# Source terms and well balanced schemes for balance laws. Application to the shallow water simulation

Philippe HELLUY, Nicolas SEGUIN

ANAM/MNC, ISITV, Université de Toulon et du Var

# Systems of balance laws

We are interested in systems of partial differential equations of the form

$$\partial_t \mathbf{w}(t,x) + \mathrm{div}_x \mathbf{f}(a(x), \mathbf{w}(t,x)) = \mathbf{s}(a(x), \mathbf{w}(t,x)) \nabla a(x),$$

where $\mathbf{w} : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}$ is the vector of the unknowns
and $a : \mathbb{R}^d \mapsto \mathbb{R}$ is a given function.

We restrict this presentation to the one-dimensional case $d = 1$:

$$\partial_t \mathbf{w} + \partial_x \mathbf{f}(a(x), \mathbf{w}) = \mathbf{s}(a(x), \mathbf{w}) a'(x).$$

We assume that the flux $\mathbf{f}$ and the source $\mathbf{s}$ are smooth functions and that

$$\partial_t \mathbf{w} + \partial_x \mathbf{f}(a(x), \mathbf{w}) = 0$$

is hyperbolic, that is $\partial_{\mathbf{w}} \mathbf{f}$ is diagonalizable in $\mathbb{R}$.

# Some examples of balance laws

A water-oil mixture in a porous medium:

$$\partial_t u + \partial_x (a(x) f(u)) = 0.$$

Gas dynamics in a nozzle with a space-depending cross-section:

$$\partial_t \rho + \partial_x (\rho u) = -\frac{\rho u}{a} a'(x),$$
$$\partial_t (\rho u) + \partial_x (\rho u^2 + p(\rho)) = -\frac{m^2}{a\rho} a'(x).$$

Shallow water flows over a non-flat topography:

$$\partial_t h + \partial_x (hu) = 0,$$
$$\partial_t (hu) + \partial_x (hu^2 + h^2/2) = -gha'(x).$$

# Numerical simulation of balance laws

Numerical difficulties:

- upwinding with respect to the source term,
- *stiff* source terms:
  characteristic time of the source $\ll$ time step for the hyperbolic part,
- robustness of the scheme (vacuum zones),
- entropy satisfying ($\partial_t S(a, \mathbf{w}) + \partial_x F(a, \mathbf{w}) \leq 0$),
- preservation of the *stationary states*, that are smooth parts verifying

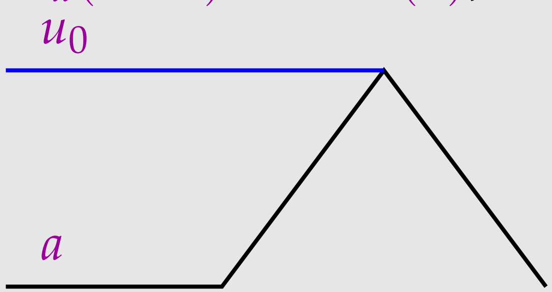
$$\partial_x \mathbf{f}(a(x), \mathbf{w}) = \mathbf{s}(a(x), \mathbf{w}) a'(x),$$

separated by stationary entropic shock waves

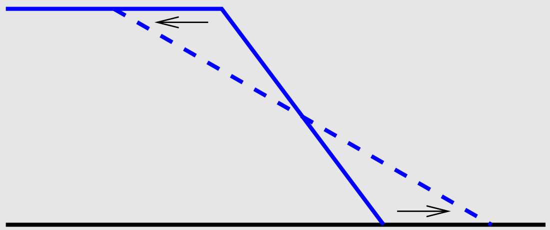
$$\mathbf{f}(a(x_0), \mathbf{w}(x_0^+)) - \mathbf{f}(a(x_0), \mathbf{w}(x_0^-)) = 0.$$

# The splitting method

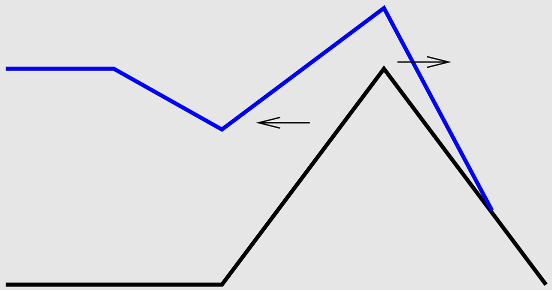
Let us study the case $\partial_t u + \partial_x(u^2/2) = -ua'(x)$, with




First, we solve the homogeneous equation $\partial_t u + \partial_x(u^2/2) = 0$:



Second, we solve the differential equation $\partial_t u = -ua'(x)$:



# The well-balanced scheme

The *well-balanced scheme* corresponds to the extension of the Godunov scheme [Godunov 59] for (systems of) balance laws, it is based on solving a nonconservative Riemann problem at each interface of the mesh.

[Isaacson, Temple 95]: projection/convection point of view for the study of the scalar balance law $\partial_t u + \partial_x f(a,u) = s(a,u)a'(x)$.

[Greenberg, LeRoux 96], [Gosse 96]: “Finite Volume” interpretation and well-balanced property.

[LeRoux 98]: extension to systems of balance laws.

[Goatin, LeFloch 03]: theoretical study of the Riemann problem for systems of balance laws.

# The well-balanced scheme: the projection/convection interpretation

For $x \in \mathbb{R}$, this scheme may be interpreted as a projection/convection algorithm [Isaacson, Temple 95].

The evolution from $t^n$ to $t^{n+1}$ follows the scheme:

1. projection $L^2$ of $(a, \mathbf{w})$ on the space of piecewise functions which are constant on the cells $(x_{i-1/2}, x_{i+1/2})$ of the mesh, $\longrightarrow (a_i, \mathbf{w}_i^n)_{i \in \mathbb{Z}}$.
2. exact solving of the Cauchy problem

$$\begin{cases} \partial_t \mathbf{w} + \partial_x \mathbf{f}(a, \mathbf{w}) = \mathbf{s}(a, \mathbf{w}) a'(x), \quad t \in (t^n, t^{n+1}), x \in \mathbb{R}, \\ \mathbf{w}(t^n, x) = \mathbf{w}_i^n \text{ for } x \in (x_{i-1/2}, x_{i+1/2}), \\ a(x) = a_i \text{ for } x \in (x_{i-1/2}, x_{i+1/2}). \end{cases}$$

This problem is composed by local Riemann problems at each interface $x_{i+1/2}$ and the time step $\Delta t = t^{n+1} - t^n$ must prevent their interaction.

# The well-balanced scheme - the Finite Volume interpretation

The well-balanced scheme may be written under the form

$$\mathbf{w}_i^{n+1} = \mathbf{w}_i^n - \frac{\Delta t}{\Delta x}\big(\mathbf{f}(a_i, \mathbf{w}_{i+1/2}^n(0^-)) - \mathbf{f}(a_i, \mathbf{w}_{i-1/2}^n(0^+))\big),$$

where $\mathbf{w}_{i+1/2}^n(x/t)$ is the self-similar solution of the Riemann problem

$$\begin{cases} \partial_t \mathbf{w} + \partial_x \mathbf{f}(a, \mathbf{w}) = \mathbf{s}(a, \mathbf{w}) a'(x), \quad t \in (t^n, t^{n+1}), x \in \mathbb{R}, \\ (a(x), \mathbf{w}(t^n, x)) = \begin{cases} (a_i, \mathbf{w}_i^n) & \text{if } x < x_{i+1/2}, \\ (a_{i+1}, \mathbf{w}_{i+1}^n) & \text{if } x > x_{i+1/2}. \end{cases} \end{cases}$$

The source term is only concentrated at each interface $x_{i+1/2}$ and we have

$$\mathbf{f}(a_i, \mathbf{w}_{i+1/2}^n(0^-)) \neq \mathbf{f}(a_{i+1}, \mathbf{w}_{i+1/2}^n(0^+)).$$

In fact, the equality of conservation holds if either $a_i = a_{i+1}$ or $\mathbf{s}(\mathbf{w}) \equiv 0$.

# Properties of the well-balanced scheme

**Proposition.** If the solution of the Riemann problem globally exists for all initial data $(a_L, \mathbf{w}_L)$ and $(a_R, \mathbf{w}_R)$, then, under the CFL condition

$$\Delta t \leq \frac{\Delta x}{2\lambda_{max}}, \quad \text{where } \lambda_{max} \text{ is the maximal speed of waves,}$$

the well-balanced scheme

- verifies a discrete version of the entropy inequality
  $\partial_t S(a, \mathbf{w}) + \partial_x F(a, \mathbf{w}) \leq 0$,
- verifies the classical discrete maximum principles (for the saturation $u$, the density $\rho$, the height of water $h$, ...),
- maintains all the discrete steady states (smooth stationary parts, stationary entropic shocks, ...).

# The Riemann problem

The datum $a$ is considered as an additional “unknown” and the system with source term becomes a nonconservative system of PDE.
We want to solve the following Riemann problem:

$$\begin{cases} \partial_t a = 0, \\ \partial_t \mathbf{w} + \partial_x \mathbf{f}(a, \mathbf{w}) - \mathbf{s}(a, \mathbf{w})\partial_x a = 0, \quad t > 0, x \in \mathbb{R}, \\ (a, \mathbf{w})(t = 0, x) = \begin{cases} (a_L, \mathbf{w}_L) & \text{if } x < 0, \\ (a_R, \mathbf{w}_R) & \text{if } x > 0. \end{cases} \end{cases}$$

For smooth solutions, we have equivalently:

$$\partial_t \begin{pmatrix} a \\ \mathbf{w} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \partial_a \mathbf{f} - \mathbf{s} & \partial_{\mathbf{w}} \mathbf{f} \end{pmatrix} \partial_x \mathbf{w} = 0.$$

In classical problems, this matrix can degenerate.
$\Longrightarrow$ failure of the linear theory of hyperbolic systems ...

# The solutions(s) of the Riemann problem

**Lemma.** The system of PDE

$$\partial_t \begin{pmatrix} a \\ \mathbf{w} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \partial_a \mathbf{f} - \mathbf{s} & \partial_{\mathbf{w}} \mathbf{f} \end{pmatrix} \partial_x \mathbf{w} = 0$$

may degenerate if the speed of one of the waves vanishes, that is if an eigenvalue of $\partial_{\mathbf{w}} \mathbf{f}$ vanishes. Such a phenomenon is called *resonance*. Moreover, the wave with the zero speed corresponds to a linearly degenerate field.

**Theorem.** [Goatin, LeFloch 03] For small data, the solution of the Riemann problem exists, but can admit up to three solutions. In the case of three solutions, the solution is no longer continuous with respect to the initial data.

**Remark.** When $|a_R - a_L| \to 0$, that is when the space step $\Delta x \to 0$, the uniqueness is recovered (the three solutions identify at the limit).

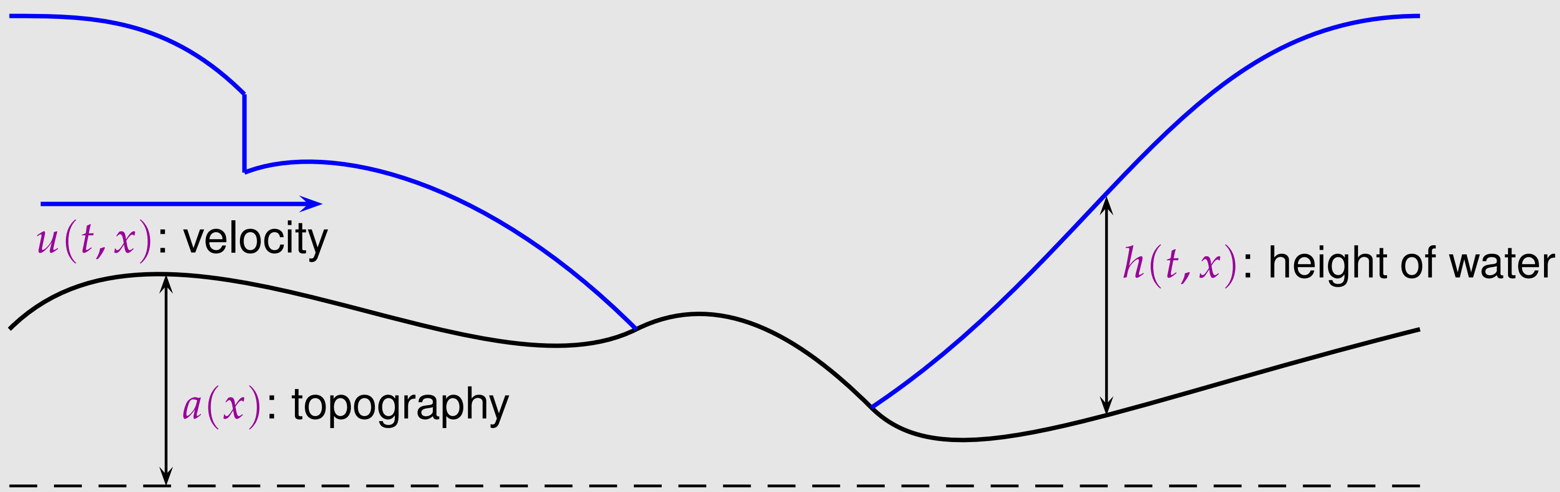
# The Saint-Venant system with topography

The shallow water flow with topography is modeled by

$$\partial_t h + \partial_x (hu) = 0,$$
$$\partial_t (hu) + \partial_x (hu^2 + gh^2/2) = -gha'(x),$$

$$\left(\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}) = \mathbf{s}(\mathbf{w}) a'(x)\right)$$

where $a$ is a given function and $g$ is the gravity constant.



# The case of the flat topography

The Saint-Venant system becomes conservative

$$(\mathcal{CL}) \quad \begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2 + gh^2/2) = 0. \end{cases}$$

$$\big(\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}) = 0.\big)$$

The *critical velocity* is defined by $c = \sqrt{gh}$.
The smooth solutions of $(\mathcal{CL})$ also are solutions of the diagonal system

$$\partial_t(u - 2c) + (u - c)\partial_x(u - 2c) = 0,$$
$$\partial_t(u + 2c) + (u + c)\partial_x(u + 2c) = 0.$$

The function $u + 2c$ is the Riemann invariant of the wave $\lambda_1 = u - c$.
The function $u - 2c$ is the Riemann invariant of the wave $\lambda_2 = u + c$.

# Parameterization of the waves

The shock waves are parametrized using the Rankine-Hugoniot jump relations

$$-\sigma(\mathbf{w}^+ - \mathbf{w}^-) + (\mathbf{f}(\mathbf{w}^+) - \mathbf{f}(\mathbf{w}^-)) = 0.$$

The 1-wave, which corresponds to $\lambda_1 = u - c$, is given by

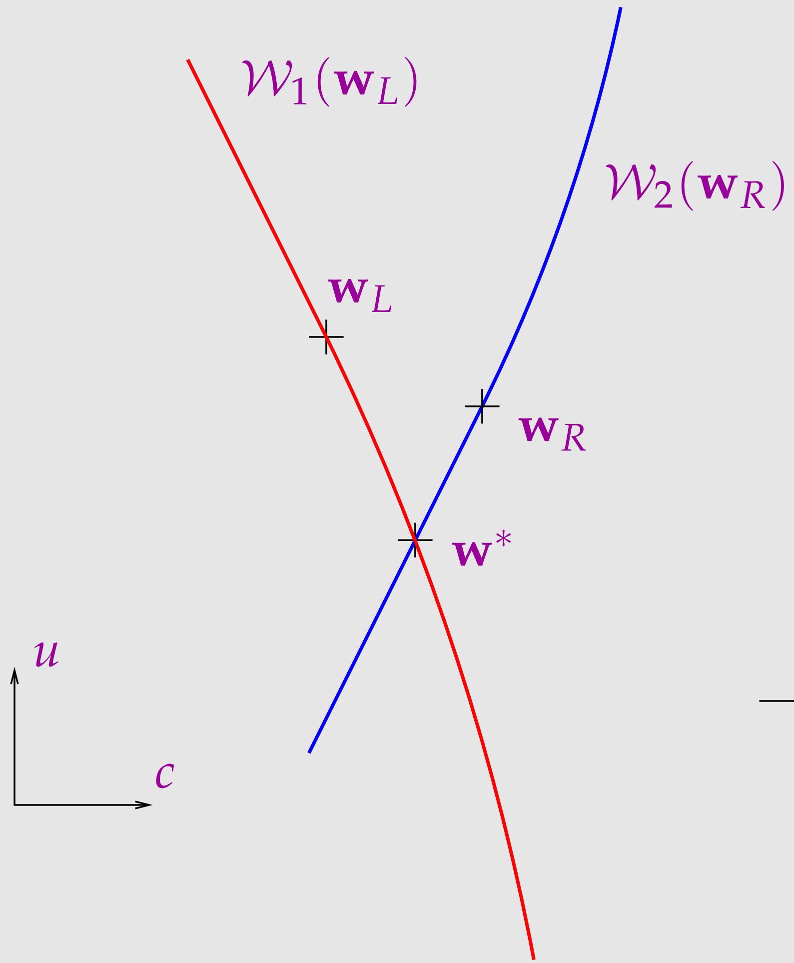
$$u = \begin{cases} u_0 - 2(\sqrt{gh} - \sqrt{gh_0}) & \text{if } h < h_0, \\ u_0 - (h - h_0)\sqrt{g\dfrac{h + h_0}{2hh_0}} & \text{if } h > h_0. \end{cases}$$

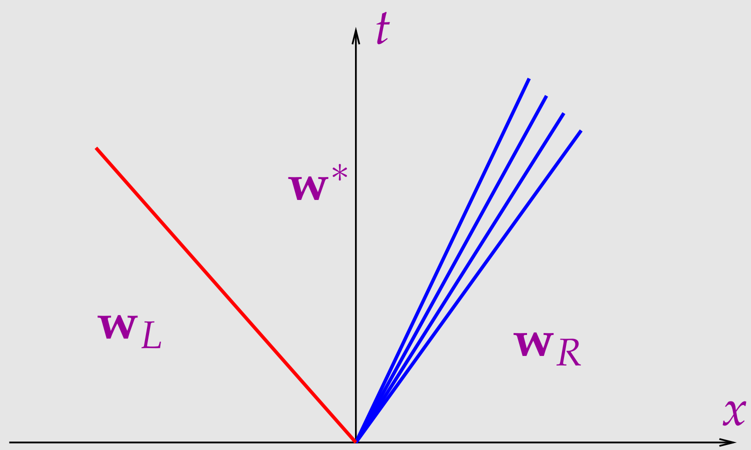
The 2-wave, which corresponds to $\lambda_2 = u + c$, is given by

$$u = \begin{cases} u_0 + 2(\sqrt{gh} - \sqrt{gh_0}) & \text{if } h < h_0, \\ u_0 + (h - h_0)\sqrt{g\dfrac{h + h_0}{2hh_0}} & \text{if } h > h_0. \end{cases}$$

# Graphical solving of the Riemann problem

Global existence and uniqueness of the solution of the Riemann problem.



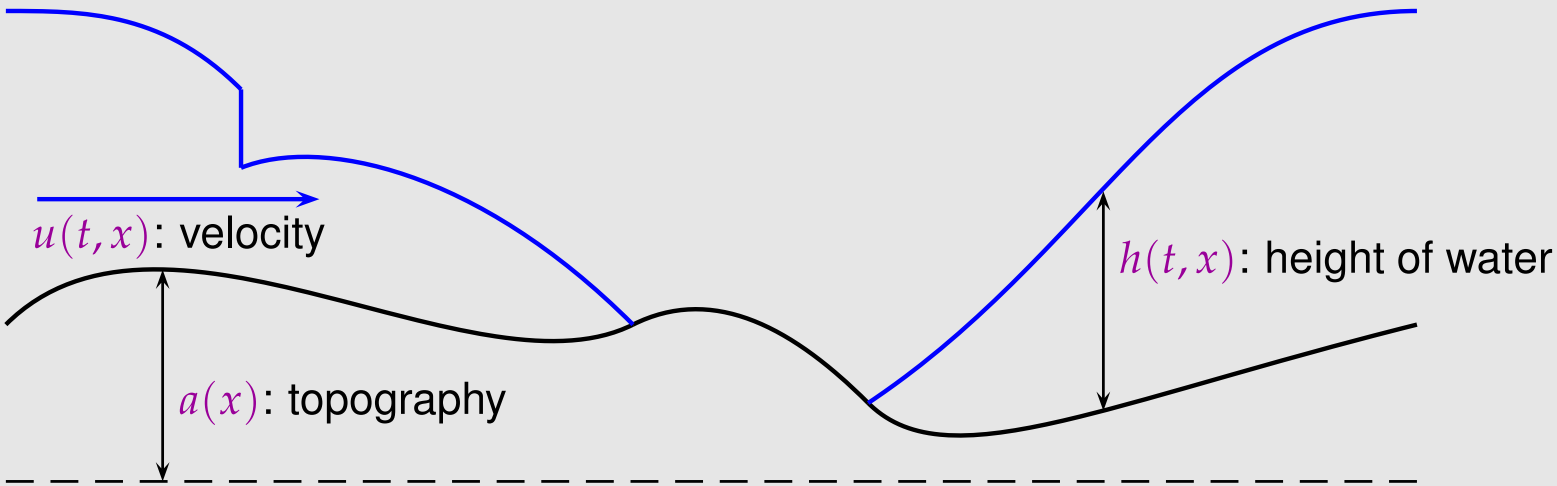
# The Saint-Venant system with topography

The shallow water flow with topography is modeled by

$$\partial_t h + \partial_x(hu) = 0,$$
$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) = -gha'(x),$$

$$\left(\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}) = \mathbf{s}(\mathbf{w})a'(x)\right)$$

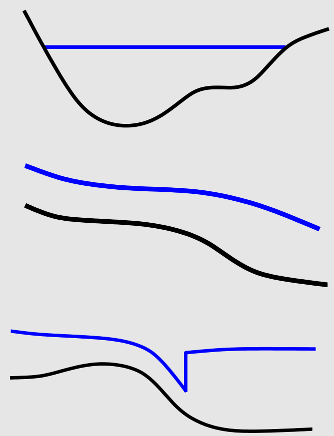
where $a$ is a given function and $g$ is the gravity constant.



# Approximation of the Saint-Venant with topography

Numerical difficulties:

- dry zones: $h = hu = 0$, (border of rivers, ...),
- preservation of the stationary states, $\partial_x \mathbf{f}(\mathbf{w}) = \mathbf{s}(\mathbf{w}) a'(x)$:
  - states at rest $h + a = Cte$ and $u = 0$,
  - smooth parts $hu = Cte$ and $u^2/2 + g(h+a) = Cte$,
  - stationary shocks $\mathbf{f}(\mathbf{w}^+) = \mathbf{f}(\mathbf{w}^-)$,



- upwinding accounting for the slope of topography,
- entropy inequality:

$$\partial_t (hu^2/2 + gh(h/2 + a)) + \partial_x (hu(u^2/2 + g(h+a))) \leq 0.$$

# Properties of the well-balanced scheme

**Proposition.** If the solution of the Riemann problem globally exists for all initial data $(a_L, \mathbf{w}_L)$ and $(a_R, \mathbf{w}_R)$, then, under the CFL condition

$$\Delta t \leq \frac{\Delta x}{2\lambda_{max}}, \quad \text{where } \lambda_{max} = \max_{i,n}\left(|u^n_{i+1/2}| + \sqrt{g h^n_{i+1/2}}\right),$$

the well-balanced scheme

- verifies a discrete version of the entropy inequality
  $$\partial_t(hu^2/2 + gh(h/2+a)) + \partial_x(hu(u^2/2 + g(h+a))) \leq 0,$$
- verifies the discrete maximum principle
  $$h^n_i \geq 0, \quad \forall i \in \mathbb{Z}, n \in \mathbb{N},$$
- maintains all the discrete steady states (smooth stationary parts, stationary entropic shocks, ...).

# The Riemann problem with topography

The topography $a$ is considered as an additional “unknown” that is independent from the time and the source term becomes a first order differential term.

We thus study the following nonconservative Riemann problem:

$$\begin{cases} \partial_t a = 0, \\ \partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}) - \mathbf{s}(\mathbf{w})\partial_x a = 0, \quad t > 0, x \in \mathbb{R}, \\ (a, \mathbf{w})(t=0, x) = \begin{cases} (a_L, \mathbf{w}_L) & \text{if } x < 0, \\ (a_R, \mathbf{w}_R) & \text{if } x > 0. \end{cases} \end{cases}$$

For smooth solutions, this system may be rewritten as

$$\partial_t \begin{pmatrix} a \\ \mathbf{w} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\mathbf{s}(\mathbf{w}) & \mathbf{f}'(\mathbf{w}) \end{pmatrix} \partial_x \begin{pmatrix} a \\ \mathbf{w} \end{pmatrix} = 0.$$

# Properties of the solution of the Riemann problem

$$\partial_t \begin{pmatrix} a \\ \mathbf{w} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\mathbf{s}(\mathbf{w}) & \mathbf{f}'(\mathbf{w}) \end{pmatrix} \partial_x \begin{pmatrix} a \\ \mathbf{w} \end{pmatrix} = 0$$

**Proposition**

- This *non conservative* system has the following eigenvalues
  $\lambda_0 = 0 \quad , \quad \lambda_1 = u - \sqrt{gh}$ et $\lambda_2 = u + \sqrt{gh}$.
- It is *non strictly hyperbolic*, which means that the eigenvectors of the system do not provide a basis of $\mathbb{R}^3$ iff $|u| = \sqrt{gh}$.
- The fields $1$ and $2$ are *genuinely non linear*, the field $0$ is *linearly degenerate*.
- The Rankine-Hugoniot jump relations for the $0$-wave are given by the $0$-Riemann invariants, that are
  $I_0^1(a, \mathbf{w}) = hu$ and $I_0^2(a, \mathbf{w}) = \frac{u^2}{2} + g(h + a)$.

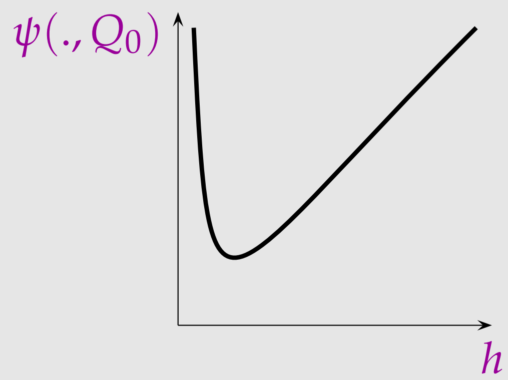
— LATEX prosper —

# Parameterization of the stationary wave, first try ...

Assume that $a_L$, $a_R$ and $\mathbf{w}_l$ are given. We seek for $\mathbf{w}_r = (h_r, Q_r)$, such that

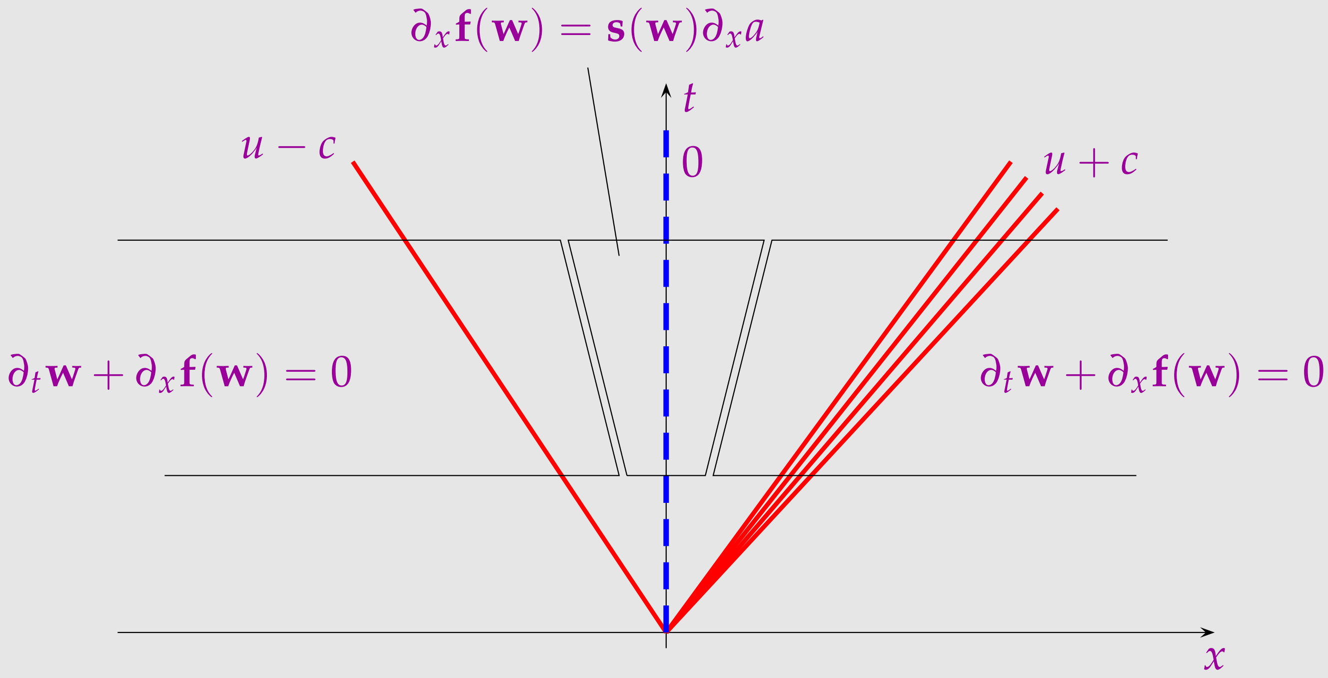
$$\begin{cases} I_0^1(a_L, \mathbf{w}_l) = I_0^1(a_R, \mathbf{w}_r), \\ I_0^2(a_L, \mathbf{w}_l) = I_0^2(a_R, \mathbf{w}_r), \end{cases} \quad \textit{i. e.} \quad \begin{cases} Q_r = Q_l, \\ \psi(h_r, Q_l) = \psi(h_l, Q_l) + g(a_L - a_R). \end{cases}$$

where $\psi(h, Q_0) = \frac{Q_0^2}{2h^2} + gh$. If we assume that $Q_0 \neq 0$, the shape of the function $\psi(., Q_0)$ is




Then, we can obtain 0, 1 or 2 solutions ...

# The self-similar solution of the Riemann problem




For $-\infty < x/t < 0^-$, the solution verifies $\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}) = 0$.

For $0^- < x/t < 0^+$, the solution verifies $\partial_x \mathbf{f}(\mathbf{w}) - \mathbf{s}(\mathbf{w})\partial_x a = 0$.

For $0^+ < x/t < +\infty$, the solution verifies $\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}) = 0$.

# Parameterization of the stationary wave, second try ...

We want to characterize the entropy weak solutions of the system

$$\partial_x \mathbf{f}(\mathbf{w}) - \mathbf{s}(\mathbf{w})\partial_x a = 0. \qquad (*)$$

Therefore, we assume that $a_L$, $a_R$ and $\mathbf{w}_l = \mathbf{w}(0^-)$ are given and we seek for $\mathbf{w}_r = \mathbf{w}(0^+)$ such that the system (*) is verified.

The topography is smoothed for $x \in [-\varepsilon; +\varepsilon]$, $a \to a_\varepsilon$:

$$a_\varepsilon(-\varepsilon) = a_L,\ a_\varepsilon(+\varepsilon) = a_R \text{ and } \mathrm{sign}(a_\varepsilon'(x)) = \mathrm{sign}(a_R - a_L).$$

The unknown $\mathbf{w}_\varepsilon$ for the problem with the smooth topography $a_\varepsilon$ must verify

$$d_x \mathbf{f}(\mathbf{w}_\varepsilon) - \mathbf{s}(\mathbf{w}_\varepsilon) d_x a_\varepsilon = 0, \text{ for } x \in (-\varepsilon; +\varepsilon),$$

complemented with the “initial” condition

$$\mathbf{w}_\varepsilon(-\varepsilon) = \mathbf{w}_l.$$

We thus want to characterize $\mathbf{w}_\varepsilon(+\varepsilon)$ ([LeRoux 98], [Seguin, Vovelle 03]).

— LaTeX prosper —

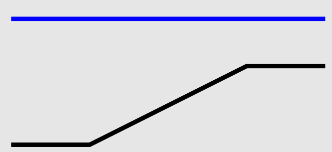
# Piecewise smooth solutions of the stationary smoothed problem

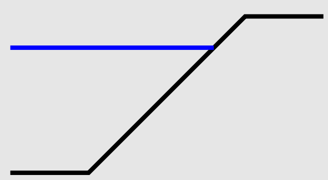
$$\begin{cases} d_x(h_\varepsilon u_\varepsilon) = 0, & x \in (-\varepsilon;\varepsilon), \\ d_x(h_\varepsilon u_\varepsilon^2 + g h_\varepsilon^2/2) + g h_\varepsilon d_x a_\varepsilon = 0, & x \in (-\varepsilon;\varepsilon), \\ a_\varepsilon(x) \ \text{given} & x \in (-\varepsilon;\varepsilon), \\ (h_\varepsilon, h_\varepsilon u_\varepsilon)(x=-\varepsilon) = (h_l, h_l u_l). \end{cases}$$

- ▷ solutions with a zero discharge $Q_l = 0$,
- ▷ smooth solutions with a non-zero discharge $Q_l \neq 0$,
- ▷ admissible discontinuities (in the Lax' sense).

Solutions with a zero discharge $Q_l = 0$, for $x \in [-\varepsilon; +\varepsilon]$:

$$\begin{aligned} h_\varepsilon(x) &= \max(h_l + a_L - a_\varepsilon(x), 0), \\ (h_\varepsilon u_\varepsilon)(x) &= 0. \end{aligned}$$





# Some definitions

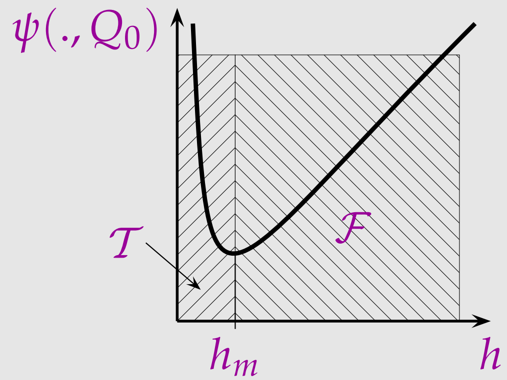
Let $c := \sqrt{gh}$ be the *critical velocity*.

We decompose $\mathbb{R}_+ \times \mathbb{R} \ni (h, hu)$ as:

- the set $\mathcal{T}$ of the *torrential* states, that are the states which verify $|u| > c$,
- the set $\mathcal{F}$ of the *fluvial* states, that are the states which verify $|u| < c$,
- the set $\mathcal{C}$ of the *critical* states, that are the states which verify $|u| = c$.

Let us recall the function $\psi(.,.)$:

$$\begin{array}{ccc} \mathbb{R}_+^* \times \mathbb{R} & \longmapsto & \mathbb{R}_+ \\ (h,Q) & \longmapsto & \psi(h,Q) = \frac{Q^2}{2h^2} + gh. \end{array}$$




The function $\psi(., Q_0))$, $Q_0 \neq 0$, is strictly convex, its minimum $h_m$ belongs to $\mathcal{C}$ ; if $h < h_m$, then $(h, Q_0) \in \mathcal{T}$ and if $h > h_m$, then $(h, Q_0) \in \mathcal{F}$.

# Smooth solutions with a non-zero discharge

Assuming that $\mathbf{w}_\varepsilon$ is smooth, then

$$\begin{cases} d_x(h_\varepsilon u_\varepsilon) = 0, \\ d_x(h_\varepsilon u_\varepsilon^2 + g h_\varepsilon^2/2) + g h d_x a_\varepsilon = 0 \end{cases} \Longleftrightarrow \begin{cases} d_x(h_\varepsilon u_\varepsilon) = 0, \\ d_x(u_\varepsilon^2/2 + g(h_\varepsilon + a_\varepsilon)) = 0. \end{cases}$$

Using $\mathbf{w}_\varepsilon(-\varepsilon) = \mathbf{w}_l$, it yields

$$\begin{aligned} &\psi(h_\varepsilon(x), Q_l) = \psi(h_l, Q_l) - g(a_\varepsilon(x) - a_L), && x \in [-\varepsilon; +\varepsilon], \\ &(h_\varepsilon u_\varepsilon)(x) = Q_l, && x \in [-\varepsilon; +\varepsilon], \\ &\text{et } \; h_\varepsilon(-\varepsilon) = h_l. \end{aligned}$$

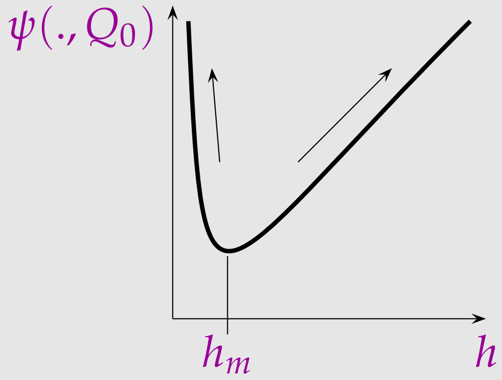
The derivative of the first equation with respect to $x$ is

$$\partial_h \psi(h_\varepsilon(x), Q_l)\, h_\varepsilon'(x) = -g\, a_\varepsilon'(x).$$

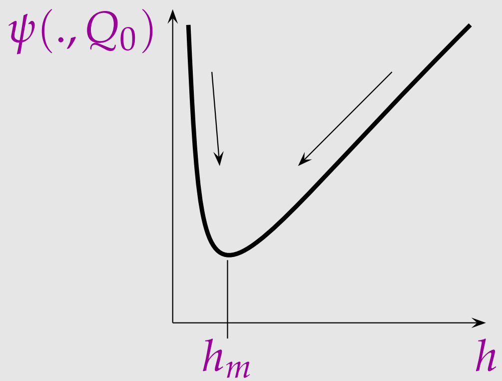
# Smooth solutions with a non-zero discharge

We have: $\partial_h \psi(h_\varepsilon(x), Q_l)\, h'_\varepsilon(x) = -g\, a'_\varepsilon(x).$

If $a_L > a_R$:




If $a_L < a_R$:




$\Longrightarrow$ A smooth solution cannot cross the critical set $\mathcal{C}$.

# Admissible discontinuities

Actually, the discontinuities of the smoothed stationary system are shock waves with a zero speed of the non-stationary system.

Let $\mathbf{w}^-$ and $\mathbf{w}^+$ be the states at the left and at the right of the discontinuity. Then, we have:

$$\mathbf{f}(\mathbf{w}^+) - \mathbf{f}(\mathbf{w}^-) = 0,$$

since $a_\varepsilon$ is smooth. Moreover, since the discontinuity is a shock wave, an entropy condition must be satisfied:

$$\lambda(\mathbf{w}^-) > 0 > \lambda(\mathbf{w}^+).$$

Therefore, the states $\mathbf{w}^-$ and $\mathbf{w}^+$ belong to the following sets:

- if the discontinuity is a 1-shock, then $\mathbf{w}^- \in \mathcal{T}$ and $\mathbf{w}^+ \in \mathcal{F}$,
- if the discontinuity is a 2-shock, then $\mathbf{w}^- \in \mathcal{F}$ and $\mathbf{w}^+ \in \mathcal{T}$.

# Piecewise smooth solutions of the stationary smoothed problem

Finally, we obtain:

- ▷ the piecewise smooth solution admits at most one discontinuity,
- ▷ the state $\mathbf{w}_\varepsilon(+\varepsilon)$ is same whatever the (monotonic) smoothed topography $a_\varepsilon$ we choose,
- ▷ the state $\mathbf{w}_\varepsilon(+\varepsilon)$ is the same whatever the thickness, that is $2\varepsilon$, is,
- ▷ the parameterization we obtain is equivalent the the ones proposed in [Isaacson, Temple 95] and [Goatin, LeFloch 03].

The difficulties for the construction of the solution of the Riemann problem:

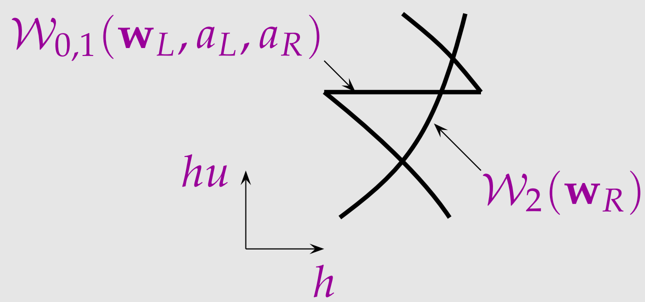
- ▷ the parameterization of the $0$-wave is not explicit,
- ▷ resonance: the $0$-wave and the $1$ and $2$-wave are not arranged in order.

# Construction of the solution of the Riemann problem

Two equivalent methods of construction of the solution :

First method : “mixed” waves [Goatin, LeFloch 03]

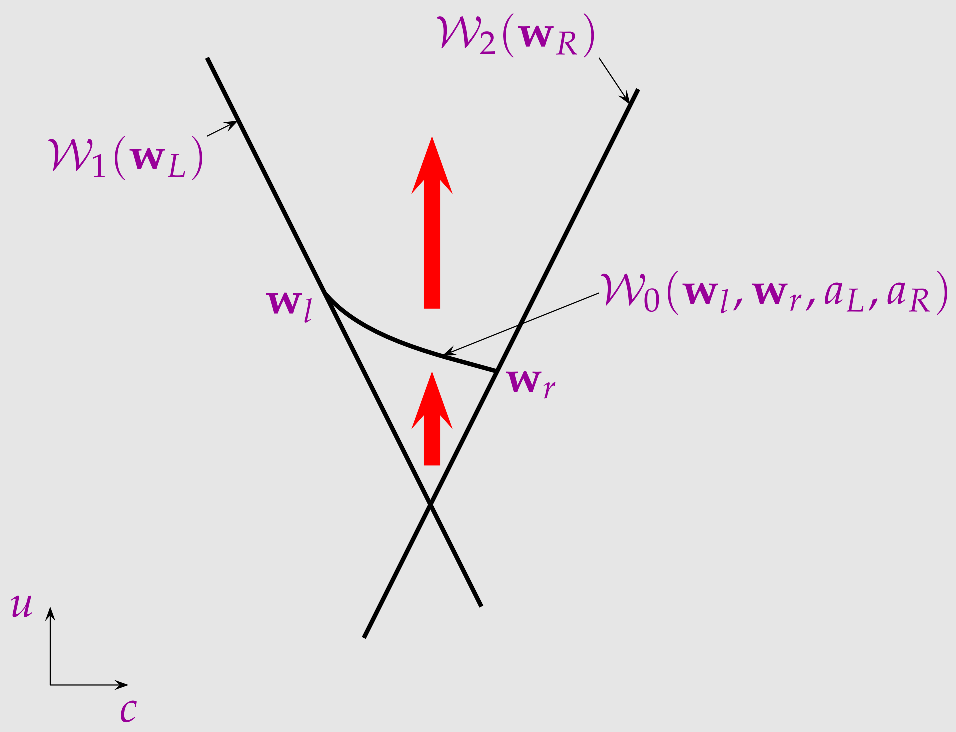
- The $0$-wave is parameterized together with one of the GNL waves.
  $\longrightarrow \mathcal{W}_{0,1}(\mathbf{w}_L, a_L, a_R)$
- The other GNL wave is parameterized.
  $\longrightarrow \mathcal{W}_2(\mathbf{w}_R)$
- We seek for the intersection of the two parameterization:
  $\mathcal{W}_{0,1}(\mathbf{w}_L, a_L, a_R) \bigcap \mathcal{W}_2(\mathbf{w}_R)$.



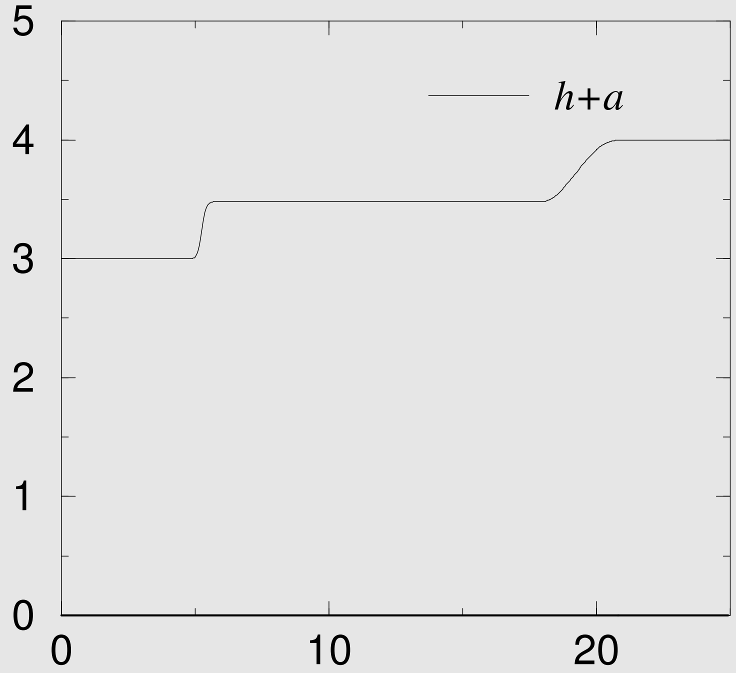
# Construction of the solution of the Riemann problem

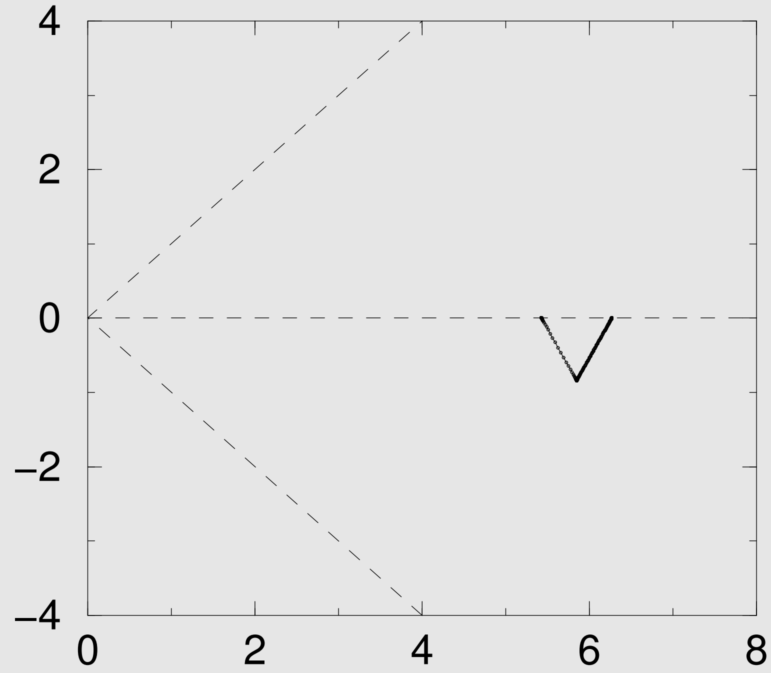
## Second method : the method of “continuation”

- We first seek the solution for a flat topography: $a_L = a_R$.
- We let increase the jump of topography, letting the structure of the solution vary in a continuous way, until obtaining $a_R - a_L$.



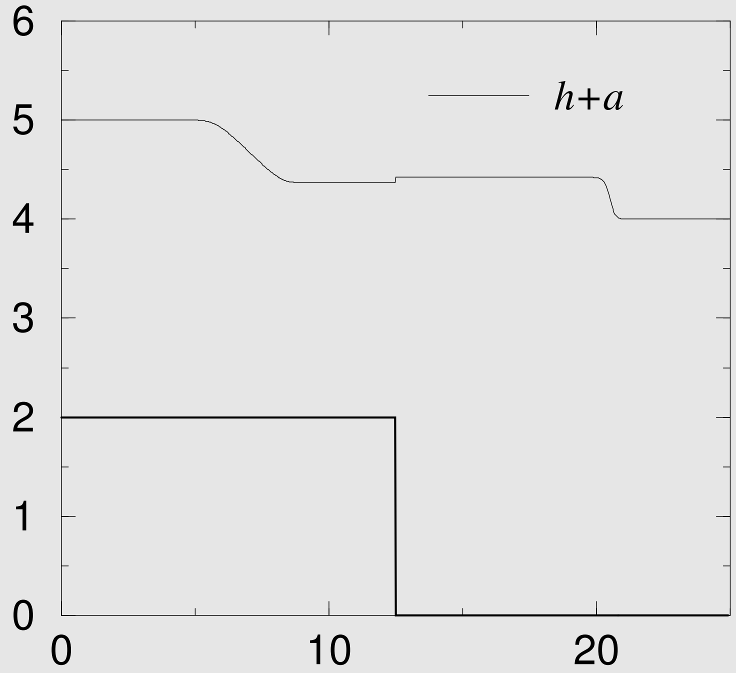
# Numerical tests: flat topography

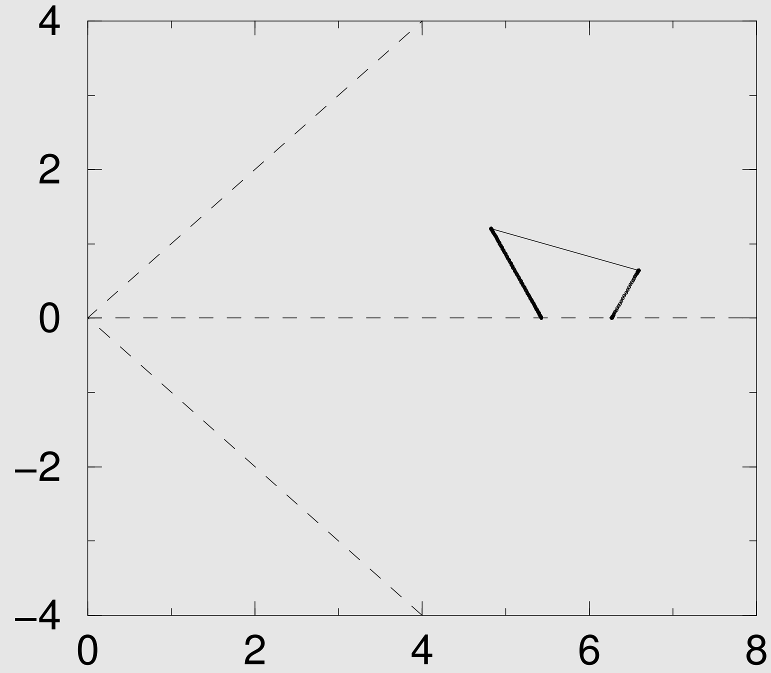



A test with a flat topography.
Left: $h+a$ vs $x$. Right: $u$ vs $c$.

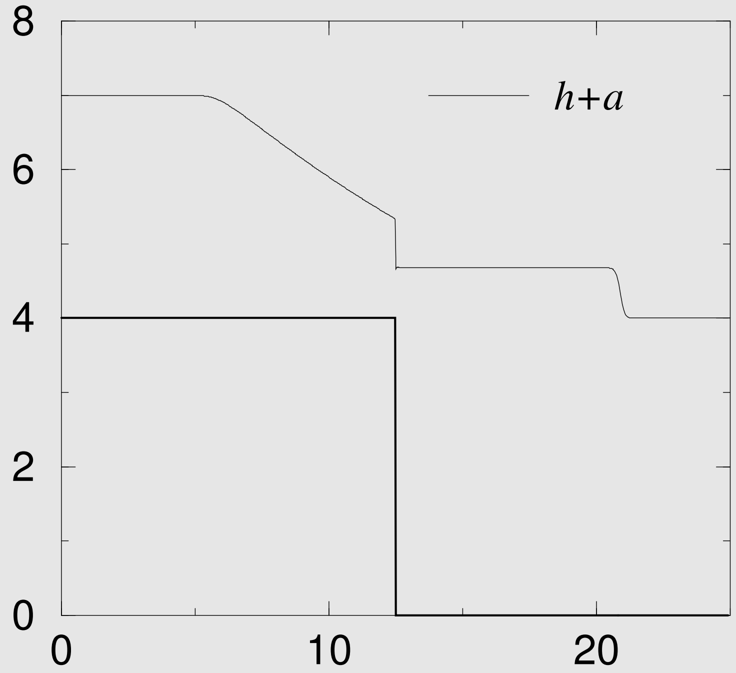
# Numerical tests: a small decreasing jump of topography

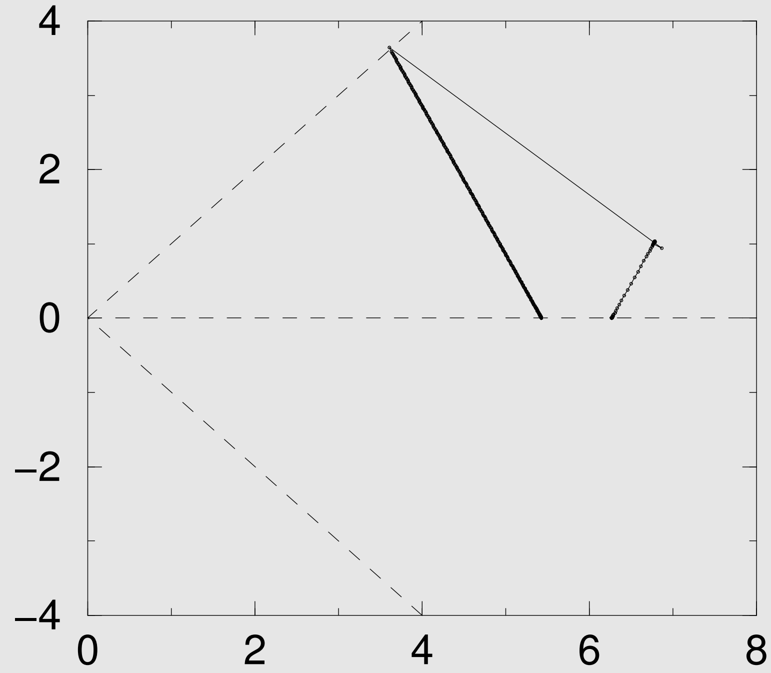



A test with a small decreasing jump of topography.
Left: $h + a$ vs $x$. Right: $u$ vs $c$.

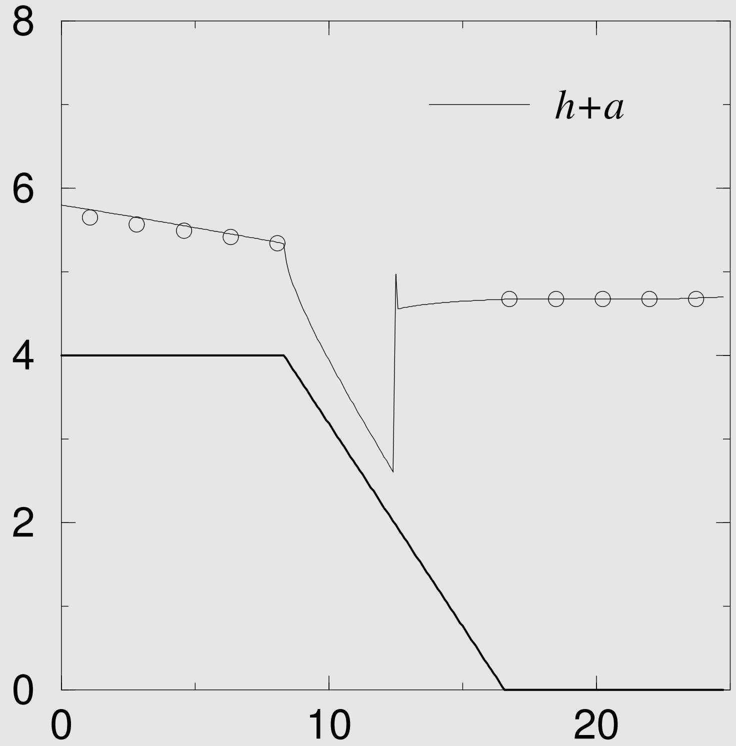
# Numerical tests: a stationary 1-shock wave

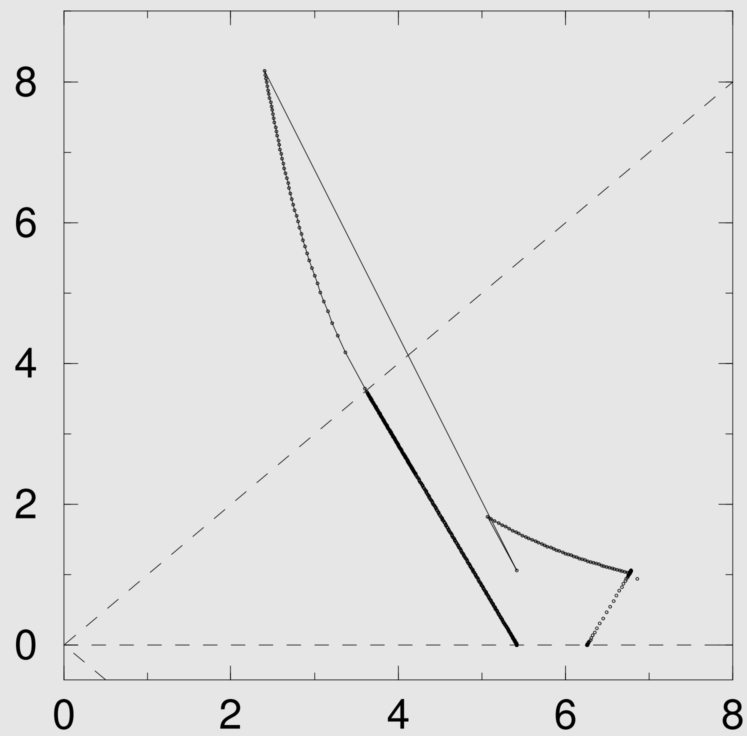



A test with a stationary 1-shock wave.
Left: $h + a$ vs $x$. Right: $u$ vs $c$.

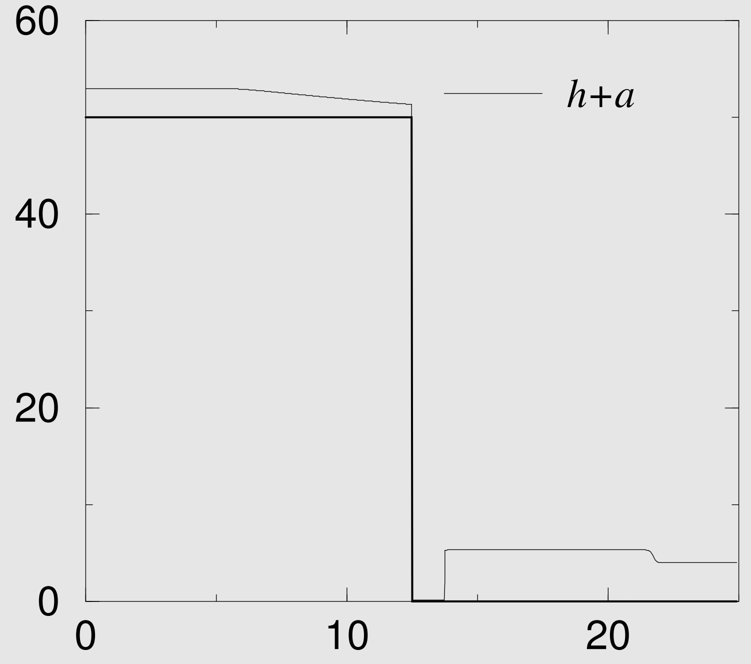
# Numerical tests: a stationary 1-shock wave

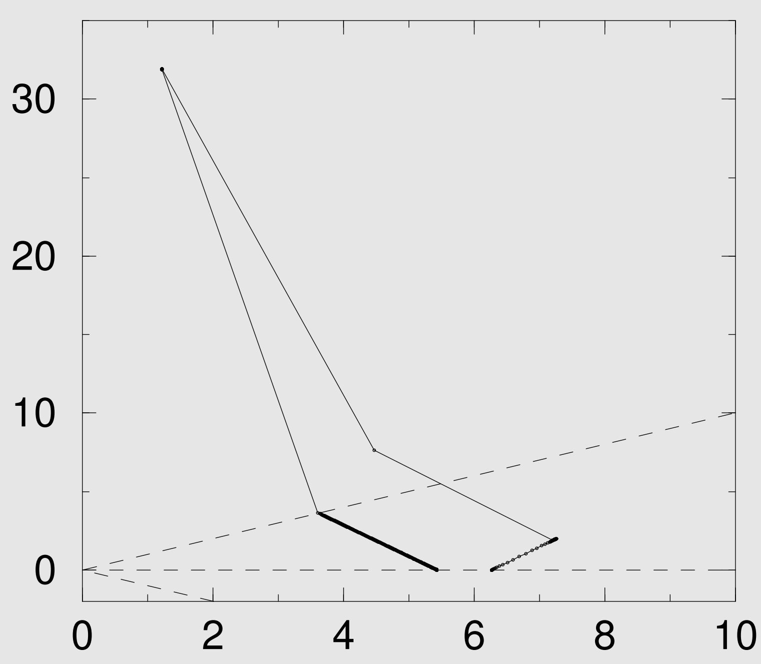



A test with a stationary 1-shock wave.
Left: $h+a$ vs $x$. Right: $u$ vs $c$.

# Numerical tests: a large decreasing jump of topography



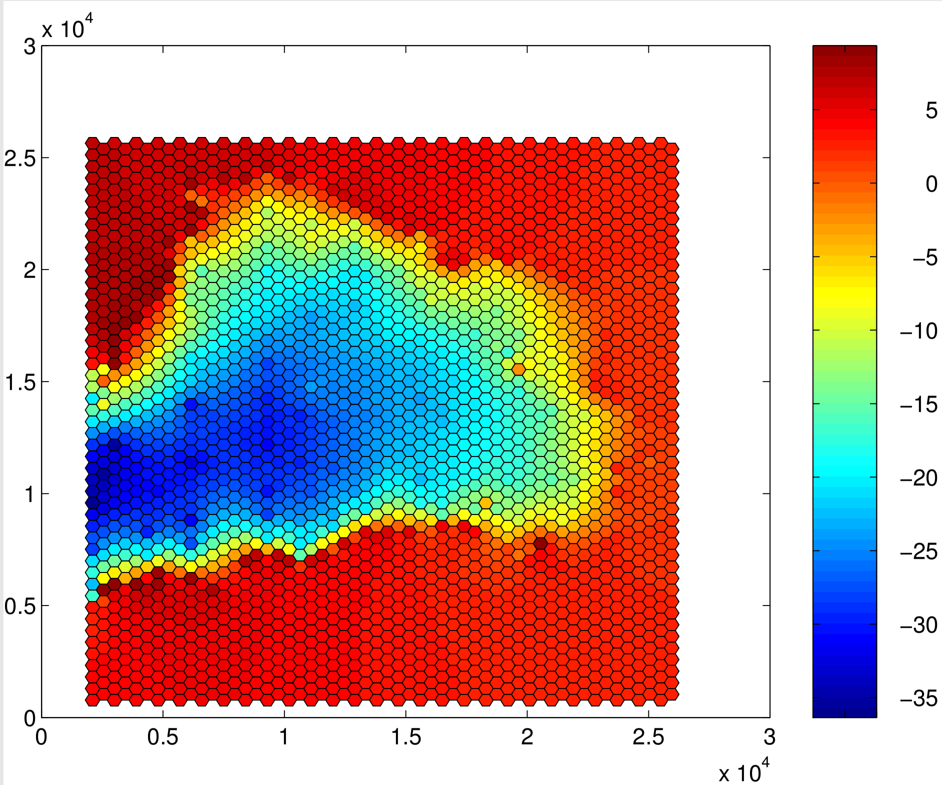




A test with a large decreasing jump of topography.
Left: $h + a$ vs $x$. Right: $u$ vs $c$.

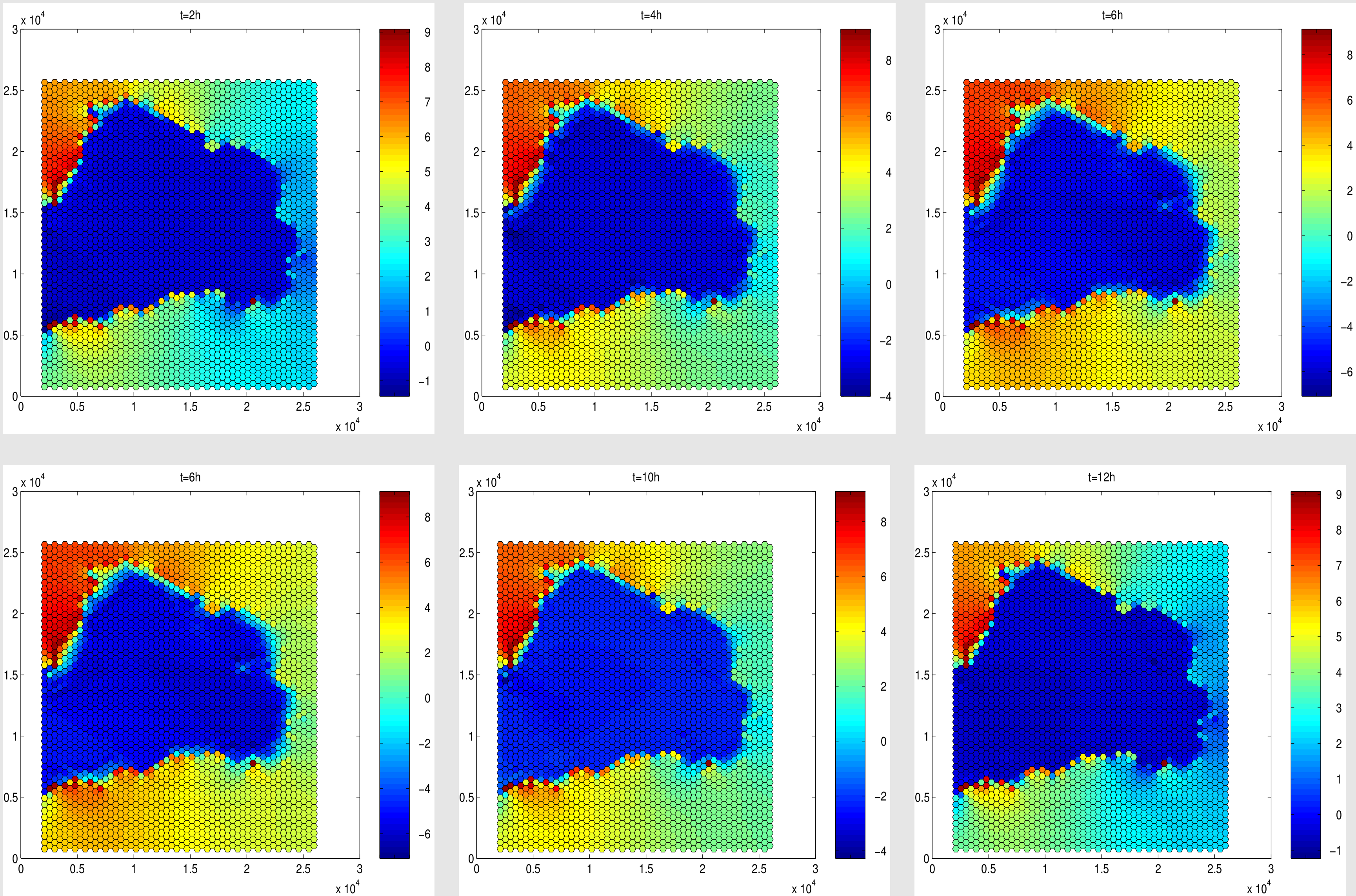
— LaTeX prosper —

# Numerical tests: the Douarnenez bay

Topography



# The Douarnenez bay – $h + a$



# Conclusion

Thus ...

- Source terms $\Longrightarrow$ additional numerical difficulties.
- Extension of the Godunov scheme: the well-balanced scheme.
  - robust and accurate,
  - maintains all the discrete steady states,
  - but the Riemann problem is very complicated ...

And now ...

- Global existence of the solution of the Riemann problem for specific systems ? [LeFloch, Thanh 03], [Baer, Nunziato 86] ...
- How to choose one of the three solutions ?
- Simple well-balanced schemes based on the previous analysis ?
- Numerical methods for resonant systems ?