NAIVE MODAL SET THEORY

It is well-known that there are natural mathematical statements that cannot be settled by the standard axioms of set theory. The Zermelo-Fraenkel axioms, with the Axiom of Choice (ZFC), are incomplete. In light of the incompleteness phenomenon, this paper explores an alternative approach to set theory that couples a naive conception of set with a notion of mathematical modality.

The ZFC axioms were proposed as an alternative to the naive conception of set, which takes an unrestricted comprehension axiom as its sole set existence principle: for any condition Φ, there exists a set of all and only those things that satisfy the condition Φ. It is well-known that the naive conception of set is inconsistent. In response to the inconsistency of the naive conception, and to the extensive incompleteness of the ZFC axioms, we argue in favour of a modal conception of set. According to this conception, sets are merely possible, or potential, with respect to their members. The basic set existence principle that we articulate is a modal version of the unrestricted comprehension axiom: for any condition Φ, it is possible that there exists a set of all and only those things that satisfy the condition Φ.

We extend the language of set theory with modal operators. When formalizing modal notions, it is most common to extend the object language with two modal operators, the familiar $\Box$ for necessity and $\Diamond$ for possibility. The object language we use has four modal operators: $\Box$, $\Diamond$, $\blacksquare$, and $\blacklozenge$. These four modal operators let us scan the universe of sets in different directions: the familiar $\Box$ and $\Diamond$ look out to the higher reaches of the set-theoretic universe, while $\blacksquare$ and $\blacklozenge$ track backwards to previous stages in the set-formation process.

Given this modal language of set theory, we explore the following basic proof-theoretic question: What can be derived from a modal version of the unrestricted comprehension axiom?

Modal Unrestricted Comprehension: $\Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge\Phi(x)]$

We show that from this modal comprehension axiom, one can derive the existence of many interesting sets, including infinite sets. We also show that this modal set theory does not lead to contradiction in the usual way, and that there is reason to think it is consistent.

We build on contemporary work in modal set theory, which, though fruitful, has not been explored to its full potential. Early investigations in modal set theory include Fine (1981) and Parsons (1983b). Recent work on modal set theory has also been done by Linnebo (2010, 2013) and Studd (2013). The current project differs from these in several ways. The most significant difference concerns the proof-theoretic strength of the modal logic involved. Linnebo, Parsons, and Studd hone in on the modal logic S4.2 as the most appropriate logic for modal set theory, while Fine deploys the modal logic S5. We argue that the modal logic of set theory need not be as strong as either of these. We develop a modal set theory using a weaker modal logic, relying on the strength of the modal unrestricted comprehension axiom to minimize incompleteness.

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