A note on the hierarchy of algebraizable logics

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Introduction – 1

Abstract Algebraic Logic (AAL): Abstract study and classification of (propositional) logics based on their relation to algebras. (Universal logic)

Algebraizable logics (Blok-Pigozzi, 1989):

- propositional logics enjoying an algebraic counterpart *in the same way as classical logic*
- logics are finitary and their algebraic counterpart are quasivarieties
- translations between logical entailment and equational consequence
- these translations are given by finite sets of formulae
- correspondence between logical and algebraic properties

AAL has extended Blok and Pigozzi's notion of algebraizability in two directions:

1. By considering *weaker links* between algebras and logics: weakly algebraizable, equivalential, protoalgebraic logics, ...
2. By dropping the finitary condition on BP-algebraizable logics, but formally keeping the same link (translations) between algebras and logics.

We concentrate on the latter.

Aim of this talk: Discuss (in)finiteness issues in the generalized notion of algebraizable logic, clarity their relations and obtain a hierarchy (classification) of algebraizable logics.

Propositional logic

Definition

A logic L in a language $\mathcal{L}$ is a relation $\vdash_{\mathrm{L}} \subseteq \mathcal{P}(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}}$ st.

- if $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Reflexivity)
- if $\Delta \vdash_{\mathrm{L}} \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Cut)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each substitution σ. (Structurality)

Observe that reflexivity and cut entail:

- if $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Monotonicity)

L is finitary iff for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{L}} \varphi$.

Algebraic counterpart – 1

$\mathcal{L}$-algebra: $\boldsymbol{A} = \langle A, \langle c^{\boldsymbol{A}} \mid c \in C_{\mathcal{L}} \rangle \rangle$, where $A \neq \emptyset$ (universe) and $c^{\boldsymbol{A}} \colon A^n \to A$ for each $\langle c, n \rangle \in \mathcal{L}$.

Algebra of formulae: the algebra $\boldsymbol{Fm}_{\mathcal{L}}$ with domain $Fm_{\mathcal{L}}$ and operations $c^{\boldsymbol{Fm}_{\mathcal{L}}}$ for each $\langle c, n \rangle \in \mathcal{L}$ defined as:

$$c^{\boldsymbol{Fm}_{\mathcal{L}}}(\varphi_1, \ldots, \varphi_n) = c(\varphi_1, \ldots, \varphi_n).$$

$\boldsymbol{Fm}_{\mathcal{L}}$ if the absolutely free algebra in language $\mathcal{L}$ with generators Var.

$\boldsymbol{A}$-evaluation: a homomorphism from $\boldsymbol{Fm}_{\mathcal{L}}$ to $\boldsymbol{A}$

$\mathcal{L}$-matrix: a pair $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ where

- $\boldsymbol{A}$ is an $\mathcal{L}$-algebra (algebraic reduct of $\mathbf{A}$)
- $F \subseteq A$ (filter of $\mathbf{A}$, set of designated elements).

Algebraic counterpart – 2

Semantical consequence: $\Gamma \models_{\mathbb{K}} \varphi$ if for each $\langle \boldsymbol{A}, F \rangle \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e, we have $e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$.

Lemma

Let $\mathbb{K}$ a class of $\mathcal{L}$-matrices. Then $\models_{\mathbb{K}}$ is a logic in $\mathcal{L}$. Furthermore if $\mathbb{K}$ is a finite class of finite matrices, then the logic $\models_{\mathbb{K}}$ is finitary.

$\mathbf{A}$ is an L-matrix iff $\mathrm{L} \subseteq \models_{\mathbf{A}}$ (i.e if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Gamma \models_{\mathbf{A}} \varphi$)

$\mathbf{MOD}(\mathrm{L})$: the class of all L-matrices.

Leibniz congruence

Let L be a logic and $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$.

Leibniz congruence: $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ iff for each formula χ and each $\mathbf{A}$-evaluation e it is the case that

$$e[p{\rightarrow}a](\chi) \in F \quad \text{iff} \quad e[p{\rightarrow}b](\chi) \in F$$

Theorem

$\Omega_{\boldsymbol{A}}(F)$ *is the largest congruence of* $\boldsymbol{A}$ *compatible with* F*.*

Algebraic counterpart – 3

An $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ is reduced, if $\Omega_{\boldsymbol{A}}(F)$ is the identity relation $\mathrm{Id}_{\boldsymbol{A}}$.
$\mathbf{MOD}^*(\mathrm{L})$: the class of all reduced L-matrices.

An algebra $\boldsymbol{A}$ is an L-algebra if there is a set $F \subseteq A$ s.t. $\langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})$.

$\mathbf{ALG}^*(\mathrm{L})$: the class of all reduced L-algebras.

Theorem (Completeness)

Let L be a logic. Then for any set Γ of formulae and any formula φ the following holds:

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi.$$

Generalized equivalences

Let $\mathbf{A} = \langle A, F\rangle$ be an arbitrary matrix and E a set of formulae in two variables.

We define a relation $\Omega^E_{\boldsymbol{A}}(F)$:

$$\langle a, b\rangle \in \Omega^E_{\boldsymbol{A}}(F) \quad \text{iff} \quad E^{\boldsymbol{A}}(a, b) \subseteq F$$

When is $\Omega^{\Leftrightarrow}$ the Leibniz congruence?

Theorem

Let L *be a logic and* $\Leftrightarrow$ *a set of formulae. TFAE:*

- $\Omega^{\Leftrightarrow}_{\boldsymbol{A}}(F)$ *is the Leibniz congruence of each* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$
- $\Omega^{\Leftrightarrow}_{\boldsymbol{A}}(F)$ *is the identity for all* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$
- L *satisfies:*

$$
\begin{array}{ll}
\text{(R)} & \vdash_{\mathrm{L}} \varphi \Leftrightarrow \varphi \\
\text{(MP)} & \varphi, \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} \psi \\
\text{(Cng)} & \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \Leftrightarrow c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n) \\
 & \textit{for each } \langle c, n\rangle \in \mathcal{L} \textit{ and each } 0 \leq i < n.
\end{array}
$$

Algebraizable logics

An equation in the language $\mathcal{L}$ is a formal expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in Fm_{\mathcal{L}}$.

A logic L is algebraizable if

1. there exists a set $\Leftrightarrow(p, q)$ of formulae st. $\Omega^{\Leftrightarrow}_{\boldsymbol{A}}(F)$ is the identity for each $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$.
2. there is a set of equations $\mathcal{T}$ in one variable such that for each $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$ and each $a \in A$ holds:

   $a \in F$ if, and only if, $\mu^{\boldsymbol{A}}(a) = \nu^{\boldsymbol{A}}(a)$ for every $\mu \approx \nu \in \mathcal{T}$.

We say that $\mathcal{T}$ is a truth definition.

L is regularly algebraizable if it further satisfies $p, q \vdash_{\mathrm{L}} p \Leftrightarrow q$.

Characterizations of algebraizable logics

$$\rho[\Pi] = \bigcup_{\alpha\approx\beta\in\Pi} (\alpha\Leftrightarrow\beta) \qquad \tau[\Gamma] = \{\alpha(\varphi)\approx\beta(\varphi) \mid \varphi\in\Gamma,\ \alpha\approx\beta\in\mathcal{T}\}$$

Theorem

Given a logic L*, TFAE:*

1. L *is algebraizable with the equivalence* $\Leftrightarrow$ *and the truth definition* $\mathcal{T}$*.*
2. *There is a set* $\mathcal{T}$ *of equations in one variable and a set* $\Leftrightarrow$ *of formulae in two variables such that:*
   1. $\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi\approx\psi$ *iff* $\rho[\Pi] \vdash_{\mathrm{L}} \rho(\varphi\approx\psi)$
   2. $p \dashv\vdash_{\mathrm{L}} \rho[\tau(p)]$
3. *There is a set* $\mathcal{T}$ *of equations in one variable and a set* $\Leftrightarrow$ *of formulae in two variables such that:*
   1. $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\tau[\Gamma] \models_{\mathbf{ALG}^*(\mathrm{L})} \tau(\varphi)$
   2. $p\approx q \mathrel{=}\!\mid\!\models_{\mathbf{ALG}^*(\mathrm{L})} \tau[\rho(p\approx q)]$

Finiteness issues

Theorem

Let L *be an algebraizable logic. Then the following hold:*

- *If* L *is finitary, then* τ *can be chosen finite*
- *If* $\models_{\mathbf{ALG}^*(\mathrm{L})}$ *is finitary, then* ρ *can be chosen finite*
- *If* L *is finitary and* ρ *is finite, then* $\models_{\mathbf{ALG}^*(\mathrm{L})}$ *is finitary.*
- *If* $\models_{\mathbf{ALG}^*(\mathrm{L})}$ *is finitary and* τ *is finite, then* L *is finitary.*

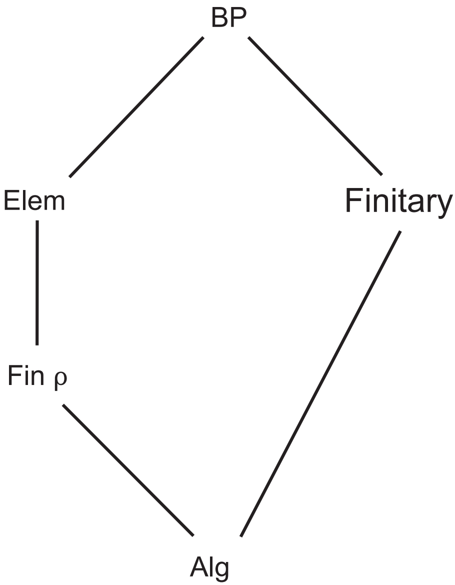
More kinds of algebraizable logics

Definition

An algebraizable logic L is

- finitely algebraizable if ρ can be taken finite
- elementarily algebraizable if $\mathbf{ALG}^*(\mathrm{L})$ is a quasivariety, i.e., $\models_{\mathbf{ALG}^*(\mathrm{L})}$ is finitary
- algebraizable in the sense of Blok-Pigozzi if it is finitary and finitely algebraizable

More kinds of algebraizable logics



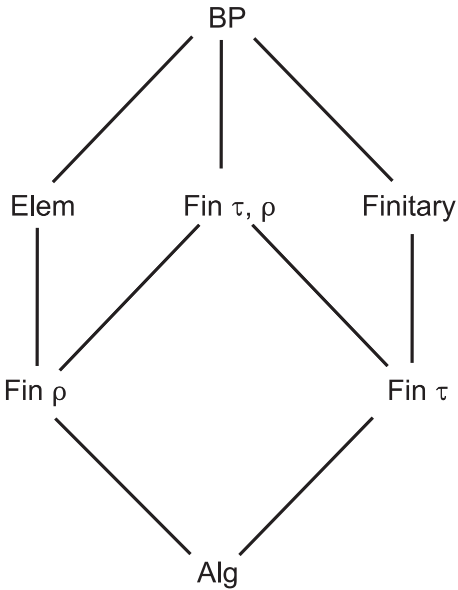
More on finiteness issues

Theorem

Let L *be an algebraizable logic. Then the following hold:*

- *If* L *is finitary, then it has a finite truth definition.*
- *If* L *is elementarily algebraizable, then* L *is finitely algebraizable.*
- *If* L *is finitary and finitely algebraizable, then* L *is elementarily algebraizable.*
- *If* L *is elementarily algebraizable with a finite truth definition, then* L *is finitary.*

Extending the hierarchy



Separating example – 1

Raftery's logic is elementarily finitely algebraizable but not finitary. It has the language $\{\Box, \leftrightarrow, \pi_1, \pi_2\}$, axioms:

$$\varphi \leftrightarrow \varphi \quad \varphi \leftrightarrow \pi_1(\varphi \leftrightarrow \psi) \quad \psi \leftrightarrow \pi_2(\varphi \leftrightarrow \psi) \quad (\varphi \leftrightarrow \psi) \leftrightarrow \Box(\varphi \leftrightarrow \psi)$$

and rules

$$\varphi, \varphi \leftrightarrow \psi \vdash \psi$$

$$\chi \leftrightarrow \delta, \varphi \leftrightarrow \psi \vdash (\chi \leftrightarrow \varphi) \leftrightarrow (\delta \leftrightarrow \psi)$$

$$\varphi \leftrightarrow \psi \vdash *\varphi \leftrightarrow *\psi \qquad * \in \{\pi_1, \pi_2, \Box\}$$

$$\varphi \vdash \pi_1(\Box^i\varphi) \leftrightarrow \pi_2(\Box^i\varphi) \qquad i \in \omega$$

$$\{\pi_1(\Box^i\varphi) \leftrightarrow \pi_2(\Box^i\varphi) \mid i \in \omega\} \vdash \varphi$$

Separating example – 2

Dellunde's logic is finitary regularly algebraizable but not finitely algebraizable. It has the language $\{\Box, \leftrightarrow\}$ and is axiomatized by:

$$\vdash \varphi \leftrightarrow \varphi$$

$$\varphi, \varphi \leftrightarrow \psi \vdash \psi$$

$$\varphi, \psi \vdash \Box^n \varphi \leftrightarrow \Box^n \psi$$

$$\varphi \leftrightarrow \psi, \varphi' \leftrightarrow \psi' \vdash \Box^n(\varphi \leftrightarrow \varphi') \leftrightarrow \Box^n(\psi \leftrightarrow \psi')$$

for each $n \in \omega$

Separating example – 3

Łukasiewicz logic $Ł_\infty$ is regularly finitely algebraizable but not finitary and not elementarily algebraizable. It has the language $\{\to, \neg\}$, axioms:

$$\varphi \to (\psi \to \varphi) \quad (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$

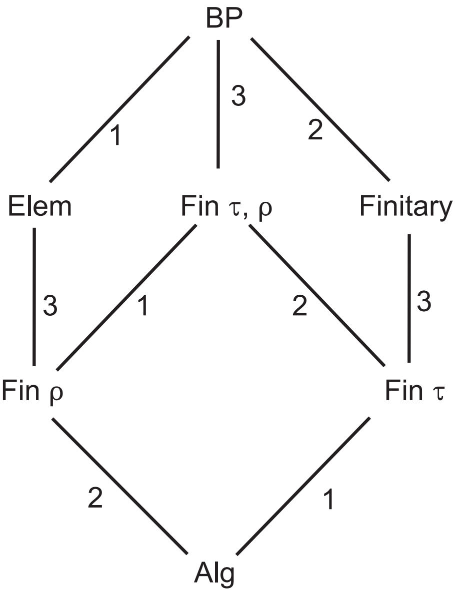
$$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi) \quad (\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

and rules:

$$\varphi, \varphi \to \psi \vdash \psi$$

$$\{i\varphi \to \psi \mid i \in \omega\} \cup \{\neg\varphi \to \psi\} \vdash \psi$$

Extending the hierarchy



Weakly implicative logics

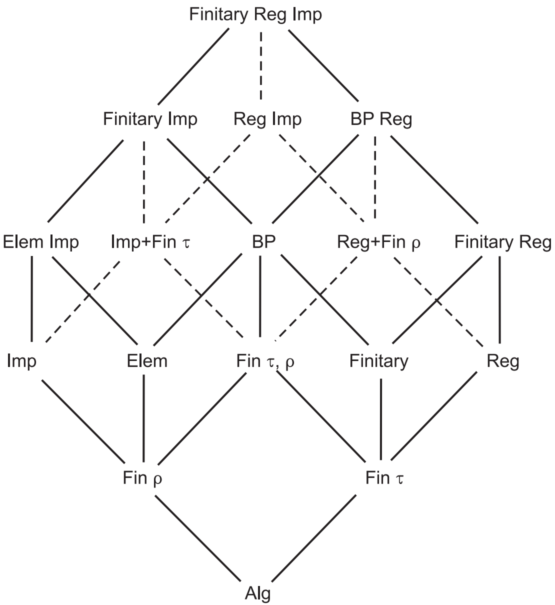
A logic L is weakly implicative if it satisfies any of the following equivalent conditions:

1. There is a set $\Leftrightarrow(p,q) = \{p \to q, q \to p\}$ of formulae s.t.

   (R) $\vdash_{\mathrm{L}} \varphi \Leftrightarrow \varphi$
   (T) $\varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi$
   (MP) $\varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi$
   (Cng) $\varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_i, \varphi, \ldots) \Leftrightarrow c(\chi_1, \ldots, \chi_i, \psi, \ldots)$
   for each $\langle c, n\rangle \in \mathcal{L}$ and each $0 \leq i < n$.

2. There exists a formula $p \to q$ st. $\Omega_{\boldsymbol{A}}^{p\to q}(F)$ is an order on $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$ and F is an upper set w.r.t. this order.

Extending the hierarchy even more



More separating examples

4 Linear logic: finitary implicative, but not regularly algebraizable.

5 $L_{\rightarrow_1,\rightarrow_2}$: regularly BP-algebraizable but not weakly implicative.

Extending the hierarchy even more