High Resolution Finite Volume Methods for Transonic Flows

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- **Mathematical models of transonic flows**
- **Numerical solution of scalar equation in 1D**
- **Extension to multidimensional case via finite volume method**
- **Numerical solution of hyperbolic systems**
- **Convection-diffusion problems**

Mathematical models of transonic flows

- **compressible viscous or inviscid flows,**

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_j)=0 \qquad \textit{mass} \qquad (1)$$

$$\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_j}(\rho u_i u_j)+\frac{\partial p}{\partial x_i}=\frac{\partial\tau_{ij}}{\partial x_j} \qquad \textit{momentum} \qquad (2)$$

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x_j}((E+p)u_j)=\frac{\partial}{\partial x_j}(u_i\tau_{ij})-\frac{\partial q_j}{\partial x_j} \qquad \textit{energy} \qquad (3)$$

ρ **- density,** u_i **- velocity vector,** p **- pressure,** E **- total energy per** m^3**,** τ_{ij} **- stress tensor,** q_j **- heat flux**

- **Newtonian fluids:** $\tau_{ij} = 2\mu S_{ij}$, $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}$.
- **Fourier's law:** $q_j = \lambda \frac{\partial T}{\partial x_j}$.
- **Equation of state:** $p = \rho RT \Rightarrow p = (\gamma - 1)(E - \frac{1}{2}\rho u_i u_i)$.

Conservative non-dimensional form

$$W_t + F(W)_x + G(W)_y = \frac{1}{Re}\left(R(W,\nabla W)_x + S(W,\nabla W)_y\right),$$

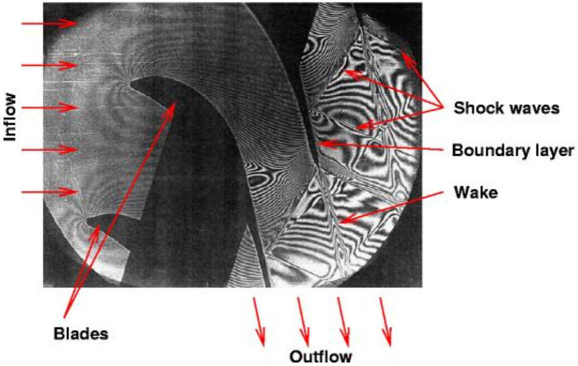
where $Re = \frac{\rho u L}{\mu}$, $Pr = \frac{c_p \mu}{\lambda}$,, $W = [\rho, \rho u, \rho v, E]^T$,

$$F(W) = [\rho u, \rho u^2 + p, \rho uv, (E+p)u]^T,\ G(W) = [\rho v, \rho uv, \rho v^2 + p, (E+p)v]^T$$

$$R(W,\nabla W) = [0, \frac{4}{3}u_x - \frac{2}{3}v_y, u_y + v_x, \ldots]^T,\ S(W,\nabla W) = [0, u_y + v_x, \frac{4}{3}v_y - \frac{2}{3}u_x, \ldots]^T$$

Transonic flow

Flow at high Re



- viscous effects mainly in vicinity solid walls,
- thin boundary layers,
- shock waves,

Inviscid flow - system of Euler equations

Obtained from system of Navier-Stokes equations by $\mu \to 0$ ($Re \to \infty$):

$$W_t + F(W)_x + G(W)_y = 0.$$

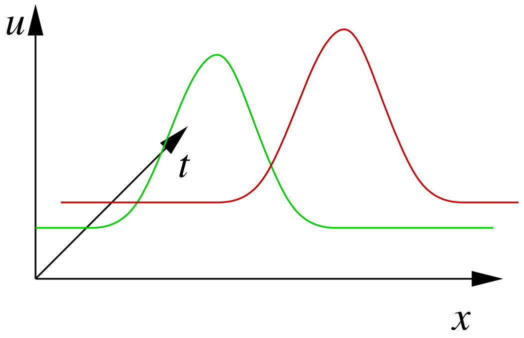
- system of equations is hyperbolic,
- no viscous effects (boundary layers, turbulence, ...),
- usually good description of flow features far from boundary,

Scalar hyperbolic equation in 1D case

$$u_t + f(u)_x = 0, \qquad \textit{equation}$$
$$u(x,0) = u_0(x). \qquad \textit{initial condition}$$

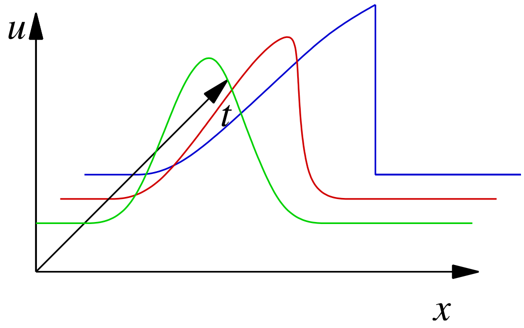
Linear case

- $f(u)_x = a u_x$, $a = const.$
- **exact solution** $u(x,t) = u_0(x - at)$.



Non-linear case

- $f(u)_x = a(u) u_x$, $a(u) \neq const.$
- **classical solution only for** $t < t_{crit}$



Theory

- ***weak solution***, $\Omega = R \times (0,\infty)$, $\phi(x,t) \in C_0^\infty$

$$\iint_\Omega (u_t\phi + f(u)_x\phi)\, dx\, dt = 0$$
$$\iint_\Omega (-u\phi_t - f(u)\phi_x)\, dx\, dt = \int_R u_0(x)\phi(x,0)\, dx.$$

- **weak solution exists, but it may be non-unique!**
- ***physically correct*** **weak solution can be chosen by the so-called** ***entropy condition***: $\mathcal{U}_t + \mathcal{F}_x \leq 0$ **in** $\mathcal{D}'$ **with** $\mathcal{U}$ **convex and** $\mathcal{F}' = \mathcal{U}' f'$,
- **the** ***entropy solution*** **is equal to viscosity vanishing limit of** u^ε **for** $\varepsilon \to 0+$, **where**

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}.$$

Numerical solution

Let $x_{i+1/2} = (i+1/2)\Delta x$ **and** $t^n = n\Delta t$**. Then:**

$$0 = \int_{x_{i-1/2}}^{x_{i+1/2}} \left(u_t + f(u)_x\right) dx =$$

$$= \frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t)\,dx - f\left(u(x_{i-1/2},t)\right) + f\left(u(x_{i+1/2},t)\right).$$

Denote $u_i(t) = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t)\,dx$**, then**

$$\frac{d}{dt}u_i = \frac{f\left(u(x_{i-1/2},t)\right) - f\left(u(x_{i+1/2},t)\right)}{\Delta x}.$$

Time integration

- $\frac{d}{dt}u_i \approx \frac{u_i(t^{n+1})-u_i(t^n)}{\Delta t}$,

Numerical flux $\mathsf{f}^n_{i+1/2}$

- $\mathsf{f}^n_{i+1/2} = \mathsf{f}(u^n_i, u^n_{i+1}) \approx f\left(u(x_{i+1/2}, t^n)\right)$
- **consistency** $\mathsf{f}(v,v) = f(v)$,
- **Lipschitz continuous** $|\mathsf{f}(u,v) - f(w)| \leq K(|u-w| + |v-w|)$,

Discrete (explicit) method

- $\mathsf{u}^{n+1}_i = \mathsf{u}^n_i - \frac{\Delta t}{\Delta x}\left(\mathsf{f}^n_{i+1/2} - \mathsf{f}^n_{i-1/2}\right)$.

Numerical solution (low order schemes)

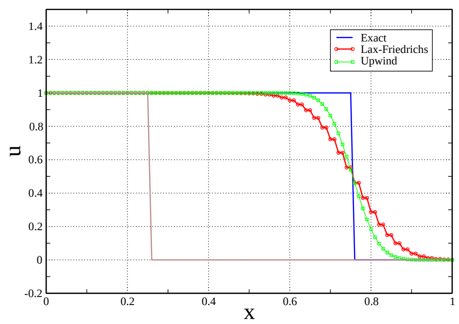
Assume $u_t + a u_x = 0$ with $a > 0$.

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right), \qquad f_{i+1/2}^n = a u_i^n, \text{ (upwind)}$$

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - \frac{a\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right), \quad f_{i+1/2}^n = \frac{a u_i^n + a u_{i+1}^n}{2} - \frac{\Delta x}{2\Delta t}(u_{i+1}^n - u_i^n), \text{ (LF)}$$

- **well known theory: stability and convergence for $\Delta t \leq \Delta x / a$, valid also in non-linear case,**
- **error for smooth data $\approx O(\Delta x)$,**
- **modified equation ($\beta > 0$, $\beta = O(1)$):**

$$u_t + a u_x = \Delta x \beta\left(\frac{\Delta t}{\Delta x}\right) u_{xx} + O(\Delta x^2).$$



Numerical solution (high order schemes)

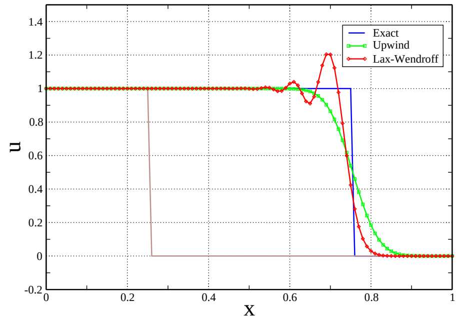
$$u(x,t^{n+1}) = u(x,t^n) + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \ldots = u(x,t^n) - \Delta t a u_x + \frac{\Delta t^2}{2} a^2 u_{xx} + \ldots,$$

$$\mathrm{u}_i^{n+1} = \mathrm{u}_i^n - \frac{a\Delta t}{2\Delta x}\left(\mathrm{u}_{i+1}^n - \mathrm{u}_{i-1}^n\right) + \frac{\Delta t^2 a^2}{2\Delta x^2}\left(\mathrm{u}_{i+1}^n - 2\mathrm{u}_i^n + \mathrm{u}_{i-1}^n\right) \quad \textbf{(Lax-Wendroff)}$$

$$\mathrm{f}_{i+1/2}^n = \frac{a\mathrm{u}_i^n + a\mathrm{u}_{i+1}^n}{2} - \frac{\Delta t a^2}{2\Delta x}\left(\mathrm{u}_{i+1}^n - \mathrm{u}_i^n\right)$$

- **stable for $\Delta t \le \Delta x / a$,**
- **theory only for linear equation!**
- **error for smooth data $\approx O(\Delta x^2)$,**
- **modified equation ($\gamma > 0$, $\gamma = O(1)$) :**

$$u_t + a u_x = -\Delta x^2 \gamma\left(\frac{\Delta t}{\Delta x}\right) u_{xxx} + O(\Delta x^3).$$



Fight against oscillations

- artificial viscosity approach: $u_t + f(u)_x = \Delta x^p \varepsilon u_{xx}$
  - ε usually depends on the first or second derivatives of the solution,
  - used very often but almost no theory!
- TVD (total variation diminishing) schemes:
  - systematic approach for constructing non-oscillatory schemes,
  - proofs of stability and convergence even for non-linear case,
  - complications with extension to 2D and 3D,
- ENO (essentially non-oscillatory) schemes:
  - no proofs, relatively simple to extend do 2D and 3D,
- other strategies: composite schemes, filtering, etc....

Artificial viscosity

- **Von Neumann & Rychtmeyer (1950)**

$$f_{i+1/2} = f^{LW}_{i+1/2} - \varepsilon^{(2)} |u^n_{i+1} - u^n_i| (u^n_{i+1} - u^n_i)$$

- **MacCormack & Baldwin (1975) (for Euler or Navier–Stokes eq.)**

$$F_{i+1/2} = F^{LW}_{i+1/2} - \varepsilon^{(2)} (|u| + c) \frac{|p_{i+1} - 2p_i + p_{i-1}|}{p_{i+1} + 2p_i + p_{i-1}} (W^n_{i+1} - W^n_i)$$

- **Beam & Warming (1976)**

$$F_{i+1/2} = F^{LW}_{i+1/2} - D^{(2)} + \varepsilon^{(4)} \left(W^n_{i+2} - 3W^n_{i+1} + 3W^n_i - W^n_{i-1} \right)$$

- **Jameson & Turkel (1981) - previous terms with special choice of $\varepsilon^{(2)}$ and $\varepsilon^{(4)}$.**

TVD schemes

- **Total variation:** $TV(\mathrm{u}^n) = \sum_i |\mathrm{u}_i^n - \mathrm{u}_{i-1}^n|$.
- **TVD property:** $TV(\mathrm{u}^{n+1}) \le TV(\mathrm{u}^n)$.
- **Harten's lemma: scheme**

$$\mathrm{u}_i^{n+1} = \mathrm{u}_i^n - C_{i-1/2}^n(\mathrm{u}_i^n - \mathrm{u}_{i-1}^n) + D_{i+1/2}^n(\mathrm{u}_{i+1}^n - \mathrm{u}_i^n)$$

has TVD property if $C_{i-1/2}^n \ge 0$**,** $D_{i+1/2}^n \ge 0$**, and** $C_{i+1/2}^n + D_{i+1/2}^n \le 1$**.**

Monotone schemes

$$\mathrm{u}_i^{n+1} = \mathrm{u}_i^n - \frac{\Delta t}{\Delta x}\left(\mathrm{f}_{i+1/2}^n - \mathrm{f}_{i-1/2}^n\right) = \mathcal{H}(\mathrm{u}_{i-1}^n, \mathrm{u}_i^n, \mathrm{u}_{i+1}^n), \text{ with } \quad \frac{\partial \mathcal{H}(v_1, v_2, v_3)}{\partial v_k} \ge 0.$$

Monotone schemes are TVD, but their accuracy is limited to first order!

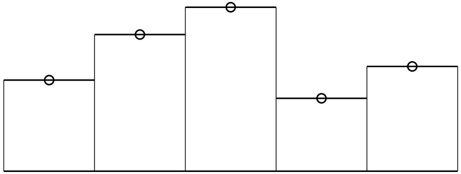
High order TVD schemes - flux limiting approach

- **Idea:** $\mathsf{f}^{low}, \mathsf{f}^{high} \Rightarrow \mathsf{f}^{TVD}_{i+1/2} = \mathsf{f}^{low}_{i+1/2} + Q_{i+1/2}\left(\mathsf{f}^{high}_{i+1/2} - \mathsf{f}^{low}_{i+1/2}\right)$.
- **Example [Causon]: upwind + Lax-Wendroff**

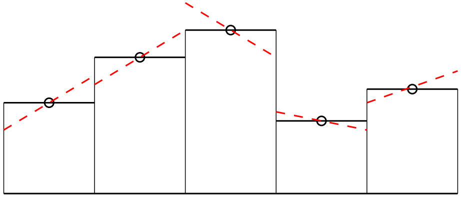
$$\begin{aligned} \mathsf{u}_i^{n+1} &= \mathsf{u}_i^n - a\frac{\Delta t}{2\Delta x}(\mathsf{u}_{i+1}^n - \mathsf{u}_{i-1}^n) + a^2\frac{\Delta t^2}{2\Delta x^2}(\mathsf{u}_{i+1}^n - 2\mathsf{u}_i^n + \mathsf{u}_{i-1}^n) + \\ &+ K(\mathsf{r}_i^+, \mathsf{r}_{i+1}^-)(\mathsf{u}_{i+1}^n - \mathsf{u}_i^n) - K(\mathsf{r}_{i-1}^+, \mathsf{r}_i^-)(\mathsf{u}_i^n - \mathsf{u}_{i-1}^n), \end{aligned}$$

$$\begin{aligned} K(\mathsf{r}_i^+, \mathsf{r}_{i+1}^-) &= \frac{|a|\Delta t}{2\Delta x}(1 - \frac{|a|\Delta t}{\Delta x})\left[1 - \Phi(\mathsf{r}_i^+) + 1 - \Phi(\mathsf{r}_{i+1}^-)\right] \\ \Phi(\mathsf{r}) &= \max\left(0, \min(2\mathsf{r}, 1)\right). \\ \mathsf{r}_i^+ &= (\mathsf{u}_i^n - \mathsf{u}_{i-1}^n)/(\mathsf{u}_{i+1}^n - \mathsf{u}_i^n) \\ \mathsf{r}_i^- &= (\mathsf{u}_{i+1}^n - \mathsf{u}_i^n)/(\mathsf{u}_i^n - \mathsf{u}_{i-1}^n). \end{aligned}$$

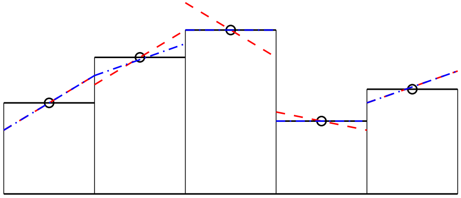
High order TVD schemes - slope limiting approach



- $f_{i+1/2} = f(u_i, u_{i+1})$,
- **first order scheme,**



- $f_{i+1/2} = f(u^L_{i+1/2}, u^R_{i+1/2})$,
- $u^L_{i+1/2} = u_i + \frac{\Delta x}{2}\sigma_i$, $u^R_{i+1/2} = u_{i+1} + \frac{\Delta x}{2}\sigma_{i+1}$,
- $\sigma_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x}$, **unstable scheme,**



- $\sigma_i = \frac{1}{\Delta x} minmod(u_{i+1} - u_i, u_i - u_{i-1})$,
- $minmod(x,y) = sign(x)\, min(|x|, sign(x)y)$

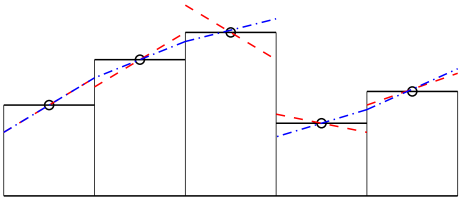
Theory of TV stability

- **Lax-Wendroff theorem + TVD property $\Rightarrow$ numerical solution u converges to a weak solution of initial value problem as Δx and $\Delta t \longrightarrow 0$,**

Properties of TVD schemes

- **TVD schemes are stable and convergent even for non-linear case and non-smooth solutions,**
- **the solution is free of spurious oscillations,**
- **the accuracy is usually reduced to low order near discontinuities and extremas,**
- **the extension to multidimensional case is limited to first order schemes!**

ENO schemes



- $\sigma_i = \frac{1}{\Delta x} \begin{cases} u_{i+1} - u_i & \textbf{if } |u_{i+1} - u_i| \le |u_{i+1} - u_i| \\ u_i - u_{i-1} & \textbf{if } |u_{i+1} - u_i| > |u_{i+1} - u_i| \end{cases}$

- **uniformly high order schemes but no more TVD,**
- **possible to extend to arbitrary order,**
- **the oscillations are (hopefully) small,**
- **sometimes problems with convergence to steady state.**

Weighted ENO schemes

- $\sigma_i = w_{i-1/2}\sigma_{i-1/2} + w_{i+1/2}\sigma_{i+1/2}$, with $\sigma_{i+1/2} = \frac{u_{i+1}-u_i}{\Delta x}$,
- $w_{i-1/2} = \frac{w(\sigma_{i-1/2})}{w(\sigma_{i+1/2})+w(\sigma_{i+1/2})}$, $w_{i+1/2} = \frac{w(\sigma_{i+1/2})}{w(\sigma_{i+1/2})+w(\sigma_{i+1/2})}$, $\Rightarrow w_{i-1/2} + w_{i+1/2} = 1$,
- $w(\sigma) = O(1)$ for small $|\sigma|$, $w(\sigma) \to 0$ for large $|\sigma|$,
- eg. $w(\sigma) = \frac{1}{1+C|\sigma|^p}$, with $C > 0$, $p > 0$.

Numerical experiment

ENO schemes for spatial discretisation, TVD RK3 for time discretisation

Linear equation: $u_t + u_x = 0$,

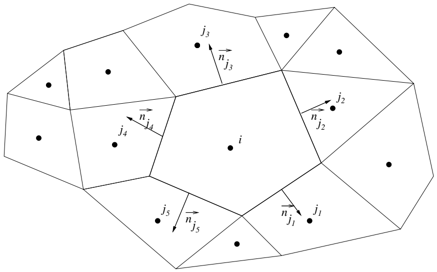
Initial condition:

$$u_0(x) = \begin{cases} \sin^2(\pi x/0.25) & \text{for } x \in (0.25, 0.5) \\ 0 & \text{otherwise.} \end{cases}$$

n	ENO	order	WENO	order
100	8802.7		5964.6	
200	3567.2	1.3	1608.2	1.9
400	1083.7	1.7	374.4	2.1
800	307.6	1.8	85.4	2.1
1600	83.8	1.9	18.3	2.2
3200	23.4	1.9	3.6	2.3

L_1 error multiplied by 10^6.

Extension to 2D - Finite Volume method



$$\int_{\Omega_i} \left(\frac{\partial u}{\partial t} + \frac{\partial f_j(u)}{\partial x_j} \right) d\mathbf{x} = 0,$$

$$\frac{d}{dt} \left(\int_{\Omega_i} u(\mathbf{x}, t)\, d\mathbf{x} \right) + \int_{\partial\Omega_i} \mathbf{f}(u) \cdot \mathbf{n}\, dS = 0,$$

Numerical flux: $\int_{\partial\Omega_i} \mathbf{f}(u) \cdot \mathbf{n}\, dS = \sum_j \int_{\Gamma_{ij}} \mathbf{f}(u) \cdot \mathbf{n}\, dS \approx \sum_j |\Gamma_{ij}| \mathsf{f}(\mathsf{u}_i, \mathsf{u}_j, \mathbf{n}_{ij}),$

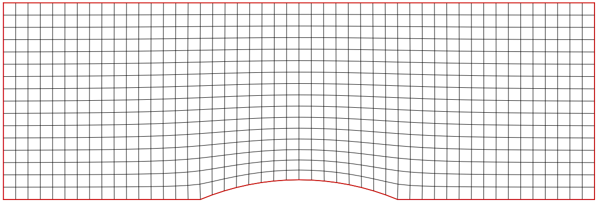
Semi-discrete finite volume method: $\frac{d\mathsf{u}_i(t)}{dt} + \frac{1}{|\Omega_i|} \sum_j |\Gamma_{ij}| \mathsf{f}(\mathsf{u}_i, \mathsf{u}_j, \mathbf{n}_{ij}) = 0.$

Discrete (explicit) finite volume method:

$$\frac{\mathsf{u}_i^{n+1} - \mathsf{u}_i^n}{\Delta t} = -\frac{1}{|\Omega_i|} \sum_j |\Gamma_{ij}| \mathsf{f}(\mathsf{u}_i^n, \mathsf{u}_j^n, \mathbf{n}_{ij}) = -R(\mathsf{u}^n)_i.$$

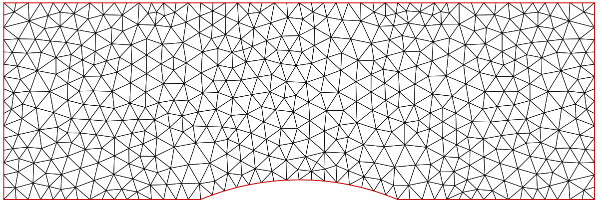
Choice of mesh topology

Structured meshes



- simple coding,
- possible to use one-dimensional schemes in each index direction,
- good results for simple geometry,
- complicated meshing for complex geometry,

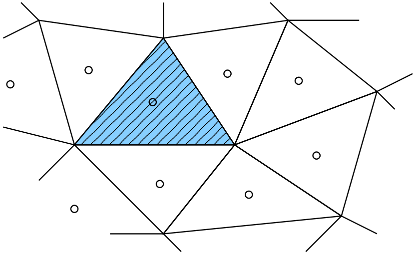
Unstructured meshes



- more complicated coding,
- ability to handle complex geometry,
- mesh adaptation,

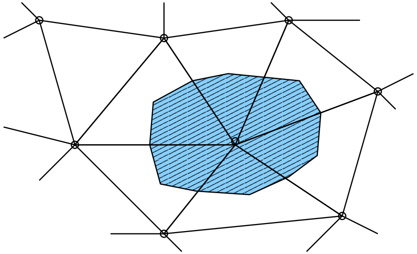
Choice of the control volume

Primary volumes



- values stored at the centers of mesh cells,
- more control volumes, less interfaces,

Dual volumes



- values stored at the nodes of mesh,
- lower # of volumes, higher # of interfaces ($\Rightarrow$ more flux evaluations)
- nodes of the mesh are not in centers of control volumes!

Monotone methods

$$\mathsf{u}_i^{n+1} = \mathsf{u}_i^n - \Delta t R(\mathsf{u}^n)_i = \mathcal{H}(\mathsf{u}^n).$$

Example: $u_t + \mathbf{a}\nabla u = 0$**,** $\mathbf{a} = const.$**, first order upwind method:**

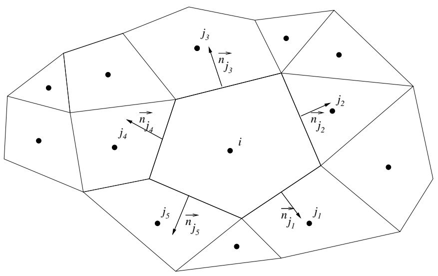
- **numerical flux** $\mathsf{f}(\mathsf{u}_i, \mathsf{u}_j, \mathbf{n}_{ij}) = \left\{ \begin{array}{ll} (\mathbf{a}\cdot\mathbf{n}_{ij})\mathsf{u}_j & \textbf{for } \mathbf{a}\cdot\mathbf{n}_{ij} < 0, \\ (\mathbf{a}\cdot\mathbf{n}_{ij})\mathsf{u}_i & \textbf{for } \mathbf{a}\cdot\mathbf{n}_{ij} \geq 0. \end{array} \right\},$
- **upwind scheme:** $\mathsf{u}_i^{n+1} = \mathsf{u}_i^n - \frac{\Delta t}{|\Omega_i|}\sum_j |\Gamma_{ij}| \left[(\mathbf{a}\cdot\mathbf{n}_{ij})^- \mathsf{u}_j^n + (\mathbf{a}\cdot\mathbf{n}_{ij})^+ \mathsf{u}_i^n \right],$
- **stability (monotonicity) condition:** $\Delta t \leq \frac{|\Omega_i|}{\sum_j |\Gamma_{ij}| (\mathbf{a}\cdot\mathbf{n}_{ij})^+},$

Properties of monotone methods:

- **convergent to weak (or entropy) solution,**
- **satisfy maximum principle,**
- **unfortunatelly, their accuracy is limited to first order.**

High order methods

- **application of one-dimensional schemes (structured meshes):**
  - **dimensional splitting:** $\mathsf{u}^{n+1} = L\mathsf{u}^n = L_I L_J \mathsf{u}^n = (I - \Delta t R_I)(I - \Delta t R_J)\mathsf{u}^n$,
  - **directional TVD (ENO):** $\mathsf{u}^{n+1} = (I - \Delta t R_I - \Delta t R_J)\mathsf{u}^n$ **where** $I - \Delta t R_{I,J}$ **define one-dimensional TVD (ENO) schemes,**
- **multidimensional reconstruction (unstructured meshes):**



$$\iint_{\Omega_i} P_i(\mathbf{x};\mathsf{u}^n)\,d\mathbf{x} = |\Omega_i|\mathsf{u}_i^n$$

$$\iint_{\Omega_j} P_i(\mathbf{x};\mathsf{u}^n)\,d\mathbf{x} = |\Omega_j|\mathsf{u}_j^n + O(|\mathbf{x}_j - \mathbf{x_i}|^p)$$

TVD schemes in 2D

time-space discretisation: $x_i = i\Delta x$, $y_j = j\Delta y$, $t_n = n\Delta t$, $\mathrm{u}^n_{i,j} \approx u(x_i, y_j, t_n)$.
explicit conservative scheme:

$$\mathrm{u}^{n+1}_{i,j} = \mathrm{u}^n_{i,j} - \frac{\Delta t}{\Delta x}\left[\mathrm{f}^n_{i+1/2,j} - \mathrm{f}^n_{i-1/2,j}\right] - \frac{\Delta t}{\Delta y}\left[\mathrm{g}^n_{i,j+1/2} - \mathrm{g}^n_{i,j-1/2}\right] \qquad \text{(**)}$$

where $\mathrm{f}^n_{i+1/2,j} = \mathrm{f}(\mathrm{u}^n_{i-p,j}, \ldots, \mathrm{u}^n_{i+q,j})$ **and** $\mathrm{g}^n_{i,j+1/2} = \mathrm{g}(\mathrm{u}^n_{i,j-p}, \ldots, \mathrm{u}^n_{i,j+q})$.

Total variation is: $TV(\mathrm{u}^n) = \Delta x \Delta y \sum_{i,j} \left|\mathrm{u}^n_{i,j} - \mathrm{u}^n_{i-1.j}\right| / \Delta x + \left|\mathrm{u}^n_{i,j} - \mathrm{u}^n_{i,j-1}\right| / \Delta y.$

It is possible to construct a multidimensional TVD scheme, but the <u>strong TV bound implies first order of accuracy</u> for any multidimensional scheme (Goodman, LeVeque).

Two-dimensional schemes of the form (**) constructed using two one-dimensionalhigh-order TVD schemes <u>are no more TVD</u>, nevertheless, they perform well in practical computations.

Schemes with weak TV bound

Assume a two-dimensional scheme of the form(**) with numerical fluxes f and g of the form: $\mathsf{f}^n_{i+1/2,j} = \mathsf{p}^n_{i+1/2,j} + \frac{\Delta x}{\Delta t}\mathsf{a}^n_{i+1/2,j}$, $\mathsf{g}^n_{i,j+1/2} = \mathsf{q}^n_{i,j+1/2} + \frac{\Delta y}{\Delta t}\mathsf{b}^n_{i,j+1/2}$, **where** p **and** q **are monotone fluxes and** a **and** b **are high order corrections.**

<u>Theorem:</u> (Coquel, LeFloch 1991) Consider a family of approximate solutions U_h **constructed by the scheme (**) on the grid with** $\Delta x \to 0$ **(**$\Delta t/\Delta x = const.$**,** $\Delta y = \Delta x = h$**). Assume that:**

$$||\mathsf{U}_h||_\infty \le C, \qquad \frac{\Delta t}{\Delta x}\max|f'(u)| \le \frac{1}{4}, \qquad \frac{\Delta t}{\Delta y}\max|g'(u)| \le \frac{1}{4}$$

$$\left|\mathsf{a}^n_{i+1/2,j}\right| \le M\Delta t^\alpha, \qquad \left|\mathsf{b}^n_{i,j+1/2}\right| \le M\Delta t^\alpha, \qquad \alpha \in \left(\frac{2}{3}, 1\right)$$

Then U_h **converges to the unique entropy solution in** L^1_{loc}**.**

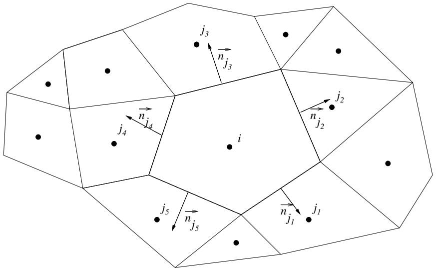
Example: 2D version of the scheme proposed by Causon

Linear equation: $u_t + au_x + bu_y = 0$

$$
\begin{aligned}
\mathsf{u}_{i,j}^{n+1} &= \mathsf{u}_{i,j}^{n} - a\frac{\Delta t}{2\Delta x}(\mathsf{u}_{i+1,j}^{n} - \mathsf{u}_{i-1,j}^{n}) + a^2\frac{\Delta t^2}{2\Delta x^2}(\mathsf{u}_{i+1,j}^{n} - 2\mathsf{u}_{i,j}^{n} + \mathsf{u}_{i-1,j}^{n}) + \\
&+ K(\mathsf{r}_{i,j}^{+}, \mathsf{r}_{i+1,j}^{-})(\mathsf{u}_{i+1,j}^{n} - \mathsf{u}_{i,j}^{n}) - K(\mathsf{r}_{i-1,j}^{+}, \mathsf{r}_{i,j}^{-})(\mathsf{u}_{i,j}^{n} - \mathsf{u}_{i-1,j}^{n}) + \\
&+ \textbf{similar terms in } j \textbf{ direction}
\end{aligned}
$$

$$
\begin{aligned}
K(\mathsf{r}_{i,j}^{+}, \mathsf{r}_{i+1,j}^{-}) &= \frac{|a|\Delta t}{2\Delta x}(1 - \frac{|a|\Delta t}{\Delta x})\left[1 - \Phi(\mathsf{r}_{i,j}^{+}) + 1 - \Phi(\mathsf{r}_{i+1,j}^{-})\right] \\
\Phi(\mathsf{r}) &= \min\left[\max(0, \min(2\mathsf{r}, 1)), \frac{M'\Delta x^{\alpha}}{|\mathsf{u}_{i+1,j}^{n} - \mathsf{u}_{i,j}^{n}|}\right], \\
\mathsf{r}_{i,j}^{+} &= (\mathsf{u}_{i,j}^{n} - \mathsf{u}_{i-1,j}^{n})/(\mathsf{u}_{i+1,j}^{n} - \mathsf{u}_{i,j}^{n}) \\
\mathsf{r}_{i,j}^{-} &= (\mathsf{u}_{i+1,j}^{n} - \mathsf{u}_{i,j}^{n})/(\mathsf{u}_{i,j}^{n} - \mathsf{u}_{i-1,j}^{n}).
\end{aligned}
$$

Multidimensional reconstruction



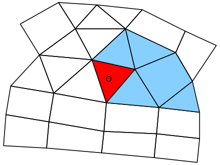
$$\iint_{\Omega_i} P_i(\mathbf{x}; \mathrm{u}^n)\, d\mathbf{x} = |\Omega_i| \mathrm{u}_i^n$$

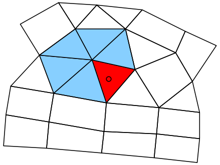
$$\iint_{\Omega_j} P_i(\mathbf{x}; \mathrm{u}^n)\, d\mathbf{x} = |\Omega_j| \mathrm{u}_j^n + O(|\mathbf{x}_j - \mathbf{x_i}|^p)$$

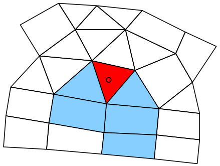
- **using "central" method for computation P_i and applying limiters [Barth],**
- **extending ENO or WENO methods**

Multidimensional reconstruction - third order ENO/WENO

$P_i(\mathbf{x}) = \sum_{|\alpha|\le 2} a_\alpha (\mathbf{x}-\mathbf{x}_i)^\alpha$, **where $\mathbf{x}_i$ is center of gravity of cell Ω_i.**

(A)  **conservation:** $\int_{\Omega_i} P_i(\mathbf{x})\,d\mathbf{x} = |\Omega_i| u_i$
accuracy: $\forall j : \int_{\Omega_j} P_i(\mathbf{x})\,d\mathbf{x} = |\Omega_j| u_j$ $\Rightarrow a_\alpha^{(A)} \Rightarrow P_i^{(A)}(\mathbf{x})$.

(B)  **conservation:** $\int_{\Omega_i} P_i(\mathbf{x})\,d\mathbf{x} = |\Omega_i| u_i$
accuracy: $\forall j : \int_{\Omega_j} P_i(\mathbf{x})\,d\mathbf{x} = |\Omega_j| u_j$ $\Rightarrow a_\alpha^{(B)} \Rightarrow P_i^{(B)}(\mathbf{x})$.

(C)  **conservation:** $\int_{\Omega_i} P_i(\mathbf{x})\,d\mathbf{x} = |\Omega_i| u_i$
accuracy: $\forall j : \int_{\Omega_j} P_i(\mathbf{x})\,d\mathbf{x} = |\Omega_j| u_j$ $\Rightarrow a_\alpha^{(C)} \Rightarrow P_i^{(C)}(\mathbf{x})$.

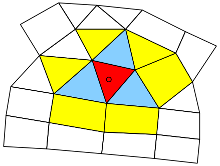
ENO reconstruction:

P_i **is selected between** $P_i^{(A)}$, $P_i^{(B)}$, $P_i^{(C)}$ **in order to minimize** $\sum_{1\le|\alpha|} |a_\alpha|$.

WENO reconstruction:

$$P_i = (w_i^{(A)} P_i^{(A)} + w_i^{(B)} P_i^{(B)} + w_i^{(C)} P_i^{(C)}) / (w_i^{(A)} + w_i^{(B)} + w_i^{(C)})$$

Multidimensional reconstruction - weighted least-square method



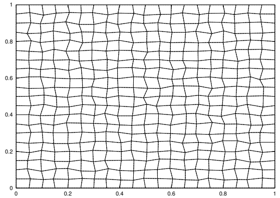
conservation: $\int_{\Omega_i} P_i(\mathbf{x})\,d\mathbf{x} = |\Omega_i|\mathrm{u}_i$
accuracy: $\forall j : \int_{\Omega_j} P_i(\mathbf{x})\,d\mathbf{x} = |\Omega_j|\mathrm{u}_j$ $\Rightarrow$ **overdetermined!**

Linear least-square method: (unstable for non-smooth data):
$\min \sum_j \left(\int_{\Omega_j} P_i(\mathbf{x})\,d\mathbf{x} - |\Omega_j|\mathrm{u}_j \right)^2$ **with respect to** $\int_{\Omega_i} P_i(\mathbf{x})\,d\mathbf{x} = |\Omega_i|\mathrm{u}_i$.

Weighted least-square method:
$\min \sum_j w_{ij}^2 \left(\int_{\Omega_j} P_i(\mathbf{x})\,d\mathbf{x} - |\Omega_j|\mathrm{u}_j \right)^2$ **with respect to** $\int_{\Omega_i} P_i(\mathbf{x})\,d\mathbf{x} = |\Omega_i|\mathrm{u}_i,$ **with data-dependent weight** $w_{ij} = \frac{1}{(\mathrm{u}_i - \mathrm{u}_j)^2/||\mathbf{x}_i - \mathbf{x}_j||^2 + \varepsilon}$.

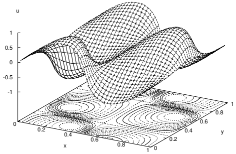
Example: WLSQR method for scalar case

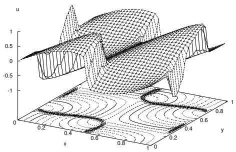
Linear problem: $u_t + u_x + u_y = 0,\ u_0(x,y) = \sin(2\pi x)\cos(2\pi y)$.



	first order		second order		third order	
$1/N$	$\|\|e\|\|_1$	order	$\|\|e\|\|_1$	order	$\|\|e\|\|_1$	order
0.1	**0.339084**	**-**	**0.141348**	**-**	**0.134682**	**-**
0.05	**0.253544**	**0.42**	**0.035086**	**2.01**	**0.021605**	**2.64**
0.025	**0.157564**	**0.68**	**0.007567**	**2.21**	**0.002843**	**2.93**
0.0125	**0.088477**	**0.83**	**0.001584**	**2.25**	**0.000377**	**2.92**

Non-linear problem: $u_t + uu_x + uu_y = 0,\ u_0(x,y) = \sin(2\pi x)\cos(2\pi y)$.





	first order		second order		third order	
$1/N$	$\|\|e\|\|_1$	order	$\|\|e\|\|_1$	order	$\|\|e\|\|_1$	order
	Smooth data ($t = 0.1$)					
0.1	**0.054867**	**-**	**0.017641**	**-**	**0.012703**	**-**
0.05	**0.040623**	**0.43**	**0.008839**	**1.00**	**0.002686**	**2.24**
0.025	**0.024009**	**0.76**	**0.001963**	**2.41**	**0.000648**	**2.05**
0.0125	**0.013414**	**0.84**	**0.000379**	**2.37**	**0.000116**	**2.48**
0.00625	**0.007095**	**0.92**	**0.000081**	**2.23**	**0.000017**	**2.77**
	Non-smooth data ($t = 0.25$)					
0.1	**0.112414**	**-**	**0.049627**	**-**	**0.047704**	**-**
0.05	**0.069466**	**0.69**	**0.018373**	**1.43**	**0.018493**	**1.36**
0.025	**0.039077**	**0.83**	**0.011098**	**0.73**	**0.009987**	**0.89**
0.0125	**0.021665**	**0.85**	**0.005554**	**1.00**	**0.004837**	**1.05**

Hyperbolic systems

$$W_t + F(W)_x = W_t + A(W)W_x = 0,$$

The system is called hyperbolic iff $A(W)$ real eigenvalues and full set of eigenvectors. Then: $A(W) = R(W)\Lambda(W)R^{-1}(W)$.

$$W_t + A(W)W_x = 0 \;\Rightarrow\; R^{-1}(W)W_t + \Lambda(W)R^{-1}W_x = 0,$$

Characteristic variables: $\delta V = R^{-1}(W)\delta W \Rightarrow V_t + \Lambda(W)V_x = 0$.

TVD schemes via flux limiting approach:

- **scalar scheme for V, projection back to W,**
- **numerical flux: $\mathsf{F}^{TVD}_{i+1/2} = \mathsf{F}^{high}_{i+1/2} - R\Psi(\Lambda)R^{-1}(\mathsf{W}_{i+1} - Wi)$.**
- **simplified method: $R\Psi(\Lambda)R^{-1} \approx \tilde{\Psi}I$.**

Numerical flux for systems of hyperbolic equations

Exact Riemann solver

- **solve exactly Riemann problems at the interfaces,**
- **usually to expensive, not possible for general hyperbolic system,**

Upwinding (Roe)

- $F_{i+1/2} = \frac{1}{2}\left(F(W_i) + F(W_{i+1})\right) - \frac{1}{2}|A_{i+1/2}|\left(W_{i+1} - W_i\right)$.

Flux splitting (VanLeer, AUSM, ...)

- $F(W) = F^+(W) + F^-(W)$ **where** $\sigma(\frac{\partial F^+}{\partial W}) \geq 0$ **and** $\sigma(\frac{\partial F^-}{\partial W}) \leq 0$.
- **numerical flux** $F_{i+1/2} = F^+(W_i) + F^-(W_{i+1})$.

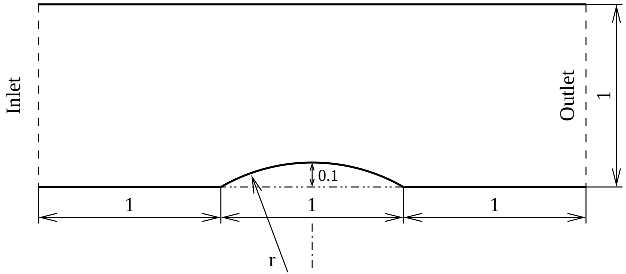
Acceleration of convergence

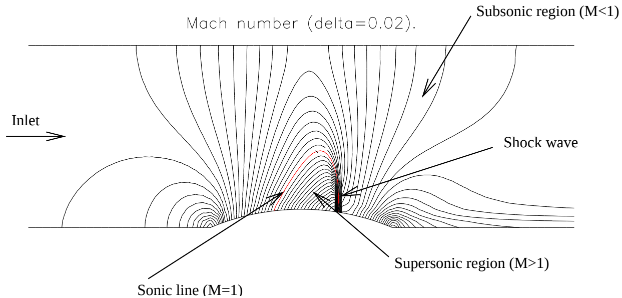
- **explicit methods $\Rightarrow$ stability condition $\Delta t \leq C\Delta x$ or even $\Delta t \leq C'\Delta x^2$ for convection-diffusion problems,**
- **convergence can be accelerated using multigrid, residual smoothing or implicit methods.**

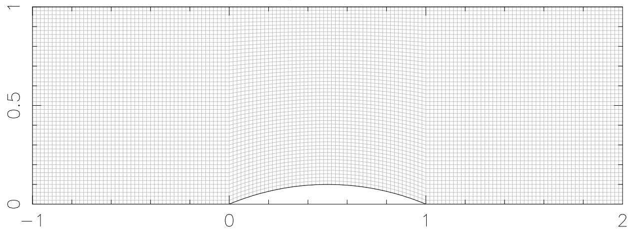
Implicit methods

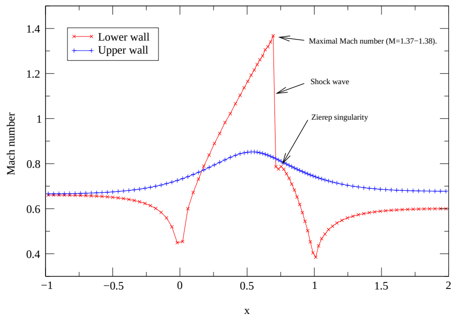
- **backward Euler method:** $\mathsf{u}^{n+1} = \mathsf{u}^n - \Delta t R(\mathsf{u}^{n+1})$**,**
- **linearized version:** $\mathsf{u}^{n+1} = \mathsf{u}^n - \Delta t \left(R(\mathsf{u}^n) + \frac{\partial R(\mathsf{u}^n)}{\partial \mathsf{u}} (\mathsf{u}^{n+1} - \mathsf{u}^n) \right) \Rightarrow$
  $\left(\frac{I}{\Delta t} + \frac{\partial R(\mathsf{u}^n)}{\partial \mathsf{u}} \right) (\mathsf{u}^{n+1} - \mathsf{u}^n) = -R(\mathsf{u}^n)$**.**
- **semi-implicit version:** $\left(\frac{I}{\Delta t} + \frac{\partial R^{(1)}(\mathsf{u}^n)}{\partial \mathsf{u}} \right) (\mathsf{u}^{n+1} - \mathsf{u}^n) = -R^{(2)}(\mathsf{u}^n)$**.**

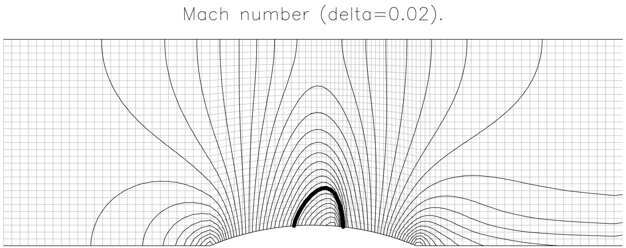
Inviscid flow through a channel

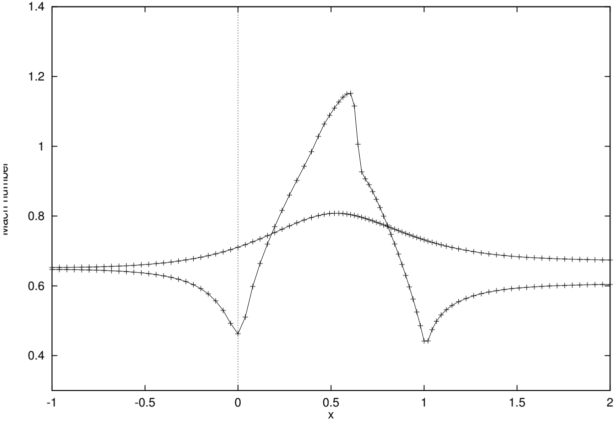




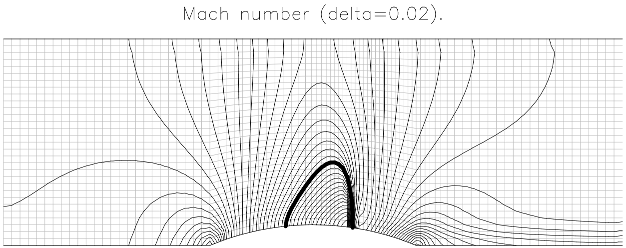


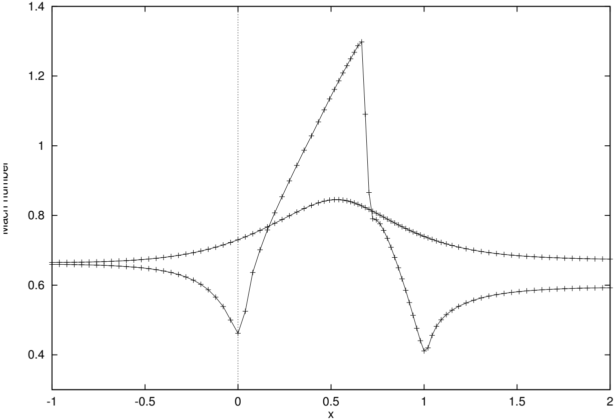




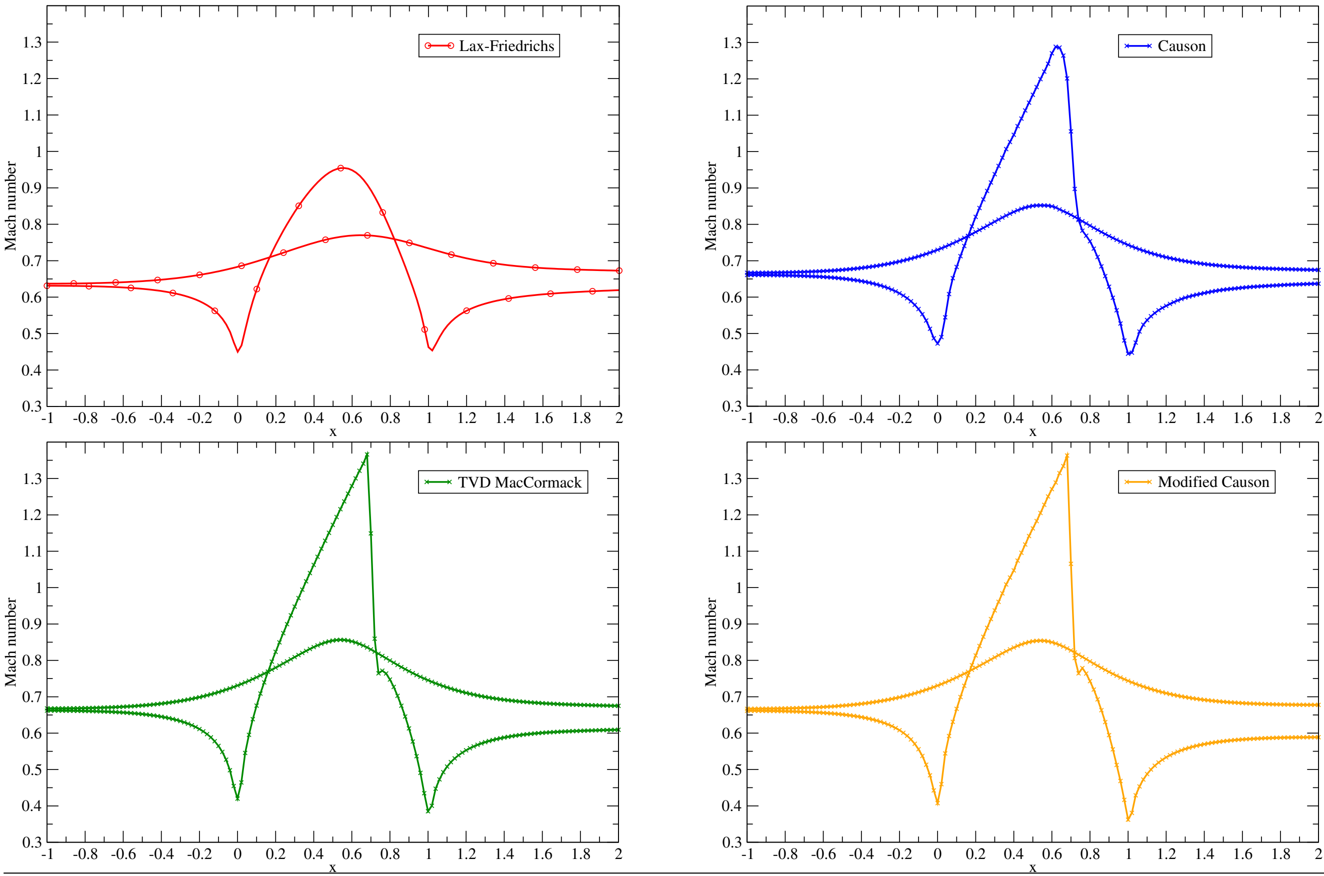


First order Osher's scheme

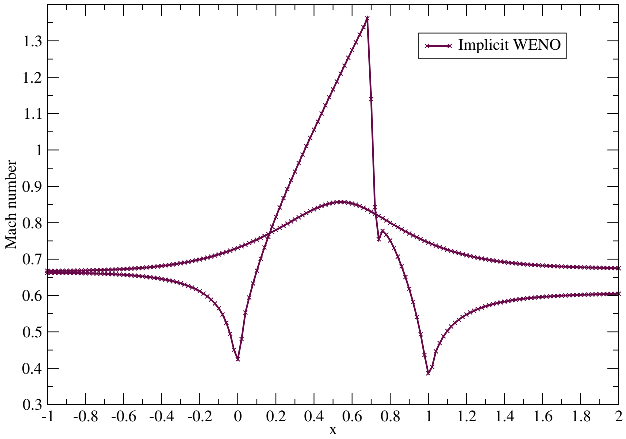


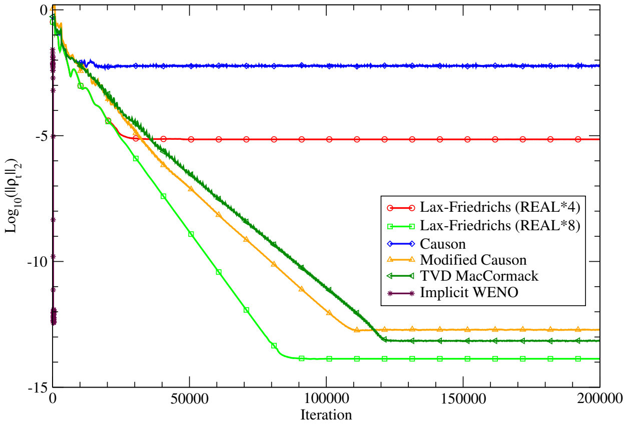


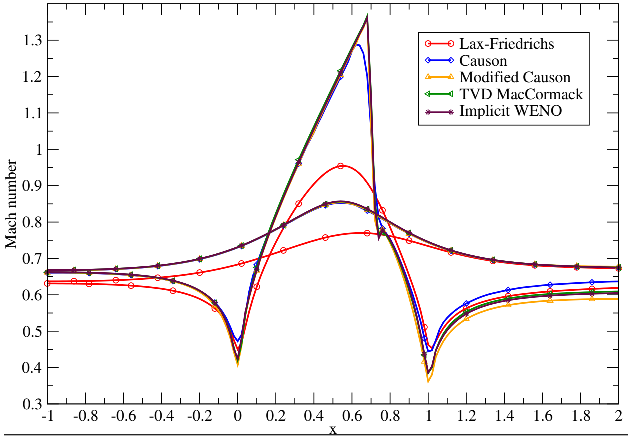
Osher's scheme, 2nd. order WENO

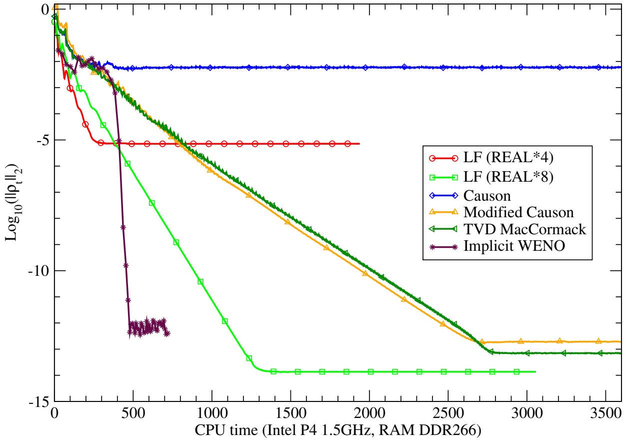


Lax-Friedrichs
Causon
TVD MacCormack
Modified Causon
Mach number
x


Implicit WENO
Mach number
x

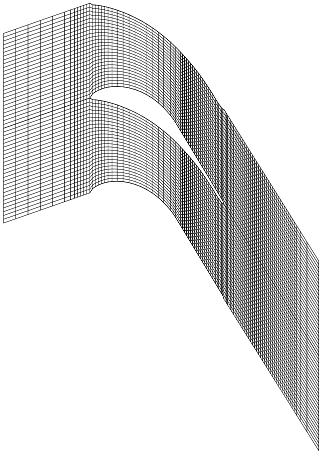

Lax-Friedrichs (REAL*4)
Lax-Friedrichs (REAL*8)
Causon
Modified Causon
TVD MacCormack
Implicit WENO
$\mathrm{Log}_{10}(\|\rho_t\|_2)$
Iteration

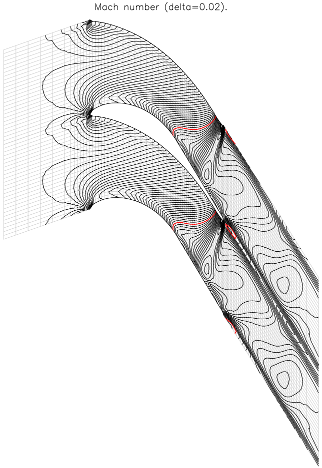

Lax-Friedrichs
Causon
Modified Causon
TVD MacCormack
Implicit WENO
Mach number
x

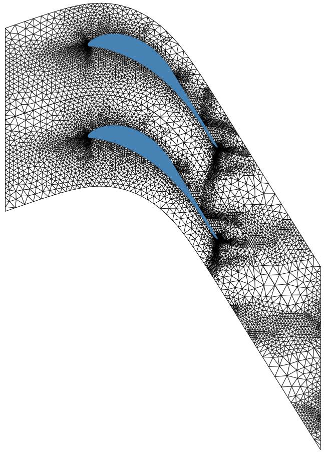

LF (REAL*4)
LF (REAL*8)
Causon
Modified Causon
TVD MacCormack
Implicit WENO
$\mathrm{Log}_{10}(\|\rho_t\|_2)$
CPU time (Intel P4 1.5GHz, RAM DDR266)

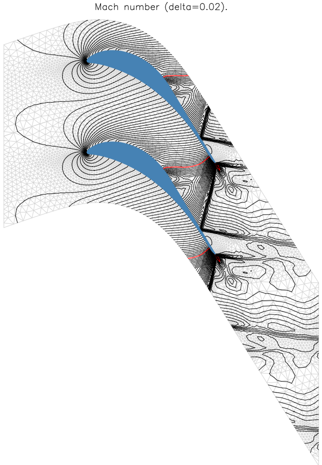
Inviscid flow through a turbine cascade

- **inlet Mach number $M_1 = 0.32$, inlet angle $\alpha_1 = 19.3^o$,**
- **outlet Mach number $M_2 = 1.18$.**





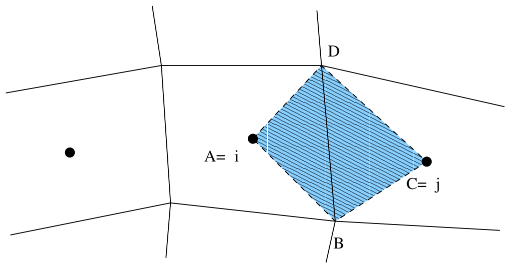




Inviscid flow through 3D channels and cascades

Convection diffusion problems

- **Model equation:** $u_t + f(u)_x + g(u)_y = \mu(u_{xx} + u_{yy})$.
- **Semi-implicit FVM:** $\frac{d\mathrm{u}_i(t)}{dt} + \frac{1}{|\Omega_i|}\sum_j |\Gamma_{ij}| \mathrm{f}(\mathrm{u}_i, \mathrm{u}_j, \mathbf{n}_{ij}) = \mu \sum_j |\Gamma_{ij}| \nabla \mathrm{u}_{ij} \cdot \mathbf{n}_{ij}$.
- **Evaluation of $\nabla \mathrm{u}_{ij}$:**



$$\nabla \mathrm{u}_{ij} = \frac{1}{|\Omega'_{ij}|}\int_{\Omega'_{ij}} \nabla u \, d\mathbf{x} = \frac{1}{|\Omega'_{ij}|}\int_{\partial\Omega'_{ij}} u\mathbf{n}\, dS \approx$$

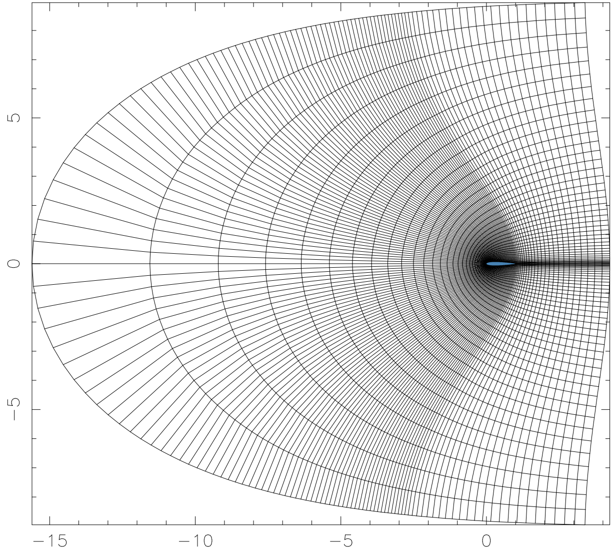
$$\approx \frac{1}{|\Omega'_{ij}|} \sum_{f \in \textbf{faces of } \Omega'_{ij}} \frac{\sum_{v \in \textbf{vertices of } f} \mathrm{u}_v}{dimension} \mathbf{n}_f |f|.$$

- **monotonicity condition for case of upwind scheme for linear eq.:**

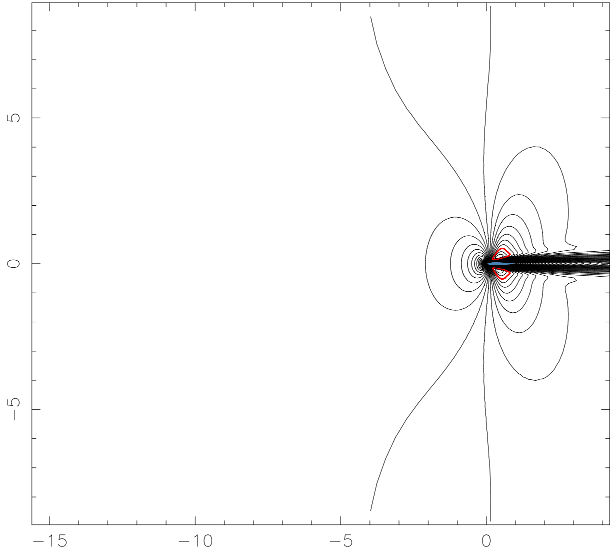
$$\Delta t \leq \frac{|\Omega_i|}{\sum_j |\Gamma_{ij}| \left[(\mathbf{a} \cdot \mathbf{n}_{ij})^+ + \frac{\mu}{(\mathbf{x}_j - \mathbf{x}_i) \cdot \mathbf{n}_{ij}} \right]}.$$

Laminar flow around NACA-0012 profile

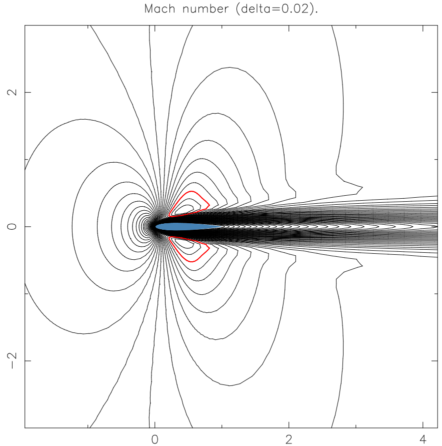
- $M_\infty = 0.85$, $\alpha_1 = 0^\circ$, $Re = 500$
- **structured mesh with** 168×40 **cells,** $\Delta y_1 \approx 0.005$**.**



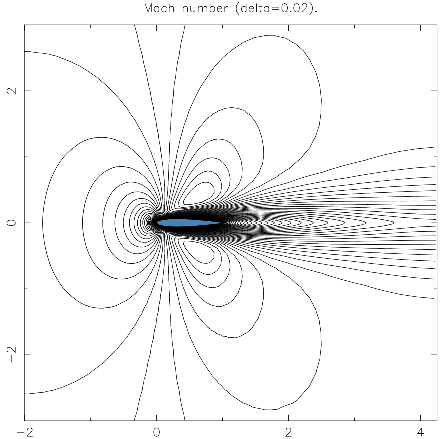
Mach number (delta=0.02).



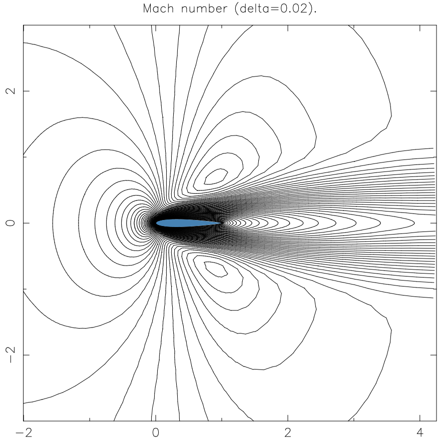
Laminar flow around NACA-0012 profile



Second order scheme, flow at $Re = 500$



First order scheme, flow at $Re = 500$



Second order scheme, flow at $Re = 50$

Viscous flow through a 3D channel