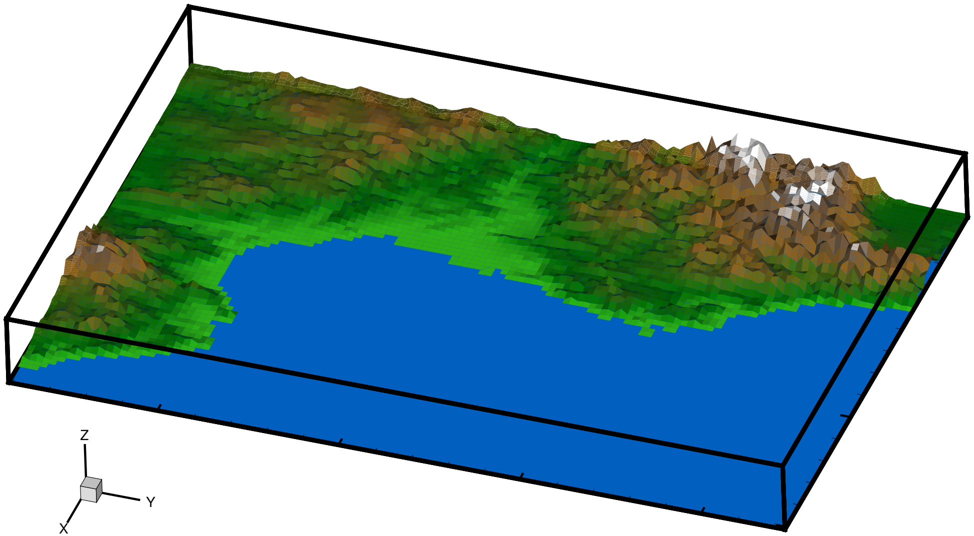
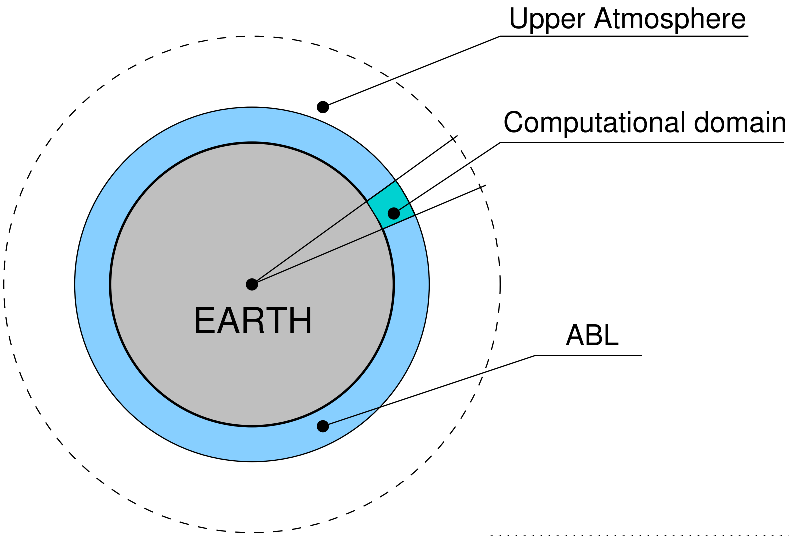
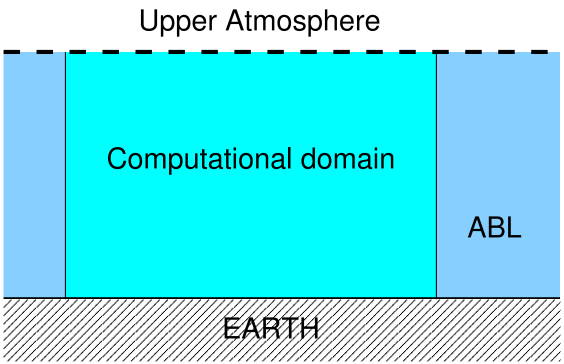
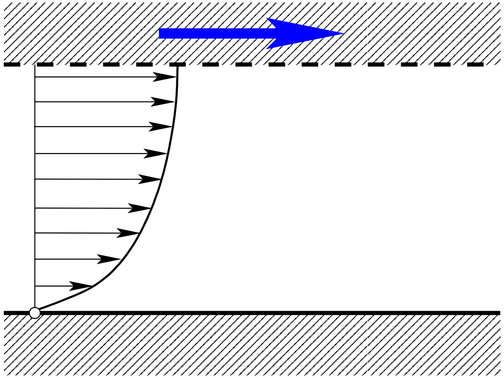


Figure 9.3: 3D domain with real terrain topography

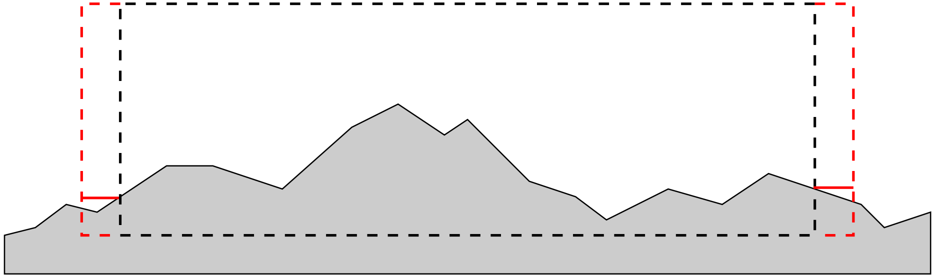
Computational domain & Boundary conditions



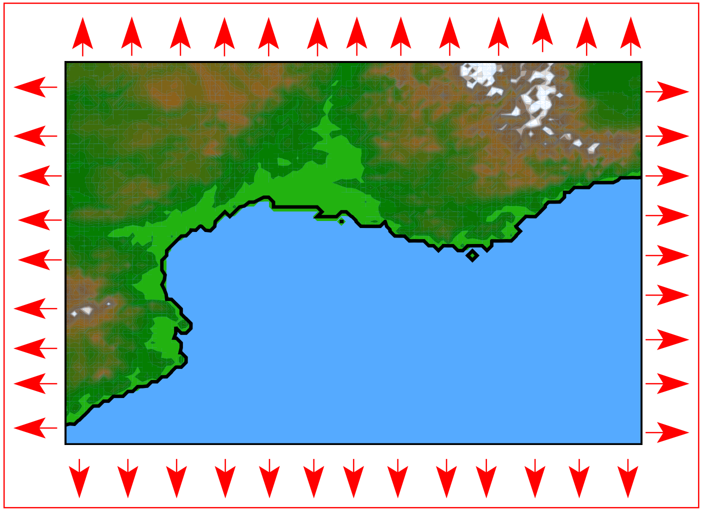




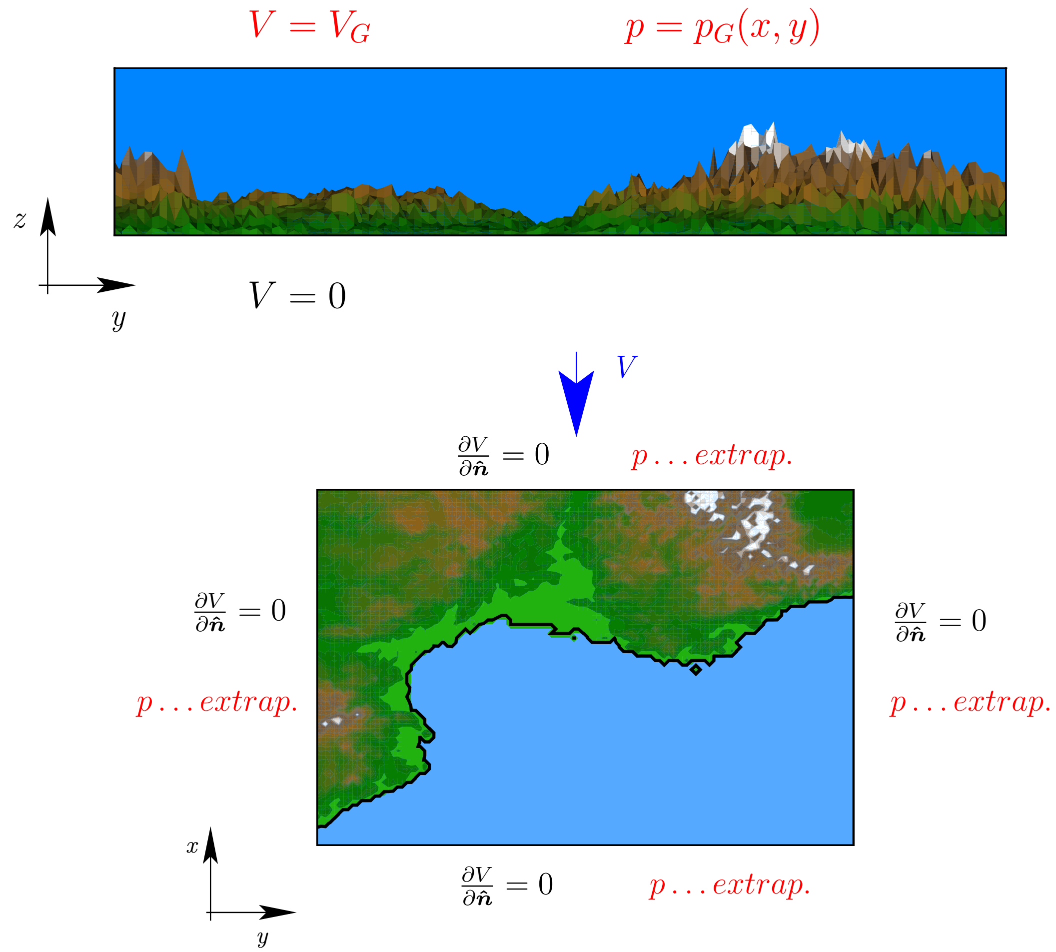
Boundary conditions implementation in complex terrain



$$\frac{\partial z_s}{\partial \hat{n}} = 0$$



Home Page
Title Page
Contents
Page 98 of 104
Go Back
Full Screen
Close
Quit


$V = V_G$
$p = p_G(x, y)$
z
y
$V = 0$
V
$\frac{\partial V}{\partial \hat{\boldsymbol{n}}} = 0$
$p \ldots extrap.$
$\frac{\partial V}{\partial \hat{\boldsymbol{n}}} = 0$
$p \ldots extrap.$
$\frac{\partial V}{\partial \hat{\boldsymbol{n}}} = 0$
$p \ldots extrap.$
x
y
$\frac{\partial V}{\partial \hat{\boldsymbol{n}}} = 0$
$p \ldots extrap.$

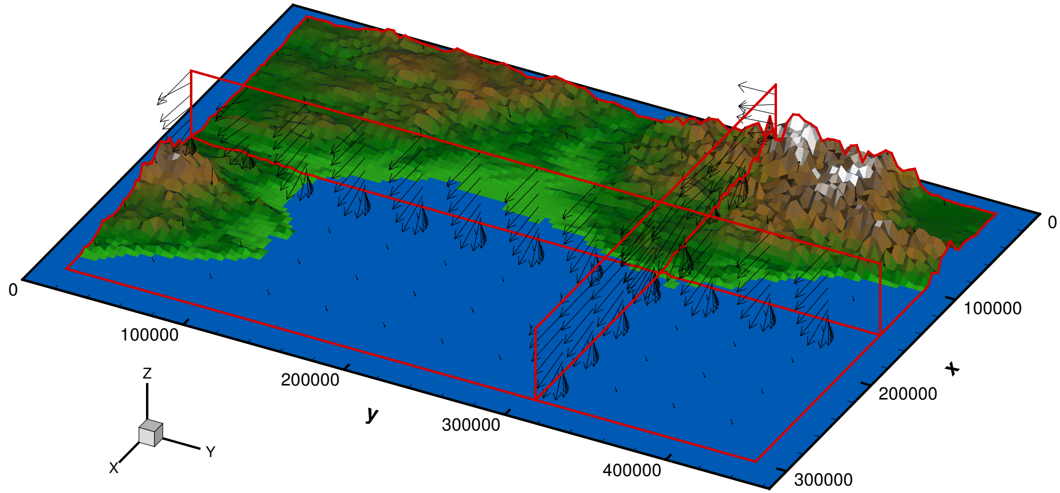


Figure 9.4: Wind velocity vectors, constant velocity at the upper boundary

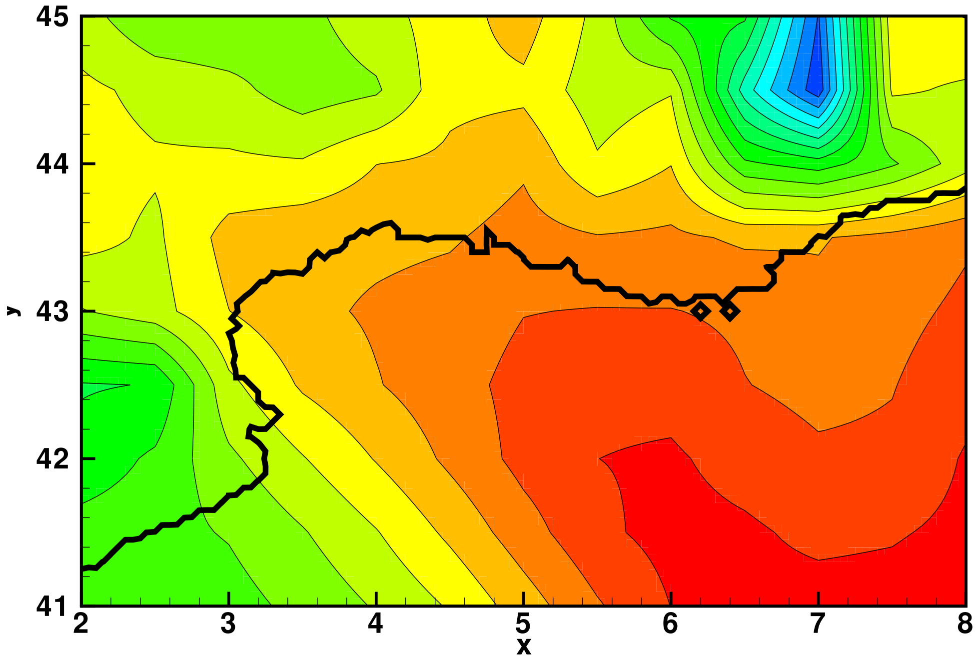


Figure 9.5: Given meteorological data (velocity field) for the level 5000 meters above the sea level.

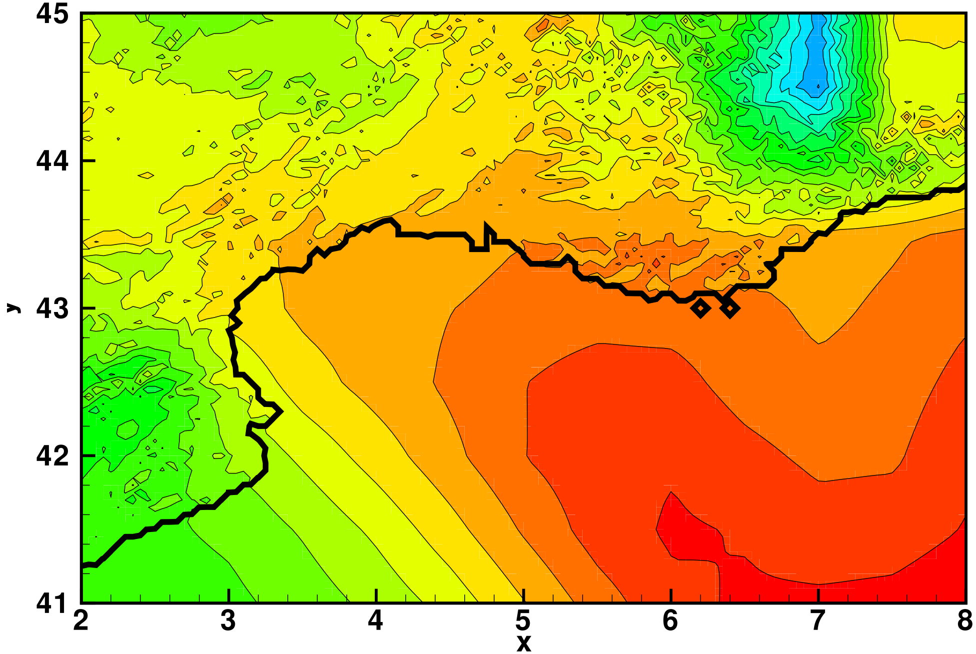
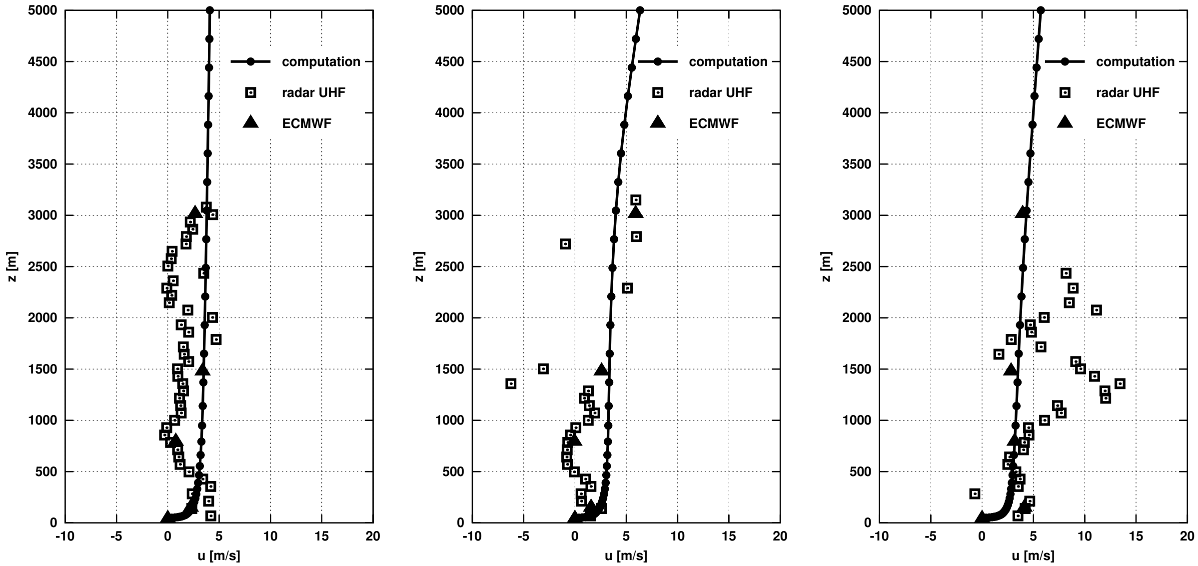
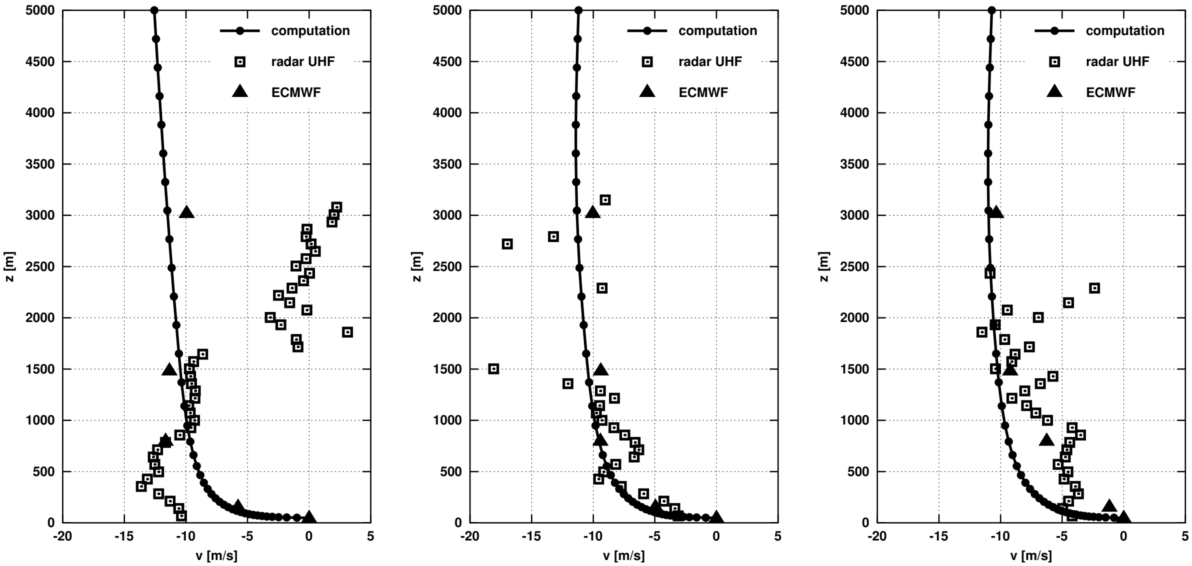


Figure 9.6: Contours of the computed near wall velocity field.



(a) 0 hours (b) 6 hours (c) 12 hours

Figure 9.7: Comparison of u-component profiles for 21. June 2001 at Saint Chamas



(a) 0 hours (b) 6 hours (c) 12 hours

Figure 9.8: Comparison of v-component profiles for 21. June 2001 at Saint Chamas