RT1-code: A mixed $RT_0 - P_0$ Raviart-Thomas finite element implementation

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Abstract

In this paper we shall describe mixed formulations - differential and variational- of a model elliptic problem, which can be interpreted as Darcy flow model. We describe * Galerkin method with finite dimensional spaces; * Local matrices and assembling; * Raviart-Thomas $RT_0 - P_0$ elements; * Edge basis and local matrices for $RT_0 - P_0$ FEM; * Model problem with corresponding local matrices, right hand side and treatment of boundary conditions; * Efficient assembling, * Use for generating saddle point systems, testing solvers and preconditioners.

Contents

1 Introduction

This report describes basis of RT1 code, which can be characterized as a code for testing solvers and preconditioners for FEM systems arising from lowest order Raviart-Thomas discretization of Darcy flow problems, see also [2, 1]. The code is characterized by

- simplicity and possibility of easy modifications,
- directly solving model problems on square domains (generalization possible),
- stochastic generation of heterogeneity,
- fast system assembling using vectorization and sparse reconstruction,
- possible testing of Krylov type solvers with both (block) matrix and matrix free (variable) preconditioners.

This report describes the finite element system generation, experiments are involved in papers, e.g. [3].

2 Problem formulation

Let us consider Darcy flow elliptic problem in the form

$$\begin{aligned} -\mathrm{div}(k\,(-g + \mathrm{grad}\,p) &= f \qquad \text{in } \Omega \\ p &= \hat{p} \qquad \text{on } \Gamma_D \\ (-k\,\mathrm{grad}\,p)\cdot n &= \hat{u} \qquad \text{on } \Gamma_N \end{aligned}$$

where $g \neq 0$ if we consider elevation changes. It can be also written in a two field form with two basic variables $p : \Omega \to R^1$ and $u : \Omega \to R^n$

$$\left.\begin{aligned} k^{-1}u + \operatorname{grad} p &= g \\ \operatorname{div}(u) &= f \end{aligned}\right\} \text{in } \Omega$$

$$\begin{aligned} p &= \hat{p} \text{ on } \Gamma_D \\ (-k \operatorname{grad} p) \cdot n &= \hat{u} \text{ on } \Gamma_N \end{aligned}$$

The variational formulation uses test functions v and q to get

$$\int_\Omega k^{-1}u \cdot v\, dx + \int_\Omega \nabla p \cdot v\, dx = \int_\Omega g \cdot v\, dx$$

$$\int_\Omega \operatorname{div}(u) q = \int_\Omega f q\, dx$$

Transformation of one mixed term then provides

$$\begin{aligned} \int\limits_\Omega \nabla p \cdot v &= \int\limits_\Omega \sum_k \frac{\partial p}{\partial x_k} v_k dx = \sum_k \left\{ \int\limits_{\partial\Omega} p\, v_k \cdot n_k - \int\limits_\Omega p \frac{\partial v_k}{\partial x_k} dx \right\} = \\ &= \int\limits_{\partial\Omega} p(v \cdot n) - \int\limits_\Omega p \operatorname{div}(v) dx \end{aligned}$$

Then the variational formulation gets the form

$$\begin{array}{lllll} \int_\Omega k^{-1}u \cdot v & -\int_\Omega \operatorname{div}(v) \cdot p & = \int_\Omega g \cdot v\, dx & -\int\limits_{\Gamma_D} \hat{p}(v \cdot n) \quad -\int\limits_{\Gamma_N} p(v \cdot n) & \forall v \\ -\int_\Omega \operatorname{div}(u) q & & = & -\int_\Omega f q & \forall q \end{array}$$

or in abstract form: find $(u, p) \in U_N \times P$

$$\begin{array}{llcll} m(u, v) & +b(v, p) & = & G(v) & \forall v \in U_0 \\ b(u, q) & & = & F(v) & \forall q \in P \end{array}$$

where

$$\begin{aligned} U &= \{v \in L_2(\Omega)^n : \ \operatorname{div}(v) \in L_2(\Omega)\} \to H(\operatorname{div}) \\ U_0 &= \{v \in U : \ v \cdot n = 0 \text{ on } \Gamma_N\} \\ U_N &= \{v \in U : \ v \cdot n = \hat{u} \text{ on } \Gamma_N\} \\ P &= \{q \in L_2(\Omega)\} \end{aligned}$$

Note that pressure BC enters $G(v) = \ldots - \int\limits_{\Gamma_0} \hat{p}(v \cdot n)$ whereas velocity BC are included in U_N .

3 Galerkin method - Mixed FEM

We start with introducing FEM spaces $U_h \subset U$, $U_{Nh} \subset U_N$, $U_{0h} \subset U_0$ and $P_h \subset P$. Then the Galerkin method is to find $(u_h, p_h) \in U_{hN} \times P_h$

$$\begin{array}{llcll} m(u_h, v_h) & +b(v_h, p_h) & = & G(v_h) & \forall v_h \in U_{0h} \\ b(u_h, q_h) & & = & F(q_h) & \forall p_h \in P_h \end{array}$$

After a choice of bases

$$\begin{aligned} U_h &= \operatorname{lin}\{\Phi_i,\, i \in I\}, \ \ P_h = \operatorname{lin}\{\Psi_j : \, j \in J\} \\ U_{Nh} &= u_N + u, \quad u \in U_{0h} \\ U_{0h} &= \operatorname{lin}\{\Phi_i : \, i \in I_0\} \\ u_N &\in \operatorname{lin}\{\Phi_i : \, i \in I \setminus I_0\}, \ u_N = \sum (\hat{u} \cdot n)(x_i)\Phi_i \end{aligned}$$

the discrete mixed problem can be written as - find $(u_h, p_h) \in U_{hN} \times P_h, \quad u_h = u_N + \sum\limits_{i\in I_0} \alpha_i \Phi_i, \quad p_h = \sum\limits_{j\in J} \beta_j \Psi_j$

$$\begin{aligned} \sum_{i\in I_0} \alpha_i m(\Phi_i, \Phi_k) + \sum_{j\in J} \beta_j b(\Phi_k, \Psi_j) &= G(\Phi_k) - m(u_N, \Phi_k) \quad \forall k \in I_0 \\ \sum_{i\in I_0} \alpha_i b(\Phi_i, \Psi_l) &= F(\Psi_l) - b(u_N, \Psi_l) \qquad \forall l \in J \end{aligned}$$

Rewritting to matrix form provides

$$\begin{array}{llll} M\underline{\alpha} & +B^T\underline{\beta} & = \underline{G} & \underline{\alpha} \in R^{n_1},\ n_1 = \#I_0 \\ B\underline{\alpha} & & = \underline{F} & \underline{\beta} \in R^{n_2},\ n_2 = \#J \end{array}$$

where $M \in R^{n_1\times n_1}$, $M_{ij} = m(\Phi_j, \Phi_i)$, $B \in R^{n_2\times n_1}$, $B_{ij} = b(\Phi_j, \Psi_i)$, $B^T \in R^{n_1\times n_2}$, $B^T_{ij} = b(\Phi_i, \Psi_j) = B_{ji}$, $G = (G_i)$, $G_i = G(\Phi_i)$, $F = (F_k)$, $F_k = F(\Psi_k)$.

4 Local matrices and assembling

Assume that Φ_i and Ψ_i are constructed as finite element basis functions above some triangulation $\mathcal{T}_h$, i.e. $\forall T \in \mathcal{T}_h$

$$\begin{aligned} \Phi_i|_T &\in \{\overline{\Phi}_1, \dots \overline{\Phi}_\rho,\ 0 = \overline{\Phi}_0\} \\ \Psi_j|_T &\in \{\overline{\Psi}_1, \dots \overline{\Phi}_\sigma,\ 0 = \overline{\Psi}_0\} \end{aligned}$$

Then

$$\begin{aligned} m(\Phi_i, \Phi_k) &= \int\limits_\Omega k^{-1}\Phi_i \cdot \Phi_k \, dx = \sum_{T\in\mathcal{T}_h} \int\limits_T K^{-1}\Phi_i \cdot \Phi_k \, dx = \sum_{T\in\mathcal{T}_h} \int\limits_T k^{-1}\overline{\Phi}_{loc(i)}\overline{\Phi}_{loc(k)} \, dx \\ b(\Phi_i, \Psi_j) &= \int\limits_\Omega (\mathrm{div}\Phi_i)\Psi_j \, dx = \sum_{T\in\mathcal{T}_h} \int\limits_T (\mathrm{div}\, \overline{\Phi}_{loc(i)})\overline{\Psi}_{loc(j)} \, dx \end{aligned}$$

where $loc_k(i) = loc_k(i, T)$ is a transformation from global index to local index of basis function on T. It can be also zero.

Vice versa, for $T \in \mathcal{T}_h$, it is possible to construct local matrices

$$\begin{aligned} M_T,\ (M_T)_{rs} &= \int\limits_T k^{-1}\overline{\Phi}_s \cdot \overline{\Phi}_r \, dx \\ B_T,\ (B_T)_{rs} &= -\int\limits_T \mathrm{div}\, \overline{\Phi}_s \cdot \overline{\Psi}_r \, dx \end{aligned}$$

and then perform the assembling of local matrices to global M, B

$$\begin{aligned} (M_T)_{rs} &\to M_{glob(T,r)glob(T,s)} = +(M_T)_{rs} \\ (B_T)_{rs} &\to B_{glob_1(T,r)glob_2(T,s)} = +(B_T)_{rs} \end{aligned}$$

Note there are two sets of basis functions $\{\Phi_i\}$, $\{\Psi_j\}$, two sets of local basis functions $\{\overline{\Phi}_i\}$, $\{\overline{\Psi}_j\}$ and two mappings

$$\begin{aligned} loc_1(i) &= loc_1(i, T),\ loc_2 \\ glob_1(r, T) &= i,\ glob_2(s, T) = j. \end{aligned}$$

5 Lowest order Raviart-Thomas finite elements

Let $\Omega \subset R^2$ be a 2D polygonal domain, $\mathcal{T}_h$ be its triangulation, $\mathcal{E}_h$ be set of edges of all elements $T \in \mathcal{T}_h$, see the situation in the following Figure 1.

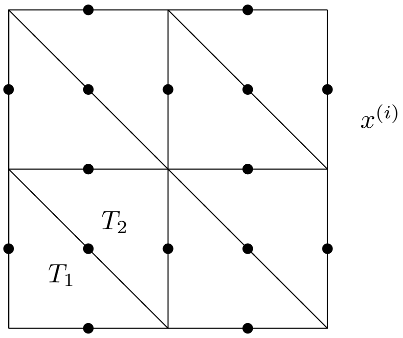


Figure 1: $\{x^{(i)}\}$ set of centres of $E_i \in \mathcal{E}_h$, $\{y^{(j)}\}$ barycentres of $T_j \in \mathcal{T}_h$

Then, we can define

$$RT_0(T) = \{v : T \to R^2,\ v(x) = \xi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},\ \xi, \eta_1, \eta_2 \in R\}$$

$$U_h = \{v : \Omega \to R^2,\ v|_T \in RT_0(T) \quad \forall T \in \mathcal{T}_h,\ v \cdot n_E \text{ is continuous over } E \in \mathcal{E}_h\}$$

$$P_h = \{q : \Omega \to R^1,\ q|_T \text{ is constant } \forall T \in \mathcal{T}_h\}.$$

Continuity of $v \cdot n_E$ guarentees $U_h \subset U$, $P_h \subset P$ is obvious. Note that $\forall E \in \mathcal{E}_h$ we define n_E (unit normal vector), independently of relation to triangles and consequently in possibly inner or outer direction, see Figure 2.

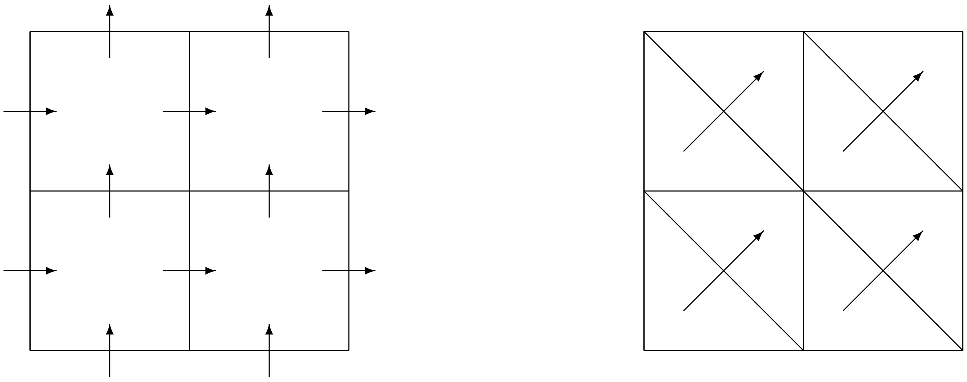


Figure 2: Prescribed normal n_E. Possible definition of $n_E, E \in \mathcal{E}_h$.

6 Local properties and local edge basis for RT(0) elements

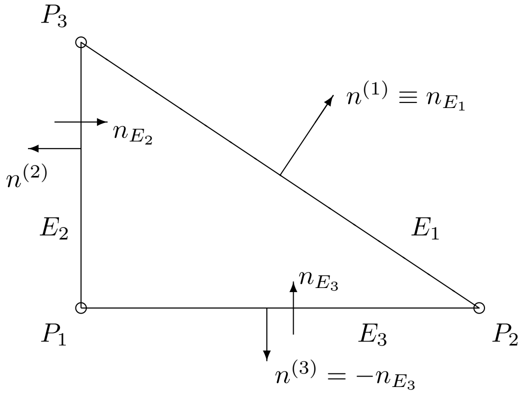


Figure 3: $T \in \mathcal{T}_h$

Lemma 6.1. *Let* $T \in \mathcal{T}_h$, $v \in RT_0(T)$. *Then* $\forall E \in \mathcal{E}_h \cup \partial T: \ v \cdot n|_E = const.$

Proof. Let $E \in \mathcal{E}_h \cup \partial T$, n_E be normal to E (can be either outer or inner to T), $x^* \in E$ be arbitrary point at E. Then

$$x \in E \ \Rightarrow \ (x - x^*) \cdot n_E = 0, \quad n_E = (n_1, n_2) \quad \Rightarrow$$

$$x_1 n_1 + x_2 n_2 = x_1^* n_1 + x_2^* n_2 = \text{const.} \quad \Rightarrow$$

$$\begin{aligned} v(x) \cdot n &= \xi x_1 n_1 + \xi x_2 n_2 + \eta_1 n_1 + \eta_2 n_2 \\ &= \xi (x_1^* n_1 + x_2^* n_2) + \eta_1 n_1 + \eta_2 n_2 = const. \end{aligned}$$

■

Lemma 6.2. *(Expression for local basis functions.) Let*

$$\overline{\Phi}_i(x) = \sigma_i \frac{|E_i|}{2|T|}(x - P_i), \quad \sigma_i = n_{E_i} \cdot n^{(i)},$$

where n_{E_i} *are global prescribed normals and* $n^{(i)}$ *are outer normals for* $T \in \mathcal{T}_h$, *see Figure 3. Then*
(i) $\overline{\Phi}_j(x) \cdot n_{E_i} = \delta_{ij}$,
(ii) $\overline{\Phi}_i \in RT_0(T)$,
(iii) $\overline{\Phi}_1, \overline{\Phi}_2, \overline{\Phi}_3$ *create a basis of* $RT_0(T)$,
(iv) div $\overline{\Phi}_i = \sigma_i \frac{|E_i|}{|T|}$.

Proof.

(i) If $i \neq j$, then $P_i \in E_j$ and $(x - P_i) \cdot n_{E_j} = 0$ for $x \in E_j$. If $i = j$ then for $x \in E_i$ the value $(x - P_i) \cdot n_{E_i}$ appears in the projection of $(x - P_i)$ to the height of T passing through P_i and therefore $|(x - P_i) \cdot n_{E_i}| = h_i$. Moreover, $\frac{1}{2} h_i |E_i| = |T|$ and $h_i = 2|T|/|E_i|$, $(x - P_i) \cdot n^{(i)} \geq 0$ - both vectors have outward direction w.r.t. T. Finally

$$(x - P_i) \cdot n_{E_i} = \sigma_i \frac{2|T|}{|E_i|}.$$

(ii) and **(iv)** are obvious

(iii) $u \in RT_0(T)$, $\quad w = u - \sum_1^3 (u \cdot n_{E_i}) \overline{\Phi}_i$. Obviously $w \cdot n_{E_i} = 0 \quad \forall E_i$. Therefore $\forall P_j : \ w(P_j) \cdot n_{E_i} = 0$ and because $\forall E_i : \ P_j \in E_i$, it holds $w(P_j) = 0 \quad \forall j = 1, 2, 3$. As w is linear polynomial, $w \equiv 0$. Proof of uniqueness: $w = \sum_1^3 \alpha_i \overline{\Phi}_i = 0 \ \Rightarrow \ w \cdot n_{E_j} = \alpha_j \overline{\Phi}_j n_{E_j} = \alpha_j = 0 \quad \forall j$.

■

7 Local matrices

Let us consider the local basis on T created by $\overline{\Phi}_1, \overline{\Phi}_2, \overline{\Phi}_3 \in RT_0(T)$ and $\Psi_1 \equiv 1$. Then $B_T \in R^{1 \times 3}$,

$$(B_T)_{1s} = \int_T (\text{div } \overline{\Phi}_s) \Psi_1 = \sigma_s \frac{|E_s|}{|T|} \cdot |T| = \sigma_s |E_s|,$$

i.e. $B_T = [\, \sigma_1 |E_1|, \, \sigma_2 |E_2|, \, \sigma_3 |E_3| \,] \in R^{1 \times 3}$. Further, $M_T \in R^{3 \times 3}$,

$$(M_T)_{rs} = \int_T k^{-1} \overline{\Phi}_s \overline{\Phi}_r \, dx = \sigma_r \sigma_s \frac{|E_r| \, |E_s|}{4|T|^2} \int_T k^{-1} (x - P_s) \cdot (x - P_r) \, dx.$$

To compute the integral $\int_T k^{-1}(x - P_s) \cdot (x - P_r) \, dx$, we can use barycentric coordinates at T,

$$x = \lambda_1(x) P_1 + \lambda_2(x) P_2 + \lambda_3(x) P_3, \qquad \lambda_1 + \lambda_2 + \lambda_3 = 1,$$

thus

$$x - P_r = \lambda_1(x)(P_1 - P_r) + \lambda_2(x)(P_2 - P_r) + \lambda_3(x)(P_3 - P_r)$$

and

$$(M_T)_{rs} = \sigma_r \sigma_s \frac{|E_r|\,|E_s|}{4|T|^2} \sum_{\alpha,\beta=1}^{3} \int\limits_T \lambda_\alpha \lambda_\beta \cdot k^{-1}(P_\alpha - P_s)\cdot(P_\beta - P_r)\,dx.$$

Assuming k constant on T and using the integration formula $\int\limits_T \lambda_\alpha \lambda_\beta = \frac{|T|}{12}(1+\delta_{\alpha\beta})$, which is a pecial case of

$$\int\limits_T \lambda_1^a\,\lambda_2^b\,\lambda_3^c\,dx = \frac{a!\,b!\,c!}{(a+b+c+2)!}\;2|T|$$

$$\int\limits_V \lambda_1^a\,\lambda_2^b\,\lambda_3^c\,\lambda_4^d\,dx = \frac{a!\,b!\,c!\,d!}{(a+b+c+d+3)!}\;6|V|$$

see e.g. [4, 5] the elements of M_T can be expressed as

$$(M_T)_{rs} = \frac{1}{48|T|}\sigma_r|E_r| \sum_{\alpha,\beta=1}^{3} (1+\delta_{\alpha\beta})k^{-1}(P_\alpha - P_s)\cdot(P_\beta - P_r)\;\sigma_s|E_s|.$$

If we define vectors $v_r,\,v_s \in R^{6\times 1}$,

$$v_r = \begin{bmatrix} P_1 - P_r \\ P_2 - P_r \\ P_3 - P_r \end{bmatrix}, \quad v_s = \begin{bmatrix} P_1 - P_s \\ P_2 - P_s \\ P_3 - P_s \end{bmatrix}, \quad p_i = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then

$$(M_T)_{rs} = \frac{1}{48|T|}\;\sigma_r|E_r|v_r^T \underbrace{\begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}}_{=^{df}\;C} \begin{bmatrix} k^{-1} & & \\ & k^{-1} & \\ & & k^{-1} \end{bmatrix} v_s\sigma_s|E_s|$$

Note that the diagonal elements are equal to elements of B_T. If we denote $C \in R^{6\times 6}$ the matrix, which appeared in the expression above and

$$V = [v_1, v_2, v_3] = \begin{bmatrix} 0 & P_1 - P_2 & P_1 - P_3 \\ P_2 - P_1 & 0 & P_2 - P_3 \\ P_3 - P_1 & P_3 - P_2 & 0 \end{bmatrix} \in R^{6\times 3},$$

then

$$(M_T) = \frac{1}{48|T|}\underbrace{\begin{bmatrix} \sigma_1|E_1| & 0 & 0 \\ 0 & \sigma_2|E_2| & 0 \\ 0 & 0 & \sigma_3|E_3| \end{bmatrix}}_{S\in R^{3\times 3}} V^T C \underbrace{\begin{bmatrix} k^{-1} & & \\ & k^{-1} & \\ & & k^{-1} \end{bmatrix}}_{L\in R^{6\times 6}} V \underbrace{\begin{bmatrix} \sigma_1|E_1| & 0 & 0 \\ 0 & \sigma_2|E_2| & 0 \\ 0 & 0 & \sigma_3|E_3| \end{bmatrix}}_{S},$$

i.e.

$$(M_T) = \frac{1}{48|T|} S V^T C L V S$$

where $S = \text{diag}\,[b_1|E_1|, b_2|E_2|, b_3|E_3|]$, $V = \begin{bmatrix} 0 & P_1 - P_2 & P_1 - P_3 \\ P_2 - P_1 & 0 & P_2 - P_3 \\ P_3 - P_1 & P_3 - P_2 & 0 \end{bmatrix}$, $L = \begin{bmatrix} k & & \\ & k & \\ & & k \end{bmatrix}^{-1} = \frac{1}{k_T}I$, if we consider the isotropic environment, $k = k_T I$ on T. For comparison see [2] formula (4.6).

Note that we constructed velocity mass matrix M. In the case of time dependent problems, we also need the pressure mass matrix $(\overline{M}_T)_{rs} = \int\limits_T \Psi_r \Psi_s = \delta_{rs}|T|$.

8 Model problem

We shall consider a model Darcy flow problems on a square domain with flow from left to right induced by the pressure gradient.

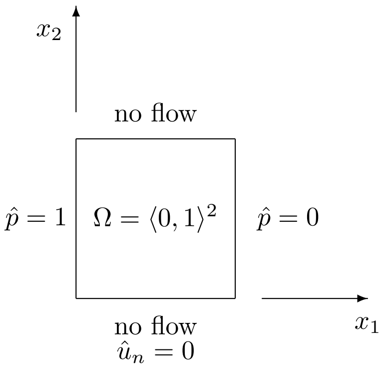


Figure 4: Model problem

The problem domain is divided into rectangular elements with the size characterized by the parameter ns = number of segments on the side.

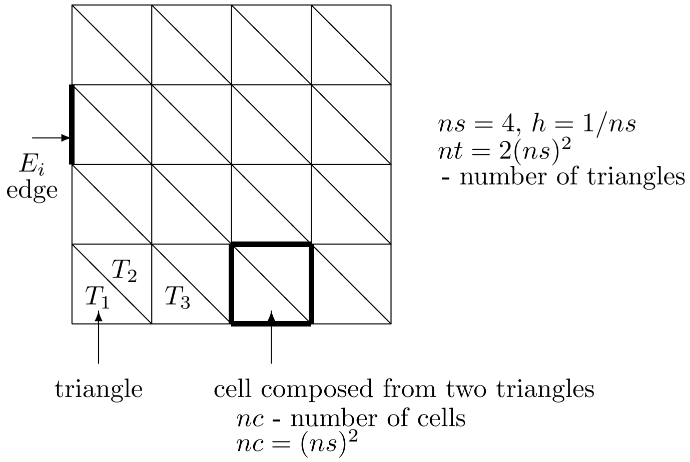
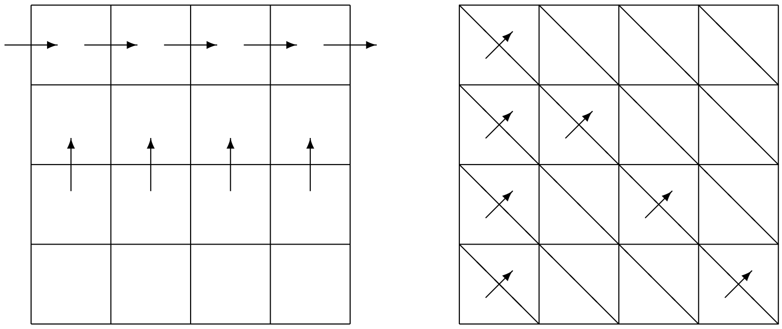


Figure 5: Discretization of the model problem.

Heterogeneity. We assume that each cell can possess a different permeability coefficient k_i, $i = 1, \ldots, nc = (ns)^2$. This can be produced by MATLAB using command sequence

```
1)    rng('default');
2)    RM = randn(ns,ns);
3)    LK = (exp(1).^(sigma*RM));
```

The first command initializes the random number generator to make the results in this example repeatable. The same sequence is generated as after restart of MATLAB. The second command generate a ns-by-ns matrix of normally distributed random numbers from $N(0, 1)$, i.e. with mean $\mu = 0$ and standard deviation 1. Then $\sigma * RM$ is a matrix of normally distributed random numbers with the mean $\mu = 0$ and standard deviation σ^2. Third command then creates matrix of conductivities such that $ln(LK)$ has normal distribution.



in the direction of coordinate system

upper right direction

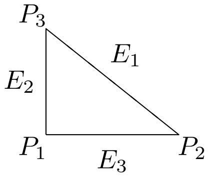
Figure 6: Global normals.

Orientation of (global) normals to element edges

Model problem - local matrices

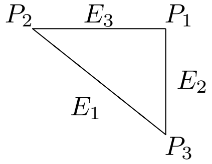
$$M_T = \frac{1}{24h^2} SV^T CLVS, \;\; L = \frac{1}{k_{cell}} I.$$

Lower triangle



$$B_T = [\sqrt{2}h, -h, -h], \; S = h \begin{bmatrix} \sqrt{2} & & \\ & -1 & \\ & & -1 \end{bmatrix}, \; V = h \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Upper triangle



$$B_T = [-\sqrt{2}h, h, h], \; S = h \begin{bmatrix} -\sqrt{2} & & \\ & 1 & \\ & & 1 \end{bmatrix} = -S_{low}, \; V_{upper} = h \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} = -V_{low}.$$

As a conclusion - the matrices $M_T = \frac{1}{24h^2} SV^T CLVS$ are the same for both lower and upper triangles.

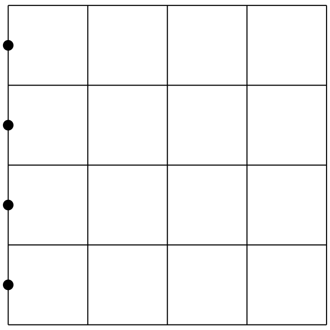
Right hand side and boundary conditions Consider the global system

$$\begin{array}{rcrcl} M\underline{\alpha} & + & B^T\underline{\beta} & = & \underline{G} \\ B\underline{\alpha} & & & = & \underline{F} \end{array}$$

where

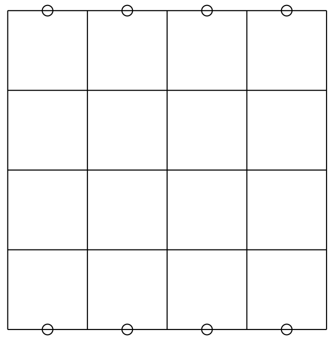
$$\underline{G}_i = \underbrace{-\int_{\Gamma_0} \hat{p}(p_i \cdot n)}_{\text{r.h.s. contribution}} - \underbrace{\sum_{k \in I \setminus I_0} \hat{u}_k m(\Phi_k, \Phi_i)}_{\text{l.h.s., in our case } \hat{u}_k = 0}$$

$$\underline{F}_j = - \underbrace{\int_\Omega f\Psi_j}_{= \int_{T_j} f = 0 \text{ in our case}} - \sum_{k \in I \setminus I_0} \hat{u}_k \underbrace{\int_\Omega \text{div } (\Phi_k)\Psi_j \, dx}_{\int_{T_j} \text{div}(\Phi_k); \; \hat{u}_k \text{are zero in our case}} \equiv 0$$



● → add $+h \cdot \hat{p}$, $\hat{p} = 1$
to the corresponding entry of $\underline{G}_i$

Figure 7: Pressure boundary conditions for the model problem.



○ → null corresp. rows, columns

Figure 8: Treatment of velocity boundary conditions: a) exclude corresponding rows and columns and rhs entries, b) or put 1 on diagonal otherwise zeros in corresponding row, columns and rhs entries

9 Assembling

Standard assembling

Algorithm 1 Standard assembling

```
define $M \equiv 0, B \equiv 0$
for 1:nt
    take $M_T, B_T$
        for $r = 1, \ldots, 3$
            for $s = 1, 2, 3$
                $M_{i(T,r)\ j(T,s)} = (M_T)_{rs}$
                $B_{\kappa(T)\ i(T,r)} = (B_T)_{1r}$
            end
        end
end
```

The standard assembing has two drawbacks: **for** cycles, which are not efficient in MATLAB, and *dense* matrix storage of the global matrix. Just replacing the global matrix declaration as *sparse* is not a good solution as it the sparse structure is not given apriori but must be constructed during the assembling process. This inefficiency can be removed by gradual recording the nonzero components and indices into one dimensional vectors X, I, J and constructin the matrix through

$$sparse(X,\ I,\ J,\ n,\ m).$$

Further improvement and loop avoiding can be done by vectorization, see [6]. The resulting code is able fast assembly very large matrices.

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