Applications of functional a posteriori error estimates to some mechanical problems

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Outline

1. Functional a posteriori error estimate for Poisson problem
2. Functional a posteriori error estimate for problems with nonlinear BC
3. Functional a posteriori error estimate for Barenblatt-Biot model
4. Flows in porous media
5. Functional a posteriori error estimate for elastoplasticity

Literature on functional a posteriori error estimates

Theory on functional a posteriori error estimates is explained in books:

Pekka Neittaanmäki and Sergey Repin, *Reliable methods for computer simulation, Error control and a posteriori estimates*, Elsevier, New York, 2004.

Sergey Repin, *A Posteriori Estimates for Partial Differential Equations* Radon Series on Computational and Applied Mathematics, de Gruyter, 2008

Explaining papers to theory and numerics to this course:

1. Sergey Repin, Jan Valdman, Functional a posteriori error estimates for problems with nonlinear boundary conditions. Journal of Numerical Mathematics 16, No. 1, 51-81 (2008)
2. Jan Valdman, Minimization of Functional Majorant in A Posteriori Error Analysis based on H(div) Multigrid-Preconditioned CG Method. Advances in Numerical Analysis, vol. 2009, Article ID 164519 (2009)
3. Sergey Repin, Jan Valdman, Functional a posteriori error estimates for incremental models in elasto-plasticity. Cent. Eur. J. Math. 7, No. 3, 506-519 (2009)
4. Jan Martin Nordbotten, Talal Rahman, Sergey Repin, Jan Valdman, A posteriori error estimates for approximate solutions of Barenblatt-Biot poroelastic model. Computational Methods in Applied Mathematics 10, No. 3, 302-315 (2010)
5. P. Neittaanmäki, S. I. Repin and J. Valdman, Functional a posteriori error estimates for elasticity problems with nonlinear boundary conditions. (in preparation)

Aposteriori error estimates

Primal problem

$$\triangle u + f = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

Let us assume that v is (numerical) approximation of u. Then it holds

Estimate of Runge

$$||\nabla(u - v)||_\Omega \leq ||\nabla v - y^*||_\Omega,$$

for $y^* \in H(\Omega, \mathrm{div})$ satisfying

$$\mathrm{div} y^* + f = 0 \quad \text{in } \Omega.$$

Aposteriori error estimates

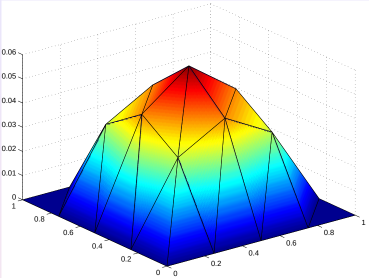
Estimate of Repin

$$||\nabla(u-v)||_\Omega \leq ||\nabla v - y^*||_\Omega + C_\Omega ||\mathrm{div}\, y^* + f||_\Omega$$

for $y^* \in H(\Omega, \mathrm{div})$. C_Ω is the constant in the Friedrichs' inequality

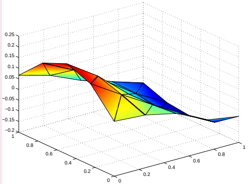
$$\|w\|_\Omega \leq C_\Omega \|\nabla w\|_\Omega \quad \forall w \in H_0^1(\Omega).$$

Example: $f(x,y) = 2x(1-x) + 2y(1-y)$

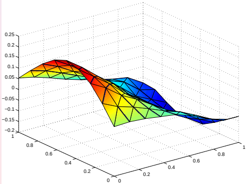


discrete solution v on coarse mesh compared to the exact solution
$u = x(1-x)y(1-y)$

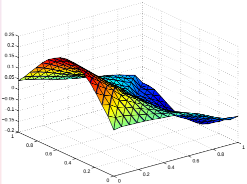
exact error^2 = 1.62e-03



majorant = 3.08e-03



majorant = 2.56e-03



majorant = 2.27e-03

Majorant minimization problem

We have

$$||\nabla v - y^*|| + C_\Omega ||\mathrm{div} y^* + f|| \\ \leq [(1+\beta)||\nabla v - y^*||^2 + (1+\frac{1}{\beta}) C_\Omega^2 ||\mathrm{div} y^* + f||^2]^{1/2}$$

for some $\beta > 0$. Therefore

Majorant minimization problem

Given $v \in H_0^1(\Omega)$ and $\beta > 0$, find the minimizer $y^* \in H(\Omega, \mathrm{div})$ of

$$\mathcal{M}(v, y^*, \beta) := (1+\beta)||\nabla v - y^*||^2 + (1+\frac{1}{\beta}) C_\Omega^2 ||\mathrm{div} y^* + f||^2 \to \min$$

Majorant minimization

The minimization of the right hand side (majorant)

$$(1+\beta)||\nabla v - y^*||^2 + (1+\frac{1}{\beta})C_\Omega^2||\mathrm{div} y^* + f||^2 \to \min$$

leads to the linear system for the discrete solution y^*:

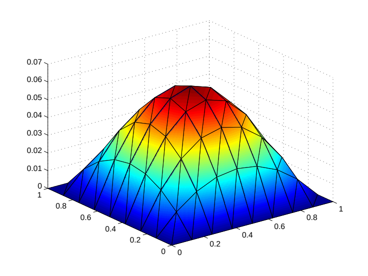
$$\left[(1+\beta)M + (1+\frac{1}{\beta})C_\Omega^2 DIVDIV\right] y^* = (1+\beta)l_1 - (1+\frac{1}{\beta})C_\Omega^2 l_2,$$

where matrices $M, DIVDIV$ represent the "mass" matrix and "divdiv" matrix defined by the equalities:

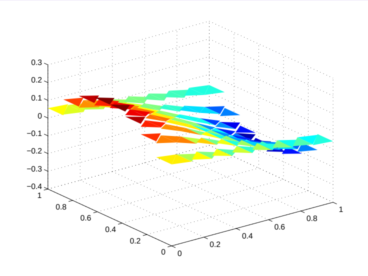
$$\int_\Omega uv\,dx = u^T Mv, \quad \int_\Omega \mathrm{div} u \;\, \mathrm{div} v\,dx = u^T DIVDIVv$$

$$(l_1)^T y^* = (\nabla v, y^*), \;\; (l_2)^T y^* = (f, \mathrm{div} y^*).$$

error^2=3.24e-03



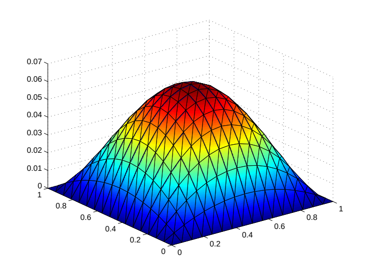
majorant=9.05e-03



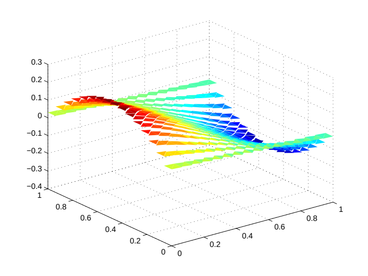
$$\mathcal{T}_2 : I_{eff} = 1.67$$

Figure: Discrete solution v (left) and y-component of the flux y (right).

error^2=8.95e-04



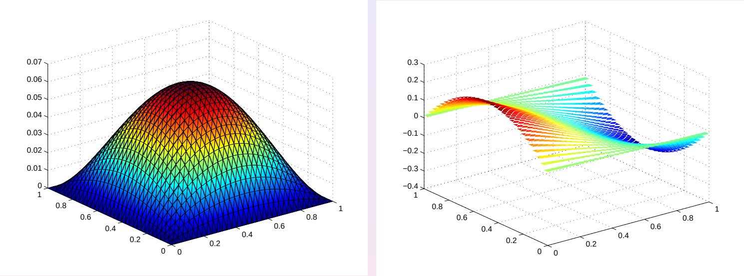
majorant=2.63e-03



$$\mathcal{T}_3 : I_{eff} = 1.71$$

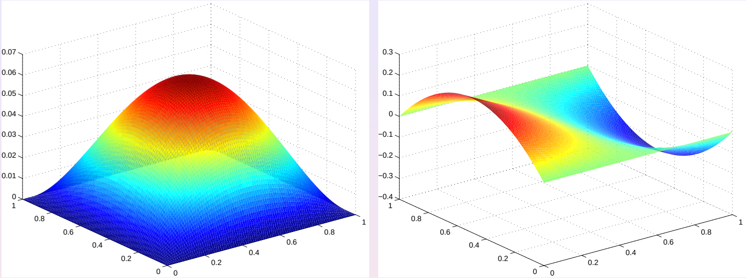
Figure: Discrete solution v (left) and y-component of the flux y (right).

error^2=2.29e-04 majorant=6.85e-04



$$\mathcal{T}_4 : I_{eff} = 1.72$$

Figure: Discrete solution v (left) and y-component of the flux y (right).



$\mathcal{T}_\infty$

Figure: Exact solution v (left) and y-component of the exact flux y (right).

Computional efficiency for Raviart-Thomas (RT0) elements

System matrix: $(1+\beta)M + (1+\frac{1}{\beta})C_\Omega^2 DIVDIV$,
here $\beta = 1$ for all levels.

problem size	without preconditioner	multigrid preconditioner	time in seconds (without setup)
5	1	1	0.00
16	4	4	0.00
56	14	8	0.02
208	51	12	0.04
800	129	14	0.08
3136	264	15	0.24
12416	529	15	0.85
49408	1097	16	4.08
197120	2191	16	18.21
787456	4401	16	77.22

Table: Number of iterations of the CG method using no preconditioner or the multigrid (V cycles) preconditioner with the additive smoother of Arnold, Falk and Winther for 1 smothing step, tolerance=1e-8, Matlab!

Size of the discrete solution and of the discrete flux

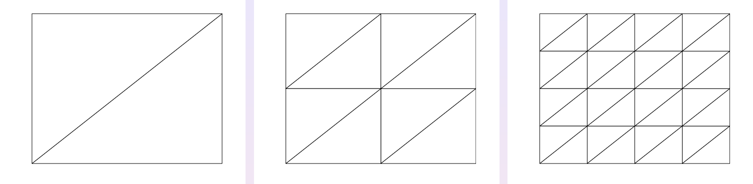


Figure: Refined meshes $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2$.

The discrete solution v is a piecewise linear nodal function (P1)
degrees of freedom on $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2$: 4, 9, 25

The discrete flux y is a lowest order Raviart Thomas function (RT0)
degrees of freedom on $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2$: 5, 16, 56

For fine triangulations it holds: number of edges = number of nodes $\cdot$ 3

Papers

1. Jan Valdman, Minimization of Functional Majorant in A Posteriori Error Analysis based on H(div) Multigrid-Preconditioned CG Method. Advances in Numerical Analysis, vol. 2009, Article ID 164519 (2009)

Problem with nonlinear BC – Classical Formulation

Minimization problem

$$\int_\Omega \left(\frac{1}{2}|\nabla v|^2 - fv\right)\,dx + \mu \int_{\Gamma_1} |v|\,d\Gamma \to \min$$

among all $v \in U := \{v \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma_1) \cap C^0(\Omega \cup \Gamma_0) : v|_{\Gamma_0} = 0\}$

Note that the variation leads to

Friction boundary condition

$$|u|\frac{\partial u}{\partial n} + \mu u = 0 \qquad \text{on } \Gamma_1$$

Problem with nonlinear bc – Classical Formulation

Friction boundary condition

$$|u|\frac{\partial u}{\partial n} + \mu u = 0 \qquad \text{on } \Gamma_1$$

Three parameter cases in our numerical examples:

1. $\mu \to +\infty$ -it implies the homogeneous Dirichlet boundary condition $u|_{\Gamma_1} = 0$.
2. $\mu = 0$ - it implies the homogeneous Neumann boundary condition $\frac{\partial u}{\partial n}|_{\Gamma_1} = 0$.
3. $\mu \in (0, +\infty)$ - this is a typical friction boundary condition.

Discrete solutions of the minimization problem

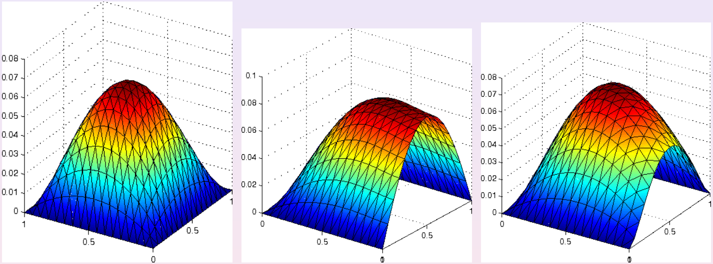


Figure: $\mu \to \infty$ (left), $\mu = 0$ (middle) and $\mu = 0.1$ (right).

Majorant estimate

Let u be an exact solution of the minimization problem and v its discrete approximation. Then it holds for all $\alpha, \beta > 0$

Estimate

$$\frac{1}{2}|||v-u|||_a^2 \leq (1+\beta) M_1(v, y^*) + \inf_{\xi^*} I_{\Gamma_1}(\gamma v, \delta_n y^*, \xi^*)$$
$$+ \frac{1}{2}\left(1+\frac{1}{\beta}\right)(1+\alpha) C_\Omega^2 \mathbf{R}_\Omega^2(y^*)$$

for arbitrary y^* function from the (flux) test space

$$Q_{\Gamma_1}^* := \{y^* \in Y^* \mid \mathrm{div} y^* \in L_2(\Omega),\ \delta_n y^* \in L_2(\Gamma_1)\}\ .$$

Majorant estimate

Note that

$$M_1 = \frac{1}{2}||\nabla v - y^*||^2_{L^2(\Omega)}, \quad \mathbf{R}_\Omega(y^*) := ||\mathrm{div} y^* + f||_{L^2(\Omega)},$$

and using the *compound functional* the boundary term is defined as

$$I_{\Gamma_1}(\gamma v, \delta_n y^*, \xi^*) := \int_{\Gamma_1} \left(j(\gamma v) + j^*(\xi^*) - (\gamma v)\, \xi^* + \frac{\theta}{2}\left|\delta_n y^* + \xi^*\right|^2 \right) d\Gamma,$$

where

$$\theta := \left(1 + \frac{1}{\beta}\right)\left(1 + \frac{1}{\alpha}\right) C^2_{\Gamma_1}, \quad j(\xi) = \mu|\xi|, \quad j^*(\xi^*) = \begin{cases} 0, & \text{if } |\xi^*| \leq \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

Estimate of the boundary term $\inf_{\xi^*} I_{\Gamma_1}(\gamma v, \delta_n y^*, \xi^*)$

Summary:

$$\inf_{\xi^*} I_{\Gamma_1}(\gamma v, \delta_n y^*, \xi^*) \leq \int_{\Gamma_1} \left(\mu|\gamma v| + \phi(\gamma v, \delta_n y^*, \mu)\right) \, d\Gamma,$$

where

$$\phi(\gamma v, \delta_n y^*, \mu) = \begin{cases} \frac{\theta}{2}(\delta_n y^* + \mu)^2 - \mu(\gamma v) & \text{if} \quad \delta_n y^* < -\mu, \\ (-\delta_n y^*)(\gamma v) & \text{if} \quad |\delta_n y^*| < \mu, \\ \frac{\theta}{2}(\delta_n y^* - \mu)^2 + \mu(\gamma v) & \text{if} \quad \delta_n y^* > \mu. \end{cases}$$

Numerical results for $\mu = 0.1$

N	majorant	$\text{error}^2/2$	I_{eff}
25	2.9e-03	1.9e-03	1.22
81	9.0e-04	5.1e-04	1.33
289	2.7e-04	1.3e-04	1.44
1089	8.7e-05	3.3e-05	1.62
4225	2.8e-05	8.2e-06	1.87
16641	9.9e-06	1.9e-06	2.24
66049	3.9e-06	3.9e-07	3.17

Table: Majorant optimization on the same mesh.

Majorant optimized using an expensive nonlinear procedure
- can be improved!

Papers

S. Repin, J. Valdman, Functional A posteriori error estimates for problems with nonlinear boundary conditions, Journal of Numerical Mathematics 16 (2008), No. 1, 51-81.

Extension to elasticity with nonlinear boundary conditions

Friction boundary condition

Minimize the displacement v in the energy

$$\int_{\Omega} \left(\frac{1}{2} C\varepsilon(v) : \varepsilon(v) - fv \right) dx + k_{\tau} \int_{\Gamma_1} |v_{\tau}| \, d\Gamma$$

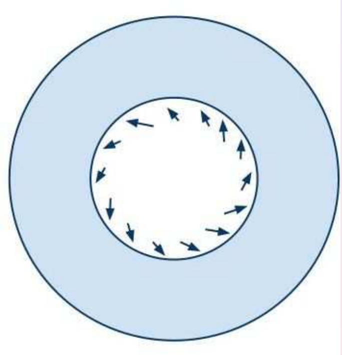
under the non-penetration condition

$$v_n = 0 \quad \text{on } \Gamma_1,$$

where $v = (v_{\tau}, v_n)$ is decomposed in the normal and tangential components on the boundary Γ_1.

Time dependent 2D symetric problem in Matlab

polar coordinates: $u = (u_r, u_\phi)$



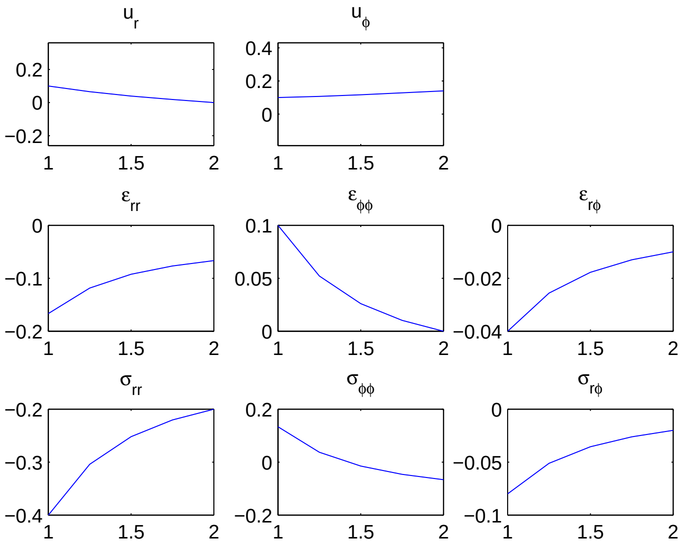
an inner radius $a = 1$,
an outer radius $b = 2$
friction parameter $k_\phi = 0.02$
Lamé parameters $\lambda = \mu = 1$

boundary conditons:

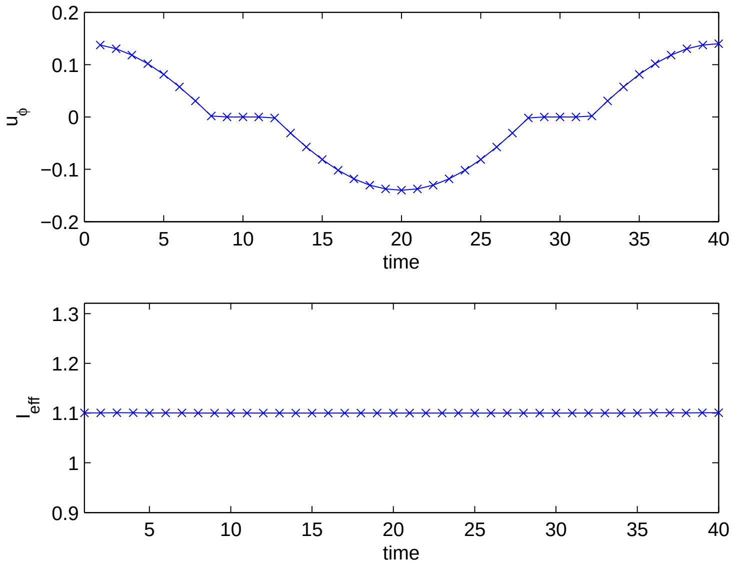
$$u_r(a) = u_\phi(a) = 0.1 \cos(\frac{t\pi}{20})$$

discrete times: $t = 0, 1, 2, \ldots, 40$

Displacements, Strains, Stresses for t_0



Slip testing, index of eficiency



Papers

P. Neittaanmäki, S. I. Repin and J. Valdman, Functional a posteriori error estimates for elasticity problems with nonlinear boundary conditions. (in preparation)

Matlab solver can be downloaded at

http://www.mathworks.com/matlabcentral/fileexchange/authors/37756

Matematics model of the Barenblatt-Biot system

Barenblatt-Biot systems representing double diffusion in elastic porous media.

$$\begin{aligned} -\nabla \cdot (\mathbb{C}\varepsilon(u)) + \alpha_1 \nabla p_1 + \alpha_2 \nabla p_2 &= f(x,t) \\ c_1 \dot{p}_1 - \nabla \cdot (k_1 \nabla p_1) + \alpha_1 \nabla \cdot \dot{u} + \kappa(p_1 - p_2) &= h_1(x,t) \\ c_2 \dot{p}_2 - \nabla \cdot (k_2 \nabla p_2) + \alpha_2 \nabla \cdot \dot{u} + \kappa(p_2 - p_1) &= h_2(x,t) \end{aligned}$$

in which u is the displacement of the solid skeleton and p_1 and p_2 are the fluid pressures in the respective components.

Mathematical analysis of this model based on the theory of implicit evolution equations in Hilbert spaces is elaborated in

R. E. Showalter and B. Momken, *Single-phase flow in composite poroelastic media*, Math. Meth. Appl. Sci. 25 (2002), no. 2, 115–139.

Static model

Static case of the Barenblatt-Biot system

$$
\begin{aligned}
-\nabla \cdot (\mathbb{C}\varepsilon(u)) + \alpha_1 \nabla p_1 + \alpha_2 \nabla p_2 &= f(x) \\
-\nabla \cdot (k_1 \nabla p_1) + \kappa(p_1 - p_2) &= h_1(x) \\
-\nabla \cdot (k_2 \nabla p_2) + \kappa(p_2 - p_1) &= h_2(x)
\end{aligned}
$$

Combining a functional a posteriori error estimate for an elasticity problem

$$
-\nabla \cdot (\mathbb{C}\varepsilon(u)) = f - \alpha_1 \nabla p_1 - \alpha_2 \nabla p_2 \tag{1}
$$

and a functional a posteriori error estimate for a double-diffusion problem

$$
-\nabla \cdot (k_1 \nabla p_1) + \kappa(p_1 - p_2) = h_1(x) \tag{2}
$$

$$
-\nabla \cdot (k_2 \nabla p_2) + \kappa(p_2 - p_1) = h_2(x) \tag{3}
$$

which describes the flow of slightly compressible fluid in a general heterogeneous medium consisting of two components.

Problem (Variational formulation)

Assume that $(h_1, h_2) \in L^2(\Omega, \mathbb{R}^2)$. *Find* $\mathbf{p} = (p_1, p_2) \in H_0^1(\Omega, \mathbb{R}^2)$, *satisfying the system of variational equalities*

$$\int_\Omega k_1 \nabla p_1 \cdot \nabla \mathrm{q}_1 + \int_\Omega \kappa (p_1 - p_2) \mathrm{q}_1 \, dx = \int_\Omega (h_1(x) \mathrm{q}_1 - k_1 \nabla \bar{p} \cdot \nabla \mathrm{q}_1) \, dx$$

$$\int_\Omega k_2 \nabla p_2 \cdot \nabla \mathrm{q}_2 + \int_\Omega \kappa (p_2 - p_1) \mathrm{q}_2 \, dx = \int_\Omega (h_2(x) \mathrm{q}_2 - k_2 \nabla \bar{p} \cdot \nabla \mathrm{q}_2) \, dx$$

for all testing functions $\mathbf{q} = (\mathrm{q}_1, \mathrm{q}_2) \in H_0^1(\Omega, \mathbb{R}^2)$.

Dirichlet boundary conditions assumed for simplicity!

Problem (Abstract variational formulation)

Find $\mathbf{p} \in Q := H_0^1(\Omega, \mathbb{R}^2)$, *such that the equality*

$$a(\mathbf{p}, \mathbf{q}) = l(\mathbf{q})$$

holds for all $\mathbf{q} \in Q$. *The bilinear form* $a(\cdot, \cdot)$ *and the linear form* $l(\cdot)$ *are*

$$\begin{aligned} a(\mathbf{p}, \mathbf{q}) &:= \int_\Omega (\Lambda\mathbf{p} : (\mathbb{A}\Lambda\mathbf{q}) + \mathbf{p} \cdot \mathbb{B}\mathbf{q}) \; dx, \\ l(\mathbf{q}) &:= \int_\Omega (h \cdot \mathbf{q} - \mathbb{C}\Lambda\mathbf{q}) \, dx, \end{aligned}$$

where $\Lambda\mathbf{q} := (\nabla \mathrm{q}_1, \nabla \mathrm{q}_2)$ *and* $\mathbb{A}$, $\mathbb{B}$ *and* $\mathbb{C}$ *are matrices formed by material dependant constants*

$$\mathbb{A} := \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \quad \mathbb{B} := \begin{pmatrix} \kappa & -\kappa \\ -\kappa & \kappa \end{pmatrix}, \quad \mathbb{C} := \begin{pmatrix} k_1 \nabla \bar{p} & 0 \\ 0 & k_2 \nabla \bar{p} \end{pmatrix}$$

and h *is the right hand side vector* $h := \begin{pmatrix} h_1 \; h_2 \end{pmatrix}^T$.

Problem (Equivalent minimization problem)

Find $\mathbf{p} \in Q = H_0^1(\Omega, \mathbb{R}^2)$ *satisfying*

$$F(\mathbf{p}) + G(\Lambda\mathbf{p}) = \inf_{\mathbf{q}\in Q}\{F(\mathbf{q}) + G(\Lambda\mathbf{q})\},$$

where

$$F : Q \to \mathbb{R}, \qquad F(\mathbf{q}) := \frac{1}{2}\int_\Omega \mathbf{q}\cdot\mathbb{B}\mathbf{q}\, dx - l(\mathbf{q}),$$

$$G : Y \to \mathbb{R}, \qquad G(\Lambda\mathbf{q}) := \frac{1}{2}\int_\Omega \Lambda\mathbf{q} : (\mathbb{A}\Lambda\mathbf{q})\, dx.$$

We need to find explicit forms of dual functionals

$$F^* : Q^* \to \mathbb{R}, \qquad F^*(\Lambda^* \mathbb{Y}^*) := \sup_{\mathbf{q} \in Q} \{\langle \Lambda^* \mathbb{Y}^*, \mathbf{q} \rangle - F(\mathbf{q})\},$$

$$G^* : Y^* \to \mathbb{R}, \qquad G^*(\mathbb{Y}^*) := \sup_{\Lambda \mathbf{q} \in Y} \{\langle\langle \mathbb{Y}^*, \Lambda \mathbf{q} \rangle\rangle - G(\Lambda \mathbf{q})\},$$

where $Y = Y^* := L^2(\Omega, \mathbb{R}^{2d})$, $\quad \Lambda^* \mathbb{Y}^* = (-\operatorname{div} y_1^*, -\operatorname{div} y_2^*)^T$
and construct the corresponding compound functionals

$$D_F : Q \times Q^* \to \mathbb{R}, \qquad D_F(\mathbf{q}, \Lambda^* \mathbb{Y}^*) := F(\mathbf{q}) + F^*(\Lambda^* \mathbb{Y}^*) - \langle \Lambda^* \mathbb{Y}^*, \mathbf{q} \rangle,$$

$$D_G : Y \times Y^* \to \mathbb{R}, \qquad D_G(\Lambda \mathbf{q}, \mathbb{Y}^*) := G(\Lambda \mathbf{q}) + G^*(\mathbb{Y}^*) - \langle\langle \mathbb{Y}^*, \Lambda \mathbf{q} \rangle\rangle.$$

By the the sum of D_F and D_G, we obtain the functional error majorant

$$M(\mathbf{q}, \mathbb{Y}^*) := D_F(\mathbf{q}, \Lambda^* \mathbb{Y}^*) + D_G(\Lambda \mathbf{q}, \mathbb{Y}^*), \tag{4}$$

which provides a guaranteed upper bound of the error:

$$\frac{1}{2} a(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q}) \leq M(\mathbf{q}, \mathbb{Y}^*) \quad \text{for all } \mathbb{Y}^* \in Y^*. \tag{5}$$

The majorant is fully computable and depends only on the approximation $\mathbf{q} \in Q$ and arbitrary variable $\mathbb{Y}^* \in Y^*$.

Lemma (dual functionals)

For $k_1, k_2 > 0$ and $\kappa > 0$, it holds

$$G^*(\mathbb{Y}^*) = \frac{1}{2}\int_\Omega \mathbb{A}^{-1}\mathbb{Y}^* : \mathbb{Y}^* \, dx,$$

$$F^*(\Lambda^*\mathbb{Y}^*) = \begin{cases} \frac{1}{4\kappa}\int_\Omega (\Lambda^*\mathbb{Y}^* + h)^2 \, dx & \text{if } \Lambda^* y_1^* + h_1 + \Lambda^* y_2^* + h_2 = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Note that the condition

$$\Lambda^*\mathbf{Y}_1^* + h_1 + \Lambda^*\mathbf{Y}_2^* + h_2 = 0$$

is weaker than two conditions

$$\Lambda^*\mathbf{Y}_1^* + h_1 = 0, \qquad \Lambda^*\mathbf{Y}_2^* + h_2 = 0,$$

which one would await from the general theory (COUPLING EFFECT!).

We obtain explicit expressions for the compound functionals

$$D_G(\Lambda \mathbf{q}, \mathbb{Y}^*) = \frac{1}{2} \int_\Omega \mathbb{A}(\Lambda \mathbf{q} - \mathbb{A}^{-1}\mathbb{Y}^*) : (\Lambda \mathbf{q} - \mathbb{A}^{-1}\mathbb{Y}^*) \, dx, \tag{6}$$

$$D_F(\mathbf{q}, \Lambda^*\mathbb{Y}^*) = \begin{cases} \frac{1}{2} \int_\Omega \mathbb{B}\mathbf{q} \cdot \mathbf{q} \, dx + \frac{1}{4\kappa} \int_\Omega (\Lambda^*\mathbb{Y}^* + h)^2 \, dx & \\ \quad \text{if } \Lambda^*\mathbf{Y}_1^* + h_1 + \Lambda^*\mathbf{Y}_2^* + h_2 = 0, & \\ +\infty \quad \text{otherwise.} & \end{cases} \tag{7}$$

and let us recall that

$$M(\mathbf{q}, \mathbb{Y}^*) := D_F(\mathbf{q}, \Lambda^*\mathbb{Y}^*) + D_G(\Lambda \mathbf{q}, \mathbb{Y}^*), \tag{8}$$

provides a guaranteed upper bound of the error:

$$\frac{1}{2} a(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q}) \leq M(\mathbf{q}, \mathbb{Y}^*) \quad \text{for all } \mathbb{Y}^* \in Y^*. \tag{9}$$

Final estimate for the coupled poro-elastic system

It holds (**q** and v are known from computations)

$$a(\mathbf{p}-\mathbf{q},\mathbf{p}-\mathbf{q}) + \|\varepsilon(\mathbf{u}-\mathbf{v})\|^2_{\mathbb{L};\Omega}$$
$$\leq 2\widehat{C}\;\; M_{\beta_1,\beta_2}(\mathbf{q},\hat{\mathbb{Y}}^*) + (1+\beta_4+\beta_5)\left\|\varepsilon(\mathbf{v}) - \mathbb{L}^{-1}\tau\right\|^2_{\mathbb{L};\Omega} +$$
$$+\left(1+\frac{1}{\beta_4}+\beta_6\right) C^2 \left\|\operatorname{div}\tau + \mathcal{F} - \alpha_1\nabla \mathrm{q}_1 - \alpha_2 \nabla \mathrm{q}_2\right\|^2_{\Omega},$$

for all $\hat{\mathbb{Y}}^* \in Y^*_{div} := \{(\mathbf{Y}^*_1, \mathbf{Y}^*_2) \in Y^* : \Lambda^*\mathbf{Y}^*_1 + \Lambda^*\mathbf{Y}^*_2 \in L^2(\Omega)\}$,
for all $\tau \in Q$,
for all $\beta_1, \dots, \beta_6 > 0$.

Here

$$\widehat{C} = 1 + C^2\left(1+\frac{1}{\beta_5}+\frac{1}{\beta_6}\right)\max\left\{\frac{1+\beta_3}{k_1}, \frac{1+\beta_3}{k_2\beta_3}\right\},$$

where $C > 0$ satisfies Friedrichs' inequality

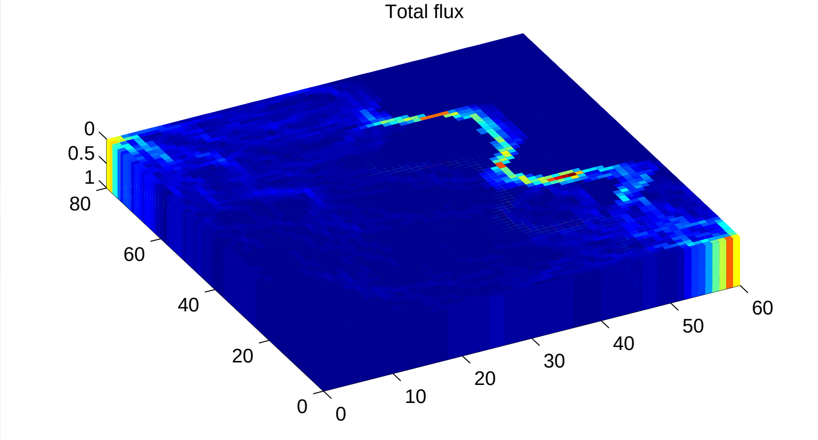
$$\|w\|_{L^2(\Omega)} \leq C\,\|\nabla w\|_{L^2(\Omega)}$$

valid for all $w \in H^1_0(\Omega)$.

Papers

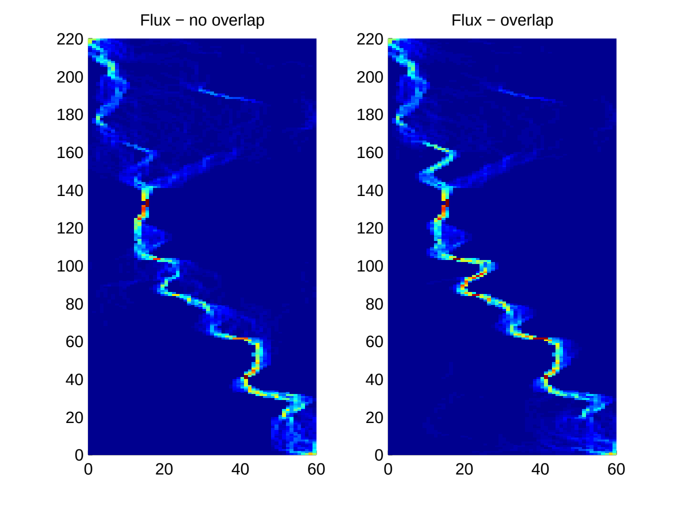
Jan Martin Nordbotten, Talal Rahman, Sergey Repin, Jan Valdman, A posteriori error estimates for approximate solutions of Barenblatt-Biot poroelastic model. Computational Methods in Applied Mathematics 10, No. 3, 302-315 (2010)

Reservoir simulator - project with SINTEF ICT



$$\begin{aligned} v &= -\lambda(\nabla p - \rho G) \\ \nabla \cdot v &= q \end{aligned}$$

Variational multiscale Method (VMS):



3D Matlab solver provided by SINTEF ICT

Similar results on mutiscale methods

MSc. thesis of Sergey Alyaev on
Adaptive Multiscale Methods Based on A Posteriori Error Estimates,
Bergen, June 2010

Basic estimate of the deviation from exact solution

For any $w \in H$ it holds

$$\frac{1}{2}|||u - v, p - q|||^2 \leq \mathcal{H}(v, q) - \mathcal{H}(u, p),$$

where $z = (u, p)$ is an exact elastoplastic solution
and $w = (v, q)$ is a discrete approximation.

where

$$|||u - v, p - q||| := \|\mathbb{C}(\varepsilon(u - v) - (p - q))\|^2_{\mathbb{C}^{-1}} + \sigma_y^2 H^2 \|q - p\|^2 .$$

Note, $H > 0$ represents a hardening parameter (done for isotropic hardening model).

Perturbed problem

Original problem

$$\mathcal{H}(v, q) := \frac{1}{2} a(v, q; v, q) - l(v) + \int_\Omega \sigma_y |q| \, dx$$

Perturbed problem

$$\mathcal{H}_\lambda(v, q) := \frac{1}{2} a(v, q; v, q) - l(v) + \int_\Omega \sigma_y \lambda : q \, dx$$

where $\lambda \in \Lambda := \{\lambda \in L^\infty(\Omega, \mathbb{R}^{d \times d}) : |\lambda| \leq 1, tr(\lambda) = 0 \;\; a.\; e. \text{ in } \Omega\}$.

$$\sup_{\lambda \in \Lambda} \mathcal{H}_\lambda(v, q) = \mathcal{H}(v, q)$$

Lagrangian

Lagrangian

$$L_\lambda(v,q;\tau,\xi) := \int_\Omega \left(\tau : (\varepsilon(v) - q) - \frac{\mathbb{C}^{-1}\tau : \tau}{2} + \xi : q - \frac{|\xi|^2}{2\sigma_y^2 H^2} - fv\right) dx + \int_\Omega \sigma_y \lambda : q \, dx,$$

where $\tau \in Q := L^2(\Omega; \mathbb{R}^{d\times d}_{sym}), \xi \in Q_0 := \{q \in Q : tr(q) = 0 \;\; a.\ e. \text{ in } \Omega\}$.

$$\sup_{\tau\in Q,\xi\in Q_0} L_\lambda(v,q;\tau,\xi) = \mathcal{H}_\lambda(v,q)$$

First estimate

It holds for all $\lambda \in \Lambda$

$$\mathcal{H}(u,p) = \inf_{v,q} \mathcal{H}(v,q) \geq \inf_{v,q} \mathcal{H}_\lambda(v,q) \geq \inf_{v,q} L_\lambda(v,q;\tau,\xi)$$

which yields the estimate

$$\frac{1}{2}|||(u-v),(p-q)|||^2 \leq \mathcal{H}(v,q) - \inf_{v,q} L_\lambda(v,q;\tau,\xi)$$

How to compute $\inf_{v,q} L_\lambda(v,q;\tau,\xi)$?

Majorant estimate for equilibrated fields

$$\frac{1}{2}|||(u-v),(p-q)|||^2 \leq \inf_{(\tau,\xi)\in Q_{f_\lambda}} \mathcal{M}(v,q,\tau,\xi,\lambda),$$

where

$$\mathcal{M}(v,q,\tau,\xi,\lambda) = \frac{1}{2}\int_\Omega \mathbb{C}(\varepsilon(v)-q-\mathbb{C}^{-1}\tau):(\varepsilon(v)-q-\mathbb{C}^{-1}\tau)\,dx + \frac{1}{2}\int_\Omega \sigma_y^2 H^2 (q - \frac{1}{\sigma_y^2 H^2}\xi)^2\,dx + \int_\Omega (\sigma_y|q| - \sigma_y\lambda : q)\,dx$$

and

$$Q_{f_\lambda} := \{(\tau,\xi)\in Q\times Q_0 : \operatorname{div}\tau + f = 0, \tau^D = \xi + \sigma_y\lambda \;\; a.\ e.\ \text{in}\ \Omega\}.$$

Structure of Functional Majorant

$\mathcal{M}(v, q, \tau, \xi, \lambda) = 0$ if and only if

$$\tau = \mathbb{C}(\varepsilon(v) - q), \tag{10}$$

$$\operatorname{div} \tau + f = 0, \tag{11}$$

$$\lambda : q = |q|, \qquad \lambda \in \Lambda, \tag{12}$$

$$\tau^D = \xi + \sigma_y \lambda, \tag{13}$$

$$\xi = \sigma_y^2 H^2 q. \tag{14}$$

These are conditions for the exact solution (u, p) of the elastoplastic minimization problem! The majorant naturally reflects properties of the original problem.

Majorant estimate for nonequilibrated fields

$$\frac{1}{2}|||(u-v),(p-q)|||^2 \leq \inf_{(\tau,\xi)\in Q_{f_\lambda}} \hat{\mathcal{M}}(v,q;\hat{\tau},\lambda,\beta,\delta),$$

where

$$\begin{aligned}\hat{\mathcal{M}}(v,q;\hat{\tau},\lambda,\beta,\delta) := \frac{1}{2}(1+\beta)\int_\Omega \mathbb{C}(\varepsilon(v)-q-\mathbb{C}^{-1}\hat{\tau}) : (\varepsilon(v)-q-\mathbb{C}^{-1}\hat{\tau})\,dx \\ +\frac{1}{2}(1+\delta)\int_\Omega \frac{1}{\sigma_y^2 H^2}(\hat{\tau}^D-\zeta)^2 dx + \int_\Omega (\sigma_y|q| - \sigma_y\lambda : q)\,dx \\ +\frac{1}{2}\left[(1+\frac{1}{\beta}) + \frac{c_2}{\sigma_y^2 H^2}(1+\frac{1}{\delta})\right] C^2 \left\|\operatorname{div}\hat{\tau}+f\right\|^2\end{aligned}$$

and $\hat{\tau} \in Q_{\mathrm{div}} := \{\tau \in Q : \operatorname{div}\tau \in L^2(\Omega,\mathbb{R}^d)\}, \quad \zeta := \sigma_y^2 H^2 q + \sigma_y \lambda.$

Papers

Sergey Repin, Jan Valdman,
Functional a posteriori error estimates for incremental models in elasto-plasticity.
Cent. Eur. J. Math. 7, No. 3, 506-519 (2009)

Thank you for your attention!