TURBULENCE MODELS
FOR INDUSTRIAL APPLICATIONS OF CFD

Jaromír Příhoda
Institute of Thermomechanics AS CR

ERCOFTAC Winter School

„Industrial and Environmental Applications of CFD Methods“

Contents

1. Introduction

- **Computational fluid dynamics** (CFD) - new scientific discipline
  - fast progress of computational technique (hardware and software)
  - demand for adequate simulation of flows in technical applications (mechanical and civil engineering, environment)
- **Numerical simulation of flow**
  - allows solution of many alternatives of projected machine and/r device
  - substitutes in some measure expensive experiments, necessary for design of machines and devices up to now
  - can not replace the need of experimental investigation of turbulent flows in any case (study of structure of turbulent flows, testing and verification of numerical models)
- **Computational programs**
  - ☐ **commercial software (FLUENT, CFX, Star-CD etc.)**
    - lead users to application without foregoing study of physical and mathematical models
    - general feature of these programs is robustness, i.e. ability to give results for any prescribed boundary conditions, obtained results can not always correspond to reality
  - ☐ **in-house programs developed at research institutions**
    - are usually fitted to solved problem, allow the solution of special tasks (for example laminar/turbulent transition, transonic flow with shock waves)

2. Physical and mathematical model of flow

- **Physical and mathematical model of fluid flow** – basis of each computational program
  - **Mathematical model**
    - numerical method, i.e. way of transformation of partial differential equations to system of algebraic equations - discretization method (finite difference method, finite volume method and/or finite element methods)
    - numerical schemes for discretization of individual terms in equations
    - choice and generation of grid
    - assignment of boundary conditions

  - **Physical model**
    - model of fluid (viscous or inviscid, newtonian or non-newtonian, laminar or turbulent, compressible or incompressible),
    - so called constitutive relations giving dependence between state variables (equation of state),
    - dependence of thermodynamic coefficients on state variables (for example Sutherland´s relation for dependence of viscosity on temperature)
    - turbulence model and model of laminar/turbulent transition respectively

  - **simplest and most used model** - laminar or turbulent flow of incompressible fluid without heat transfer

3. Fundamental equations of flow

basic system of movement equations is formed by so called conservation laws – conservation of mass, momentum, and energy

- **conservation of mass (equation of continuity)**

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho U_j\right) = 0$$

- **conservation of momentum (Navier-Stokes equations)**

$$\frac{\partial}{\partial t}\left(\rho U_i\right) + \frac{\partial}{\partial x_j}\left(\rho U_i U_j\right) = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i$$

total acceleration pressure friction volume
(local + convective) force force force

x_i - Cartesian coordinates (i=1,2,3)
t - time
U_i - velocity component in direction x_i
P - static pressure

ρ - fluid density
μ - dynamic viscosity
μ_b - so called bulk viscosity $\mu_b = 2/3\ \mu$

$$\tau_{ij} = \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) + \delta_{ij}\,\mu_b\,\frac{\partial U_k}{\partial x_k}$$ - tensor of viscous stresses

f_i - volume acceleration (for example gravitational acceleration)

- **conservation of energy**

$$\frac{\partial}{\partial t}\left(\rho h\right)+\frac{\partial}{\partial x_j}\left(\rho U_j h\right)=\frac{\partial P}{\partial t}+U_j\frac{\partial P}{\partial x_j}+\tau_{ij}\frac{\partial U_i}{\partial x_j}-\frac{\partial Q_j}{\partial x_j}$$

total change of enthalpy h	work of pressure forces	energy dissipation	heat transfer

$Q_j=-\lambda\frac{\partial T}{\partial x_j}$ - vector of heat flux (Fourier law)

h=c_pT	- enthalpy	c_p	- specific heat at constant pressure
T	- temperature	λ	- thermal conductivity

- **constitutive relations**

equation of state (for perfect gas) $\frac{P}{\rho}=RT$ R - perfect-gas constant

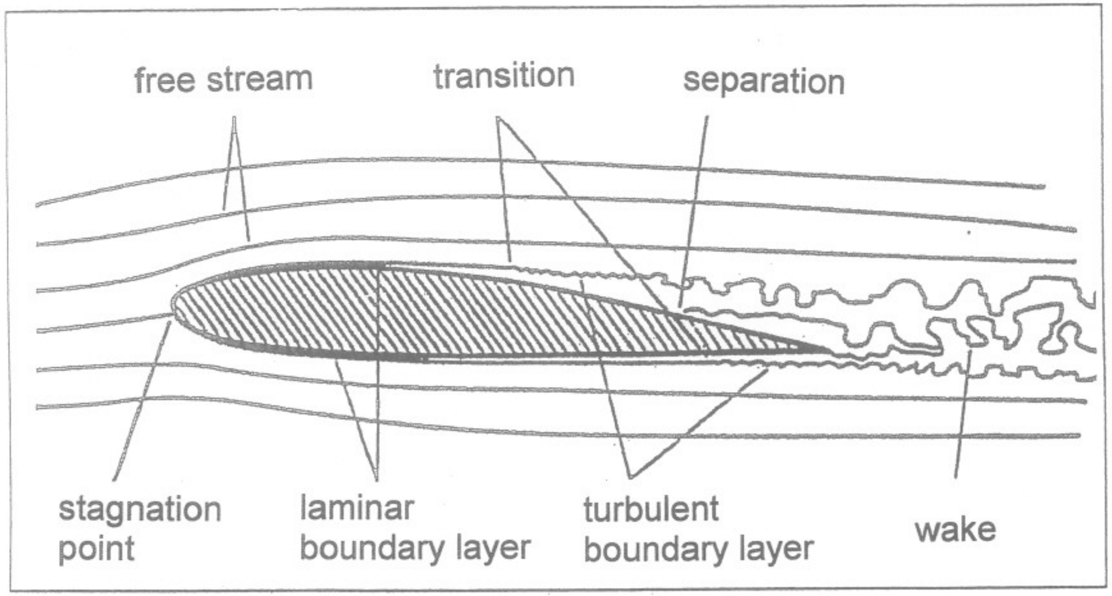
relations for thermodynamic parameters (fluid properties) $\mu,\lambda,c_p=f\left(P,T\right)$

- **simplest model** - flow of incompressible fluid without heat transfer (considered further)
  - **system of governing equations** - continuity equation and Navier-Stokes equations with constant density and viscosity
  - **laminar flow** - closed system of equations for unknown velocity components and pressure
  - **turbulent flow** - Navier-Stokes equations are valid for instantaneous values

4. Closure of equations of motion

a) separation of flow field into the free stream and boundary layer

- ☐ Prandtl (1904) – classical approach for fluids with very small viscosity ($Re=U_eL/\nu >> 1$)
- free stream - Euler equations (N-S equations with neglected viscous terms)
- boundary layer - Prandtl equations completed by a turbulence model (simplified N-S equations provided that $\delta \ll L$ a $\partial / \partial x \ll \partial / \partial y$)



b) solution of full Navier-Stokes equations

- validity of Navier-Stokes equations for turbulent flow – flow is variable in time and space (non-deterministic variability of flow parameters)

Solution of the full Navier-Stokes equations

i) direct numerical simulation (DNS – Direct Numerical Simulation)

- solution of unsteady Navier-Stokes equations using supercomputers
  - not very large Reynolds numbers, simple geometry (flow around flat plate, channel flow)
  - enormous demands on computer memory and speed (can be used for more complex boundary conditions)
- importance for the analysis of the structure of turbulent flow and for testing of turbulence models
  - numerical simulation allows determination of arbitrary parameters of turbulent flow

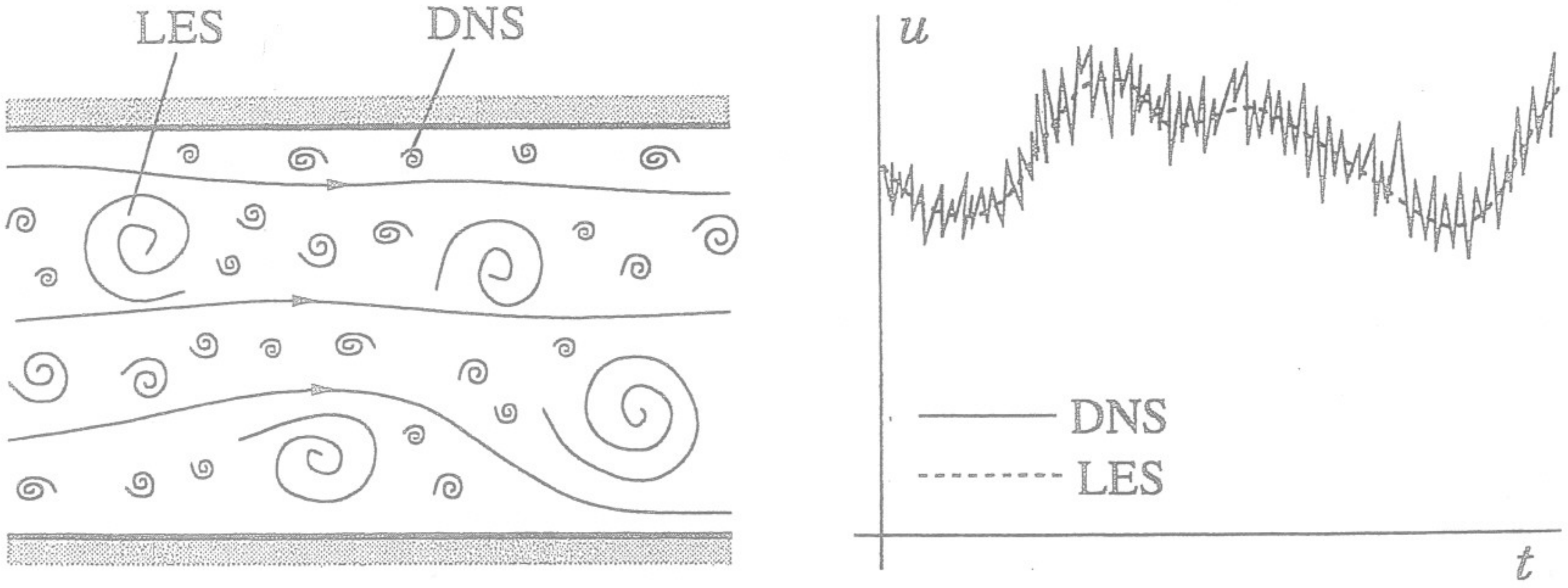
ii) solution of averaged Navier-Stokes equations – Reynolds equations

- statistical approach – instantaneous value as the sum of a mean and a fluctuating part
  - solution of steady Reynolds equations containing the tensor of Reynolds turbulent stresses
- closure of the system equations, i.e. suitable expression of Reynolds stresses is made using **a turbulence model** (statistical models)

iii) simulation of the motion of large eddies (LES – Large Eddy Simulations)

- combination of the both foregoing methods
  - solution of unsteady Navier-Stokes equations for large eddies and solution of averaged Navier-Stokes equations closed by turbulence model pro smallest eddies (so called „subgrid“ model)

Schematic illustration of turbulent motion



a) motion of turbulent eddies

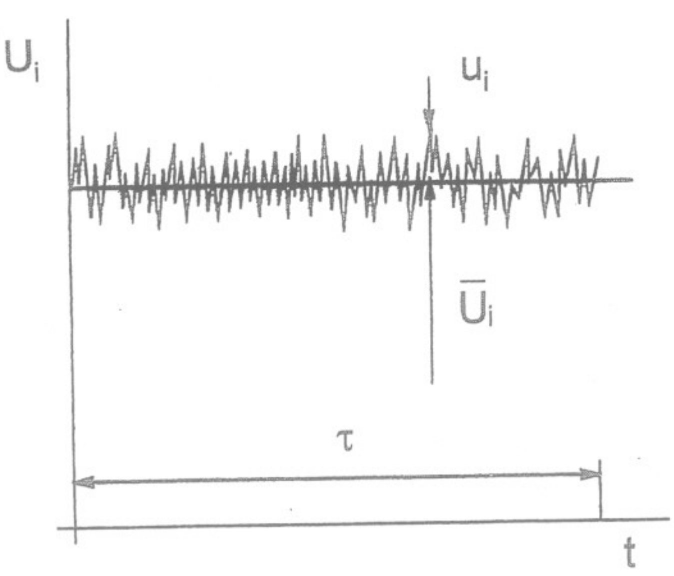
b) time behaviour of velocity component

DNS	- direct numerical simulation
LES	- simulation of large eddy motion

5. Statistical turbulence models

- **splitting into mean and fluctuating motion** - Reynolds (1895)

$$U_i = \overline{U}_i + u_i \qquad \overline{u}_i = 0 \qquad \overline{U}_i = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} U_i dt$$



- **averaged equations of motion**

continuity equation

$$\frac{\partial \overline{U}_i}{\partial x_i} = 0$$

Reynolds equations

$$\frac{\partial \overline{U}_i}{\partial t} + \overline{U}_j \frac{\partial \overline{U}_i}{\partial x_j} = g_i - \frac{1}{\rho} \frac{\partial \overline{P}}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\nu \frac{\partial \overline{U}_i}{\partial x_j} - \overline{u_i u_j} \right)$$

viscous **turbulent**
stress **stress**

- **equations of turbulent motion**

equation for the velocity fluctuation u_i

$$\frac{\partial u_i}{\partial t} + \overline{U}_k \frac{\partial u_i}{\partial x_k} + u_k \frac{\partial \overline{U}_i}{\partial x_k} + \frac{\partial \left(u_i u_k \right)}{\partial x_k} - \frac{\partial \overline{u_i u_k}}{\partial x_k} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_k^2} = 0$$

equations for components of turbulent stress

$$\frac{\partial\overline{u_i u_j}}{\partial t}+\overline{U}_k\frac{\partial\overline{u_i u_j}}{\partial x_k}=-\overline{u_k u_j}\frac{\partial\overline{U}_i}{\partial x_k}-\overline{u_k u_i}\frac{\partial U_j}{\partial x_k}-\frac{\partial}{\partial x_k}\left[\overline{u_i u_j u_k}+\overline{\frac{p}{\rho}\left(\delta_{ik}u_j+\delta_{jk}u_i\right)}-\nu\frac{\partial\overline{u_i u_j}}{\partial x_k}\right]-$$

(i) (ii) (iii)$_a$ (iii)$_b$

$$-2\nu\overline{\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\right)}+\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}$$

(iv) (v)

where are

(i) advection - transport of turbulent stress by mean flow

(ii) production - origin of turbulent stress by interaction with mean flow

(iii)$_a$ turbulent diffusion - transport of turbulent stress due to velocity and pressure fluctuations

(iii)$_b$ viscous diffusion - transport of turbulent stress due to viscosity

(iv) dissipation - transport of energy between turbulent and molecular motion

(v) redistribution - transport of energy between individual components of turbulent motion

approximation of marked terms is necessary for the closure of the system of governing equations

Ways of the closure of averaged Navier-Stokes equations

approximation of Reynolds stress

hypothesis on turbulent viscosity

$$-\overline{uv} = \nu_t \frac{\partial \overline{U}}{\partial y}$$

a) ν_t = f(mean motion)

$$\nu_t = L_m^2 \left| \frac{\partial \overline{U}}{\partial y} \right|$$

- mixing length L_m

b) ν_t = f(turbulent motion)

$$\nu_t = C_\mu V_t L_t$$

equations for Reynolds stress

a) transport equations for $\overline{u_i u_j}$

$$\frac{D\overline{u_i u_j}}{Dt} = P_{ij} + D_{ij} + \Pi_{ij} - \varepsilon_{ij}$$

b) algebraic equations for $\overline{u_i u_j}$

$$a_{ij} = \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij} = \mathrm{fce}\left(\frac{P_{ij}}{\varepsilon}, \frac{P_k}{\varepsilon}, S_{ij}, \Omega_{ij}\right)$$

characteristic scales

- velocity scale $V_t = \sqrt{k}$
- length scale L_t $\quad \varepsilon = \frac{k^{3/2}}{L_t}$

Turbulence models with turbulent viscosity

- ☐ Boussinesq (1897) – analogy with molecular momentum transfer

$$-\overline{uv} = \nu_t \frac{\partial \overline{U}}{\partial y}$$ - boundary layer

- ☐ Harlow, Nakayama (1967)

$$-\overline{u_i u_j} = \nu_t \left(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i} \right) - \frac{2}{3}\delta_{ij} k$$ - complex turbulent shear flow

a) determination of turbulent viscosity by means of mean motion - algebraic models

- ☐ Prandtl (1925) – theory of mixing length

$$\nu_t = L_m^2 \left| \frac{\partial \overline{U}}{\partial y} \right|$$ algebraic relation $L_m/\delta = f\left(y/\delta\right)$

- ☐ Cebeci, Smith (1968) – boundary layer
- ☐ Baldwin, Lomax (1978) – shear flow

- **inner region**

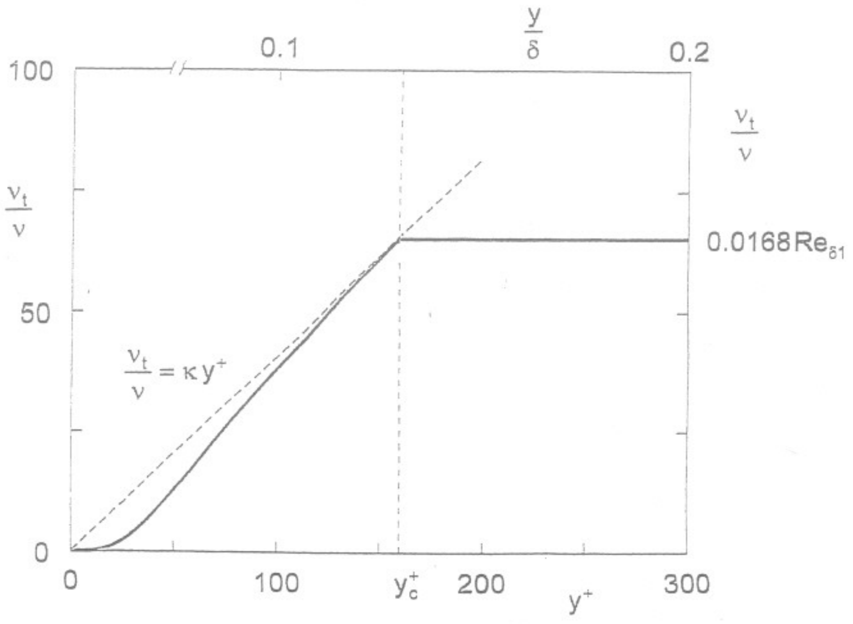
$$L_m = F_D \kappa y = \left[1 - \exp\left(-y^+/A\right)\right] \kappa y$$

- ☐ van Driest (1956) F_D - dumping function

- **outer region**

$$\nu_t = \alpha F_k \delta_1 U_e = \alpha \left[1 + 5.5\left(y/\delta\right)^6\right]^{-1} \delta_1 U_e$$

- ☐ Klebanoff (1954) F_k - intermittence function



b) determination of turbulent viscosity by means of turbulent motion

- Prandtl (1945) turbulent viscosity $\nu_t = C_\mu V_t L_\mu$

- **velocity scale** $V_t = \sqrt{k}$

transport equation for turbulent energy $k = \frac{1}{2}\overline{u_i u_i}$

$$\frac{\partial k}{\partial t} + \bar{U}_k \frac{\partial k}{\partial x_k} = P_k + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_k}\right] - \varepsilon$$

advection production viscous diffusion turbulent diffusion dissipation

$$P_k = -\overline{u_i u_j}\frac{\partial \bar{U}_i}{\partial x_j} = \left[\nu_t\left(\frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}k\right]\frac{\partial \bar{U}_i}{\partial x_j}$$

- production of turbulent energy

- **length scale**

- ***algebraic relation*** **(one-equation model)**

  usually used near the wall only (two-layer model)

- Chen, Patel (1988)

$$L_\mu = C_\mu y\left[1 - \exp\left(-\mathrm{Re}_y / A_\mu\right)\right] \qquad L_\varepsilon = C_L y\left[1 - \exp\left(-\mathrm{Re}_y / A_\varepsilon\right)\right]$$

$$\varepsilon = \frac{k^{3/2}}{L_\varepsilon} \qquad \mathrm{Re}_y = \frac{\sqrt{k}\, y}{\nu} \qquad A_\mu = 70 \qquad A_\varepsilon = 5$$

- ***transport equations* - so called two-equation models**
  - length scale is usually determined by the dissipation rate
  - frequently used k-ε model
- Rotta (1951) - assumption on local symmetry of the smallest eddies

$$\varepsilon = \nu\overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2} = \frac{k^{3/2}}{L_t}$$

turbulent viscosity $\nu_t = C_\mu \frac{k^2}{\varepsilon}$

- Launder, Spalding (1974)

transport equation for the dissipation rate ε

$$\frac{\partial \varepsilon}{\partial t} + \bar{U}_k \frac{\partial \varepsilon}{\partial x_k} = C_{\varepsilon 1}\frac{\varepsilon}{k}P_k + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_k}\right] - C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

model constants $C_\mu = 0.09 \quad C_{\varepsilon 1} = 1.45 \quad C_{\varepsilon 2} = 1.9 \quad \sigma_k = 1.0 \quad \sigma_\varepsilon = 1.3$

  - basic version of the model for large turbulent Reynolds numbers $Re_t = \frac{\nu_t}{\nu} \gg 1$
  - model is valid in the certain distance from the wall only (for $y \to 0$ is $Re_t \to 0$)

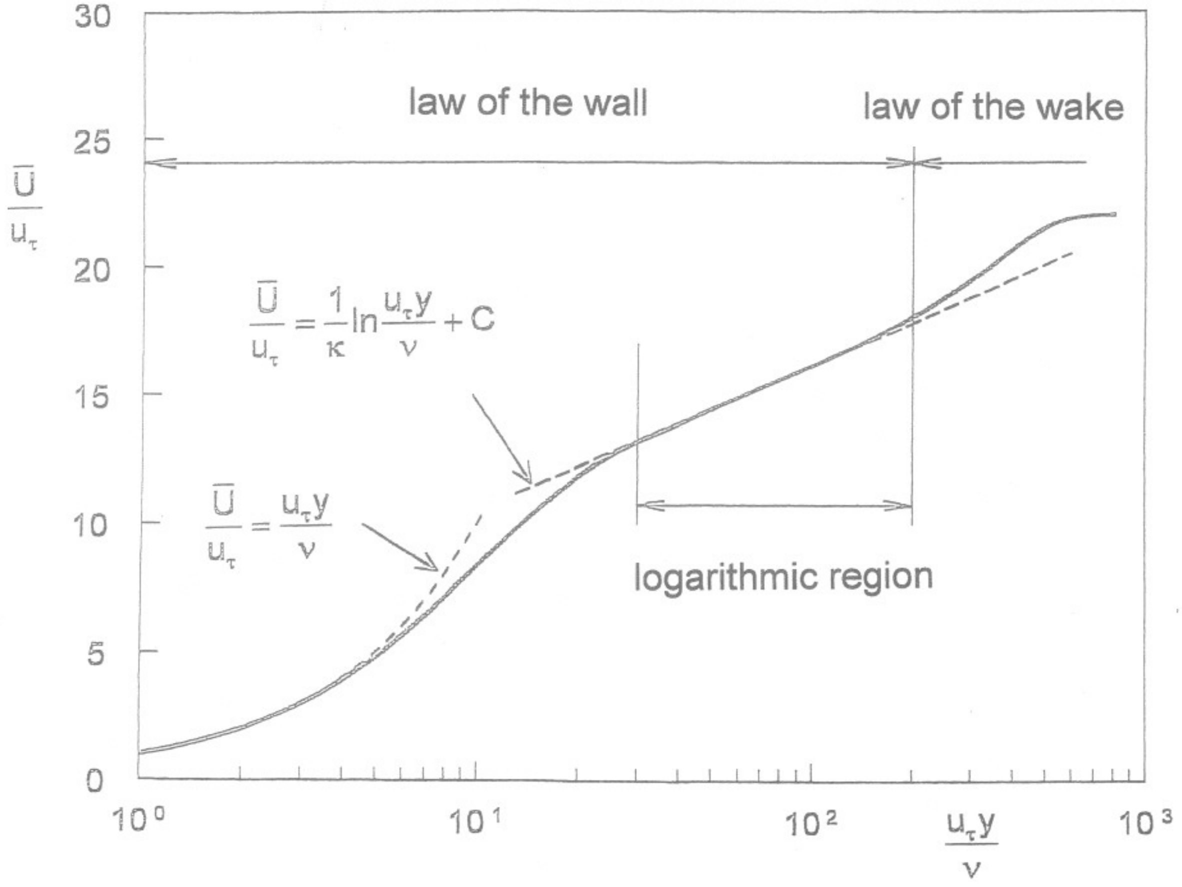
Application of turbulence models near walls (for small values Re_t)

a) Application of universal features of the boundary layer - so called **wall functions**

- boundary conditions on the wall ($y = 0$) replaced by boundary conditions in the point y_c

Mean velocity profile in the boundary layer

$$30 < y_c^+ = \frac{u_\tau y_c}{\nu} < 200$$



universal law of the wall

$$\frac{\bar{U}}{u_\tau} = f\left(\frac{u_\tau y}{\nu}\right)$$

friction velocity

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

Universal features of turbulent flow in the logarithmic region

- analysis of the boundary layer on the flat plate

experiment – Klebanoff (1954)
direct numerical simulation – Spalart (1988)

$$\frac{\overline{U}}{u_\tau} = \frac{1}{\kappa}\ln\frac{u_\tau y}{\nu} + C \qquad \frac{\partial\overline{U}}{\partial y} = \frac{u_\tau}{\kappa y} \qquad \kappa=0.41 \qquad C=5 \div 5.2$$

- turbulent shear stress

$$-\overline{uv} = \nu_t \frac{\partial\overline{U}}{\partial y} \approx u_\tau^2 \quad ➔ \quad \nu_t = u_\tau \kappa y$$

- assumption of the balance of turbulent energy (production = dissipation rate)

$$P_k = -\overline{uv}\frac{\partial\overline{U}}{\partial y} \approx \varepsilon \quad ➔ \quad \varepsilon = \frac{u_\tau^3}{\kappa y}$$

- turbulent viscosity

$$\nu_t = C_\mu \frac{k^2}{\varepsilon} \quad ➔ \quad k = \frac{u_\tau^2}{\sqrt{C_\mu}}$$

- ratio of turbulent stress and turbulent energy

$$\frac{-\overline{uv}}{k} \approx 0.3 \quad ➔ \quad C_\mu = 0.09$$

b) Modification of the model for low turbulent Reynolds numbers

- ❑ Launder, Sharma (1974), Patel, Rodi, Scheuerer (1985)
  - original model constants depend on the turbulent Reynolds number Re_t

    so called **dumping functions** – express the effect of the vicinity of the wall on turbulent fluctuations
  - non-isotropic character of dissipation near the wall is supposed

$$\varepsilon = \underbrace{\tilde{\varepsilon}}_{\text{isotropic part of dissipation rate}} + \underbrace{D}_{\text{correction}}$$

- turbulent viscosity

$$\nu_t = C_\mu f_\mu \frac{k^2}{\tilde{\varepsilon}}$$

- transport equation for turbulent energy

$$\frac{\partial k}{\partial t} + \bar{U}_k \frac{\partial k}{\partial x_k} = P_k + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_k}\right] - \varepsilon$$

- transport equation for dissipation rate

$$\frac{\partial \tilde{\varepsilon}}{\partial t} + \bar{U}_k \frac{\partial \tilde{\varepsilon}}{\partial x_k} = C_{\varepsilon 1} f_1 \frac{\tilde{\varepsilon}}{k} P_k + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \tilde{\varepsilon}}{\partial x_k}\right] - C_{\varepsilon 2} f_2 \frac{\tilde{\varepsilon}^2}{k} + E$$

Modification of the model for low turbulent Reynolds numbers (cont.)

- dumping functions – express the effect of the wall on turbulent velocity fluctuations

$$f_\mu = \exp\left[-\frac{3.4}{(1+Re_t/50)^2}\right] \qquad f_1=1 \qquad f_2 = 1-0.3\exp\left(-Re_t^2\right)$$

- correction functions

$$D = 2\nu\left(\frac{\partial\sqrt{k}}{\partial y}\right)^2 \qquad\qquad E = 2\nu\nu_t\left(\frac{\partial^2 U}{\partial y^2}\right)^2$$

increases dissipation rate near the wall	increases production of dissipation rate (decreases the maximum of turbulent energy)

- model constants C_μ=0.09 $C_{\varepsilon 1}$=1.44 $C_{\varepsilon 2}$=1.92 σ_k=1 σ_ε=1.3

- boundary conditions on the wall (y=0) $\overline{U}_i = k = \tilde{\varepsilon} = 0$

  - with growing distance from the wall $f_\mu, f_1, f_2 \to 1$ $D,E \to 0$ (basic version of k-ε model)
  - application of the model needs at least 80-100 grid points across the boundary layer

c) Two-layer turbulence models – combination of two models

i) combination of k-L/k-ε turbulence models

- ❑ Chen, Patel (1988)

- **one-equation k-L model near the wall**

turbulent viscosity $\nu_t = C_\mu \sqrt{k} L_\mu$

transport equation for turbulent energy

$$\frac{\partial k}{\partial t} + \bar{U}_k \frac{\partial k}{\partial x_k} = P_k + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_k}\right] - \varepsilon$$

dissipation rate

$$\varepsilon = \frac{k^{3/2}}{L_\varepsilon}$$

algebraic relation for length scales

$$L_\mu = C_L y \left[1 - \exp\left(-\mathrm{Re}_y / 70\right)\right] \qquad L_\varepsilon = \frac{C_L y}{1 + 5.3/\mathrm{Re}_y} \quad \mathrm{Re}_y = \frac{\sqrt{k}\, y}{\nu}$$

Norris, Reynolds (1975)

$C_L = 2.5$

- **standard k-ε model far from the wall**
- **linking of both models**

❑ Rodi (1991) $\frac{\nu_t}{\nu} = 30$ or $F_\mu = 1 - \exp\left(-\mathrm{Re}_y / A_\mu\right) = 0.95$ i.e. $\mathrm{Re}_y = 210$

  - combination of two models gives good results for flows with separation as well
  - model needs less grid points across the boundary layer (at least 40 points)

ii) combination k-ω/k-ε turbulence models

- application of dissipation rate for the determination of the length scale near the wall is not advantageous

❑ Wilcox (1988) - so called specific dissipation rate $\omega = \varepsilon / k$

- **k-ω model**
- turbulent viscosity

$$\nu_t = \frac{k}{\omega}$$

- transport equation for turbulent energy

$$\frac{\partial k}{\partial t} + \bar{U}_j \frac{\partial k}{\partial x_j} = P_k - \beta^* \omega k + \frac{\partial}{\partial x_j}\left[\left(\nu + \sigma_k \nu_t\right)\frac{\partial k}{\partial x_j}\right]$$

- transport equation for specific dissipation rate

$$\frac{\partial \omega}{\partial t} + \bar{U}_j \frac{\partial \omega}{\partial x_j} = \alpha_\omega \frac{\omega}{k} P_k - \beta \omega^2 + \frac{\partial}{\partial x_j}\left[\left(\nu + \sigma_\omega \nu_t\right)\frac{\partial \omega}{\partial x_j}\right] + \left[C_D\right]$$

original k-ω model - gives acceptable results near walls even without dumping functions

- is sensitive to prescription of boundary conditions for ω in free stream

- **modification of k-ω model**

❑ Kok (2000) – term C_D expresses the cross diffusion

$$C_D = \sigma_d \frac{1}{\omega} \max\left\{\frac{\partial k}{\partial x_i}\frac{\partial \omega}{\partial x_i}, 0\right\}$$

ii) combination of k-ω/k-ε turbulence models (cont.)

❑ Menter (1994) - two-layer k-ω/k-ε model (in the form of k-ω model)
– combination of advantages of both models

- **BSL (baseline) model**

turbulent viscosity $$\nu_t = \frac{k}{\omega}$$

  - k-ω model near the wall
  - standard k-ε model in free stream (transformed because of the simpler numerical realisation into the form of k-ω model using the blending function F_1)

- **SST (shear stress transport) model**

turbulent viscosity - includes the transport of turbulent shear stresses

$$\nu_t = \frac{a_1 k}{\max\left(a_1\omega; F_2 \frac{\partial \bar{U}}{\partial y} \right)}$$

❑ Bradshaw (1967) - constant $a_1 = 0.3$ (in the boundary layer $a_1 = -\overline{uv}/k$)

- **model constants** $\Phi = F_1\Phi_1 + \left(1 - F_1\right)\Phi_2$

Φ - constant of the new model
Φ_1 - constant of the original k-ω model
Φ_2 - constant of the transformed k-ε model

- **blending functions F_1 and F_2**

$F_1 = F_2 = 1$ near the wall, $F_1 = F_2 = 0$ in free stream

Turbulence models with equations for Reynolds stresses

a) models with transport equations for Reynolds stresses

- transport equation for Reynolds stresses

$$\frac{D\overline{u_i u_j}}{Dt} = P_{ij} + D_{ij} + \Phi_{ij} - \varepsilon_{ij}$$

- Launder, Reece and Rodi (1975), Hanjalic, Launder (1976)

- approximation of unknown terms

turbulent diffusion - so called gradient approximation (analogy with viscous diffusion)

$$\overline{u_i u_j u_k} = -\text{konst.}\frac{k}{\varepsilon}\overline{u_k u_l}\frac{\partial \overline{u_i u_j}}{\partial x_l}$$

dissipation – conversion of turbulent energy in heat (assumption of local isotropy of smallest eddies)

- Rotta (1951)

$$\varepsilon_{ij} = 2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} = \frac{2}{3}\delta_{ij}\varepsilon \qquad \text{where is} \qquad \varepsilon = \nu\overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2} = \frac{k^{3/2}}{L_\varepsilon}$$

redistrition – exchange of energy between individual components of velocity fluctuations

$$\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} = -C_1\frac{\varepsilon}{k}\left(\overline{u_i u_j} - \frac{2}{3}\delta_{ij}k\right) - C_2\left(P_{ij} - \frac{2}{3}\delta_{ij}P_k\right)$$

tendency to isotropy
of turbulent stresses

tendency to isotropy
of production of turbulent stresses

a) models with transport equations for Reynolds stresses (cont.)

Launder, Reece a Rodi (1975)

$$\frac{D\overline{u_i u_j}}{Dt} = -\left(\overline{u_j u_k} \frac{\partial \bar{U}_i}{\partial x_k} + \overline{u_i u_k} \frac{\partial \bar{U}_j}{\partial x_k} \right) + C_s \frac{\partial}{\partial x_k} \left(\frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l} \right) - \frac{2}{3} \delta_{ij} \varepsilon - C_1 \frac{\varepsilon}{k} \left(\overline{u_i u_j} - \frac{2}{3} \delta_{ij} k \right) - C_2 \left(P_{ij} - \frac{2}{3} \delta_{ij} P_k \right)$$

equation for turbulent energy k - velocity scale

$$\frac{Dk}{Dt} = -\overline{u_k u_l} \frac{\partial \bar{U}_k}{\partial x_l} + C_s \frac{\partial}{\partial x_k} \left(\frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial k}{\partial x_l} \right) - \varepsilon$$

equation for dissipation rate ε - length scale $\varepsilon = \frac{k^{3/2}}{L_\varepsilon}$

$$\frac{D\varepsilon}{Dt} = -C_{\varepsilon 1} \frac{\varepsilon}{k} \overline{u_k u_l} \frac{\partial \bar{U}_k}{\partial x_l} + C_\varepsilon \frac{\partial}{\partial x_k} \left(\frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial \varepsilon}{\partial x_l} \right) - C_{\varepsilon 2} \frac{\varepsilon^2}{k}$$

model constants

$C_1 = 1.5$	homogeneous turbulence	$C_{\varepsilon 1} = 1.45$	logarithmic law of the wall
$C_2 = 0.6$	homogeneous turbulence	$C_{\varepsilon 2} = 2.0$	decay of turbulence behind a grid
$C_s = 0.1$	numerical optimization	$C_\varepsilon = 0.13$	numerical optimization

b) models with algebraic equations for Reynolds stresses

- assumption of small changes of the ratio of Reynolds stresses and turbulent energy

$$a_{ij} = \frac{\overline{u_i u_j}}{k} - \frac{2}{3}\delta_{ij} \approx \text{konst.}$$ a_{ij} - parameter of asymmetry of turbulent stresses

from transport equations for turbulent stresses $\overline{u_i u_j}$ and for turbulent energy k

$$\frac{D\overline{u_i u_j}}{Dt} - D_{ij} = \frac{\overline{u_i u_j}}{k}\left(\frac{Dk}{Dt} - D_k\right) \quad ➔ \quad \frac{u_i u_j}{k}\left(P_k - \varepsilon\right) = P_{ij} - \varepsilon_{ij} + \Phi_{ij}$$

(advection – diffusion)$_{ij}$ ≈ (advection – diffusion)$_k$

model depends on the approximation of redistribution Φ_{ij}

❑ **implicit model** - Rodi (1976)

$$\Phi_{ij} = -C_1\frac{\varepsilon}{k}\left(\overline{u_i u_j} - \frac{2}{3}\delta_{ij}k\right) - C_2\left(P_{ij} - \frac{2}{3}\delta_{ij}P_k\right)$$ Launder, Reece a Rodi (1975)

$$\frac{\overline{u_i u_j}}{k} = \frac{2}{3}\delta_{ij} + \frac{1-C_2}{C_1}\frac{\frac{P_{ij}}{\varepsilon} - \frac{2}{3}\delta_{ij}\frac{P_k}{\varepsilon}}{1 + \frac{1}{C_1}\left(\frac{P_k}{\varepsilon} - 1\right)}$$ - difficult numerical solution

$$P_{ij} = -\overline{u_k u_j}\frac{\partial \overline{U}_i}{\partial x_k} - \overline{u_k u_i}\frac{\partial U_j}{\partial x_k}$$ - production of turbulent stresses

$$P_k = -\overline{u_i u_j}\frac{\partial \overline{U}_i}{\partial x_j}$$ - production of turbulent energy

b) models with algebraic equations for Reynolds stresses (cont.)

☐ **explicit model** - Gatski, Speziale (1993)

$$\Phi_{ij} = -C_1 \varepsilon a_{ij} + C_2 k\, 2 S_{ij} + C_3 k \left(a_{ik} S_{jk} + a_{jk} S_{ik} - \frac{2}{3} \delta_{ij} a_{kl} S_{kl} \right) + C_4 k \left(a_{ik} \Omega_{jk} + a_{jk} \Omega_{ik} \right)$$

$$S_{ij} = \frac{1}{2} \left(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i} \right)$$ - strain-rate tensor

$$\Omega_{ij} = \frac{1}{2} \left(\frac{\partial \overline{U}_i}{\partial x_j} - \frac{\partial \overline{U}_j}{\partial x_i} \right)$$ - rotation tensor

$$a_{ij} = G_1 S_{ij} + G_2 \left(S_{ik} \Omega_{kj} + S_{jk} \Omega_{ki} \right) + G_3 \left(S_{ik} S_{kj} - \frac{1}{3} \delta_{ij} S_{mn} S_{nm} \right)$$

with parameters $G_1, G_2, G_3 = f\left(\frac{k}{\varepsilon}, \frac{P_k}{\varepsilon}, S_{ij} S_{ij}, \Omega_{ij} \Omega_{ij} \right)$

- determination $\frac{P_k}{\varepsilon}$ – either $\frac{P_k}{\varepsilon} \approx$ const. (equilibrium state) or by means of turbulent viscosity

- both models are completed by transport equations for turbulent energy and for dissipation rate

$$\frac{Dk}{Dt} = -\overline{u_k u_l} \frac{\partial \overline{U}_k}{\partial x_l} + C_s \frac{\partial}{\partial x_k} \left(\frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial k}{\partial x_l} \right) - \varepsilon$$

$$\frac{D\varepsilon}{Dt} = -C_{\varepsilon 1} \frac{\varepsilon}{k} \overline{u_k u_l} \frac{\partial \overline{U}_k}{\partial x_l} + C_\varepsilon \frac{\partial}{\partial x_k} \left(\frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial \varepsilon}{\partial x_l} \right) - C_{\varepsilon 2} \frac{\varepsilon^2}{k}$$

6. Examples of solution of turbulent shear flows

Solution of averaged Navier-Stokes equations using CFX software

a) *Two-dimensional wall jet on a circular cylinder* (Příhoda, Sedlář 2002)

Three-dimensional rectangular configuration (slot height $h/R=0.04$, slot aspect ratio $b/h=50$)

- ***Turbulence models***
  - standard k-ε model
  - Menter´s two-layer BSL k-ω/k-ε model
  - Menter´s SST model modelling transport of turbulent shear stress
- ***Computation domain***
  - divided into several zones
  - structured grids with local refinement (near walls and zone boundaries), 134325 nodes
- ***Investigation of the effect of the Reynolds number*** ($Re_h=U_v h/\nu$ from 3500 to 21000)

  Coanda effect – the difference between the wall pressure and the ambient pressure
  due to streamline curvature
- ***Two flow regimes***

  subcritical – distribution of wall jet parameters dependent on Re_h
  supercritical – only very slight dependence on Re_h
  position of flow separation approaches the value $\alpha_{sep} \approx 250$ deg

Solution of averaged Navier-Stokes equations using CFX software (cont.)

b) ***Turbulent flow in curved channel of the squared cross-section*** (Příhoda, Sedlář 2003)

- ***Channel of squared cross-section*** a x a = 0.2 x 0.2 m
  - inlet part (L_1=15a)
  - bend (curvature angle 90 deg, curvature radius R/a=1)
  - outlet part (L_2=23a)
- ***Simulations for water flow*** in the range Um=0.5 up to 2 m/s (Re= 97000 up to 388000)
- ***Turbulence models***
  - standard k-ε model
  - Menter´s k-ω/k-ε SST model
  - RSM model proposed by Launder, Reece and Rodi (1975)
- ***Structured grid*** refined in the bend and near all walls, especially near the inner wall where separation occurs
- ***Effect of the Reynolds number*** on the development of secondary flows in curved channel and on the position and extent of flow separation
- ***Determination of energy losses*** and their main causes (streamline curvature, secondary flow, separation)
- ***Comparison with experiments*** in a water channel using PIV method and with data for energy losses – best results obtained for two-layer SST model

Important years for Computational Fluid Dynamics

Navier (1827), Stokes (1845)	- derivation of Navier-Stokes equations
Boussinesq (1877)	- hypothesis of turbulent viscosity
Reynolds (1895)	- averaging of Navier-Stokes equations
Prandtl (1904)	- boundary layer theory
Prandtl (1925)	- mixing length theory (algebraic model)
Kolmogorov (1942)	- two-equation turbulence model
Chou (1945), Rotta (1951)	- foundations of Reynolds Stress Models
Smagorinski (1963), Deardorff (1970)	- foundations of Large Eddy Simulation models
Launder, Spalding (1972)	- modern two-equation k-ε turbulence model
Pope (1975)	- foundations of algebraic RSM model
Launder, Reece, Rodi (1976)	- modern RSM model
Rodi (1976)	- algebraic RSM model (implicit)
Moin, Kim (1982)	- application of LES (2D channel)
Spalart (1988)	- application of DNS (2D flat plate)
Speziale, Sarkar, Gatski (1991)	- algebraic RSM model (explicit)

7. Conclusion

- The aim of the lecture was to demonstrate the fundamentals of current methods for modelling of turbulent flows with the emphasis on the physical assumptions used in models and on the possibilities of their practical application.
- The numerical simulation of flow allows the solution of complicated cases of turbulent flow important for technical applications not only in the field of construction of machines and devices (especially in turbomachinery and aeronautics) but in environment and biomechanics as well.
- Existing models are only an approximation of turbulent flow (much or less appropriate) – their validity is restricted to a certain category of flow fulfilling used assumptions and corresponding to used model constants.
- The most general turbulence model should not be always the most suitable one – more complicated models desirable require the assignment of more boundary conditions (often unknown).
- The used mathematical and physical models participate in the success of numerical simulation of turbulent flow by the same part – collaboration of specialists of both branches is a necessary condition for success (but not sufficient).