A NOTE ON NAIVE SET THEORY IN AN EXPANSION OF LP

The project of dialetheic set theories is based on a simple motivation: we want to see how much mathematics we can do by keeping the following claim of Graham Priest in his celebrated '*In Contradiction*':

> I wish to claim that (Abst) and (Ext) are true, and in fact that they analytically characterise the notion of set. ([4, p.30])

Here, (Abst) and (Ext) are $\exists y \forall x (x \in y \leftrightarrow \beta)$ and $\forall x (x \in z \leftrightarrow x \in y) \rightarrow z = y$ respectively where β is any formula which does not contain y free, and $\rightarrow$ and $\leftrightarrow$ are a suitable conditional and a suitable biconditional respectively. The challenge here is to find an appropriate conditional that serves well for our purpose.

Roughly speaking, there are two approaches to meet the challenge. One is to take the conditional to be a non-contractive relevant arrow. This approach had been facing a lot of difficulties until recently when Zach Weber proved some interesting results in [7, 8]. The other approach is to take the material biconditional defined in terms of paraconsistent negations. This line of research was initiated by Greg Restall in [5] in which naive set theory based on Logic of Paradox (**LP** hereafter) of Priest is shown to be non-trivial. The cost we pay here is that Modus Ponens will be lost. However, as remarked in [3], we have the benefit of being able to have stronger connectives in the language, such as classical negation, without losing the non-triviality of the theory, which is not possible in the first approach.

Based on these, the aim of the paper is to explore some consequences of naive set theories in an expansion of **LP**, called **dLP** (dialetheic **LP**). **dLP** is obtained by adding the consistency operator (or equivalently classical negation), introduced and studied in [2, 1], and a connexive conditional, following the idea of Heinrich Wansing presented in [6]. As a result, a smooth proof theory for the logic is available, and the logic becomes fully expressible in the sense that the matrix, adequate with respect to the logic, is functionally complete. After a brief review of the semantics and proof theory for **dLP**, I will introduce some naive set theories based on **dLP** which are non-trivial. Note here that we may formulate several theories depending on the formulation of (Ext). This is because we have at least three conditionals, i.e., a non-detachable material conditional, a detachable and connexive conditional, and a detachable and non-connexive conditional. I will compare these by considering the intuitive reading, and show some of the consequences of the axioms.

I then turn to consider possible paths towards stronger theories. Special attention is payed for the truth-untruth behavior of the set membership relation. This is due to the fact that once we formulate (Abst) with the help of the material biconditional, we do have the truth-false behavior of the set membership relation (recall that $\alpha \equiv \beta$ is true iff α and β are both true or both false in **LP**, where $\equiv$ is the material biconditional defined in **LP**), but not the truth-untruth behavior of the set membership relation. The problem is to specify some axioms that (i) do not make the theory trivial but (ii) are sufficiently strong to describe truth-untruth behavior of the set membership relation. If we understand **ZFC** based on classical logic as describing truth-untruth behavior of the set membership relation, then since the separation axiom is the one that modifies (Abst) formulated in terms of detachable conditional, we may think of adding the separation axiom. However, the separation axiom as it is will cause a problem in this context, since we can prove the existence of the universal set which gives us back Russell's paradox leading to triviality. Based on this, I will present some possible axioms, including some variants of the separation axiom, and explore some consequences.

References

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