The multiverse of naive set theory

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In this talk I will explore the idea of developing a multiverse perspective on *naive set theory*, i.e. collection theory based on the extensionality axiom

$$\forall x \forall x (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

and the unrestricted comprehension axiom schema

$$\exists x \forall y (y \in x \leftrightarrow A(y)).$$

The classical set theoretic multiverse is elaborated by Hamkins (cf. [1]) in the context of **ZF**-set theory. The multiverse perspective on set theory is based on the independence results in classical set theory. Many important sentences in classical set theory have been proven to be independent of the axioms of **ZF** (thanks to the forcing techniques). Up to this point no convincing arguments have been put forward to decide whether *the* set theoretic universe verifies these sentences are not. The multiverse view investigates set theory as the theory of a collection of interrelated set theoretic universes, in which there are two distinct universes for each independent sentence A: one in which A is true and one in which it is false. A modal logic is proposed as the syntactic tool to describe the multiverse.

This new perspective could also be extremely fruitful for the development of naive set theory. An inconsistency tolerant logic can make a distinction between several models of the above mentioned axioms. Some of these models are heavily inconsistent (in the extreme case: all sets have an inconsistent membership relation), in others the inconsistencies are more isolated (e.g. **ZF**-like sets may behave consistent). Arguably, none of these universes can a priori be excluded as a valuable set theoretic universe as they all represent a coherent (albeit inconsistent) notion of collection and membership. However, a posteriori, some can be seen to be more useful than others, because of their expressiveness and discriminatory value. Some of them can found[1] a good part of classical mathematics, others can found Quine's New Foundation mathematics, and yet others can found finitistic mathematics. It seems at least heuristically useful to stop focussing on finding *the* one true set theory and start developing syntactic tools to investigate the relations between all these inconsistent but well defined universes of naive set theory.

Dynamic proofs from the adaptive logic program (applied to naive set theory in [2]) can enable the explorer of the naive set theoretic universe to look for less inconsistent universes in a syntactic way. These universes are structured in a way that is similar to the universes of classical set theory and thus are better suited to found parts of actual mathematics.

References

[1] Joel David Hamkins. The set-theoretic multiverse. *Review of Symbolic Logic*. 5:416-449, 2012.

[2] Peter Verdée. Non-Monotonic Set Theory as a Pragmatic Foundation of Mathematics. *Foundations of Science*. 18(4):655–680, 2013.

[1] The verb 'to found' is here used in a non-metaphysical, reductionist sense: a theory or model founds another theory if the (non-)theorems of the latter can be translated into (non-)theorems of the former.