Towards Paraconsistent Games via Topologies

Can BAŞKENT

Department of Computer Science, University of Bath

can@canbaskent.net   canbaskent.net/logic

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Outlook of the Talk

- Motivation
- Paraconsistent Social Software
- Paraconsistent Games

What do I mean by Paraconsistent Games?

Paraconsistency can be given a variety of justifications from a logical and mathematical perspectives.

However, it can also be approached from game theory. Rational agents can make inconsistent decisions, may have inconsistent preferences.

Game Semantics

Game semantics is perhaps the first simple step to combine games and logic.

Hintikkan classical game semantics assume that the game is a determined, two-player, zero-sum game.

Which logics can change this game structure?

What is the game for LP, FDE, Relevant logics etc.?

Inconsistent Preferences

Paraconsistent Preferences

The paraconsistent preference relation $\preceq$ can be axiomatized as follows.

(i) For any action a, $a \preceq a$,

(ii) For all actions a, b, c, $a \preceq b$ and $b \preceq c$ imply $a \preceq c$,

(iii) For all actions a, b, either $a \preceq b$ or $b \preceq a$ or $a \not\preceq b$ or $b \not\preceq a$,

Inconsistent Preferences

Stronger Paraconsistent Preferences

The strong paraconsistent preference relation $\propto$ is axiomatized as follows.

(i) For any action a, $a \propto a$ and $a \not\propto a$;

(ii) For all actions a, b, c, $a \propto b$ and $b \propto c$ imply $a \propto c$, and $a \not\propto b$ and $b \not\propto c$ imply $a \not\propto c$;

(iii) For all actions a, b, either $a \propto b$ or $b \propto a$ or $a \not\propto b$ or $b \not\propto a$;

Rationality and Inconsistent

Inconsistent Games

Inconsistent games, then, depend on

- Inconsistent preferences
- Inconsistent utilities (?)
- Irrational players (?)
- Inconsistent beliefs and epistemics

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- **Inconsistent beliefs and epistemics**

Social Software

The term social software was coined by Rohit Parikh in his 2002 paper (Parikh, 2002). Social software can be viewed as a research program which studies the construction and verification of social procedures by using tools in logic and computer science. By definition, it relates closely to a variety of neighboring fields including game theory, social choice theory and behavioral economics.

Social Software can be seen as a very broad and loose conceptualization of computational game theory.

However, social software has not been considered from a non-classical logical perspective (B., 2016).

Which Society?

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Social Software: Some Examples

People lie, cheat, make mistakes, and misunderstand each other, they happen to be wrong in their thoughts and actions, and all of these situations (and possibly many more) require an inconsistency-friendly framework for expressive power and normative predictions.

So, social procedures/protocols/interactions do require inconsistency-friendly (also sometimes, incompleteness-friendly) frameworks.

Paraconsistent Social Software

Example (Parikh, 2002)

Two horsemen are on a forest path chatting about something. A passerby, the mischief maker, comes along and having plenty of time and a desire for amusement, suggests that they race against each other to a tree a short distance away and he will give a prize of $100. However, there is an interesting twist. He will give the $100 to the owner of the slower horse.

Paraconsistent Social Software

The solution for this game "game" requires classical negation.

When there are > 2 players, it gets more complicated and the negation behaves as permutation (Olde Loohuis & Venema, 2010).

This is when we need paraconsistency.

Paraconsistent Social Software

In order to analyze a variety of interesting social procedures and phenomena, we may need to use a variety of different logics.

And social software, in all its richness, seems to provide an ideal domain to test the strengths (and weaknesses) of different formalisms.

Rich formalisms in non-classical logics, the extensive research in behavioral economics and the way it discusses the pluralities in rational and social behavior, and finally alternative economic theories open up new avenues for social software and relate it to a broader audience.

Inconsistent Obligations

"Ordinarily the rules of a game do not tell us how to proceed with the game after the rules have been violated. In such a case, we may: (1) go back to the point at which the rule was broken, correct the mistake, and resume the game; (2) call off the game; or (3) conclude that since one rule has been broken, others may now be broken, too.

But these possibilities are not open to us when we have broken a rule of morality. Instead we are required to consider the familiar duties associated with blame, confession, restoration, reparation, punishment, repentance, and remedial justice, in order to be able to answer the question: 'I have done something I should not have done-so what should I do now?' (Or even: 'I am going to do something I shouldn't do-so what should I do after that?') For most of us need a way of deciding, not only what we ought to do, but also what we ought to do after we fail to do some of the things we ought to do."

(Chisholm, 1963)

Game Theoretical Rationality

Von Neumann - Morgenstern idea of rationality is problematic.

Rational agents, who sacrifice, do not opt in to maximize their utilities and follow their deontological commitments, present difficulties for classical understanding of game theoretical rationality.

Inconsistent Games

From "ECON-ned"

The dominant economic paradigm, neoclassical economics, became ascendant in part because it offered a theory of behavior that could be teased out in elegant formulation. Yet it rests on assumptions that are patently ridiculous: that individuals are rational and utility-maximizing (which has become a slippery notion as to be meaningless), that buyers and sellers have perfect information, that there are no transaction costs, that capital flows freely.

(Smith, 2010)

Inconsistent Games

From "Logic of Life"

Fundamental to von Neumanns approach was the assumption that both players were as clever as von Neumann himself. (...) The second problem is that game theory becomes less useful if your opponent is fallible. If player two is not an expert, player one should play to exploit his mistakes rather than defend against brilliant strategies that will never be found. The worse the opponent, the less useful the theory is.

(Harford, 2009)

Paraconsistent Epistemic Games

The Brandenburg-Keisler paradox (BK paradox) is a two-person self-referential paradox in epistemic game theory (Brandenburger & Keisler, 2006).

The following configuration of beliefs is impossible:

The Paradox

Ann believes that Bob assumes that Ann believes that Bob's assumption is wrong.

The paradox appears if you ask whether "Ann believes that Bob's assumption is wrong".

Notice that this is essentially a 2-person Russell's Paradox.

Formalism

Topological semantics appears to be the first semantics suggested for modal logic in 1938 by Tsao-Chen (Tsao-Chen, 1938). Picking up from Tsao-Chen's work, McKinsey (later with Tarski) incorporated various other algebraic and topological tools into modal logic, always remaining within the limits of classical logic (McKinsey, 1941; McKinsey & Tarski, 1944; McKinsey, 1945; McKinsey & Tarski, 1946).

The strength of topological semantics arguably comes from its versatility. Topological primitives can be used to give meaning for intuitionistic, paraconsistent and modal logics allowing us to analyze topological spaces from a semantical view point (Mortensen, 2000; B., 2013).

Model

Brandenburger and Keisler use belief sets to represent the players' beliefs.

The model (U^a, U^b, R^a, R^b) that they consider is called a *belief structure* where $R^a \subseteq U^a \times U^b$ and $R^b \subseteq U^b \times U^a$.

The expression $R^a(x, y)$ represents that in state x, Ann believes that the state y is possible for Bob, and similarly for $R^b(y, x)$. We will put $R^a(x) = \{y : R^a(x, y)\}$, and similarly for $R^b(y)$.

At a state x, we say Ann believes $P \subseteq U^b$ if $R^a(x) \subseteq P$.

Semantics

A modal logical semantics for the interactive belief structures can be given.

We use two modalities $\Box$ and $\heartsuit$ for the belief and assumption operators respectively with the following semantics.

$$\begin{array}{lll} x \models \Box^{ab}\varphi & \textit{iff} & \forall y \in U^b . R^a(x,y) \ \textbf{\textit{implies}}\ y \models \varphi \\ x \models \heartsuit^{ab}\varphi & \textit{iff} & \forall y \in U^b . R^a(x,y) \ \textbf{\textit{iff}}\ y \models \varphi \end{array}$$

Note the bi-implication in the definition of the assumption modality!

What is a Topology?

Definition

The structure $\langle S, \sigma \rangle$ is called a topological space if it satisfies the following conditions.

1. $S \in \sigma$ and $\emptyset \in \sigma$
2. σ is closed under finite unions and arbitrary intersections

Collection σ is called a topology, and its elements are called *closed* sets.

Paraconsistent Topological Semantics

Use of topological semantics for paraconsistent logic is not new. To our knowledge, the earliest work discussing the connection between inconsistency and topology goes back to Goodman (Goodman, 1981).

In classical modal logic, only modal formulas produce topological objects.

However, if we stipulate that:

extension of *any* propositional variable to be a closed set (Mortensen, 2000), we get a paraconsistent system.

Problem of Negation

Negation can be difficult as the complement of a closed set is not generally a closed set, thus may not be the extension of a formula in the language.

For this reason, we will need to use a new negation symbol $\sim$ that returns the closed complement (closure of the complement) of a given set.

Topological Belief Models

The language for the logic of topological belief models is given as follows.

$$\varphi := p \mid \sim\varphi \mid \varphi \wedge \varphi \mid \Box_a \mid \Box_b \mid \boxplus_a \mid \boxplus_b$$

where p is a propositional variable, $\sim$ is the paraconsistent topological negation symbol which we have defined earlier, and $\Box_i$ and $\boxplus_i$ are the belief and assumption operators for player i, respectively.

Topological Belief Models

For the agents *a* and *b*, we have a corresponding non-empty type space A and B, and define closed set topologies τ_A and τ_B on A and B respectively. Furthermore, in order to establish connection between τ_A and τ_B to represent belief interaction among the players, we introduce additional constructions $t_A \subseteq A \times B$, and $t_B \subseteq B \times A$. We then call the structure $F = (A, B, \tau_A, \tau_B, t_A, t_B)$ a paraconsistent topological belief model.

A state $x \in A$ *believes* $\varphi \subseteq B$ if $\{y : t_A(x, y)\} \subseteq \varphi$. Furthermore, a state $x \in A$ *assumes* φ if $\{y : t_A(x, y)\} = \varphi$.

Semantics

For $x \in A$, $y \in B$, the semantics of the modalities are given as follows with a modal valuation attached to F.

$$
\begin{array}{lll}
x \models \Box_a \varphi & \textit{iff} & \exists Y \in \tau_B \textit{ with } t_A(x, Y) \to \forall y \in Y . y \models \varphi \\
x \models \boxplus_a \varphi & \textit{iff} & \exists Y \in \tau_B \textit{ with } t_A(x, Y) \leftrightarrow \forall y \in Y . y \models \varphi \\
y \models \Box_b \varphi & \textit{iff} & \exists X \in \tau_A \textit{ with } t_B(y, X) \to \forall x \in X . x \models \varphi \\
y \models \boxplus_b \varphi & \textit{iff} & \exists X \in \tau_A \textit{ with } t_B(y, X) \leftrightarrow \forall x \in X . x \models \varphi
\end{array}
$$

The Result

Theorem

The BK sentence is satisfiable in some paraconsistent topological belief models.

Counter-model satisfies the paradoxical sentence at the (inconsistency-friendly) boundary (B., 2015).

Further Non-Classicity

It is also possible to analyze the paradox from

- non-classical set theory,
- co-heyting algebras,
- category theory,
- product topologies.

(B., 2015)

Yablo-style Reformulation

Yablo's Paradox, on the other hand, is a *non*-self referential paradox unlike the Brandenburger - Keisler paradox (Yablo, 1993). Yablo considers the following sequence of sentences.

$$
\begin{aligned}
S_1 &: \forall k > 1, S_k \text{ is untrue,} \\
S_2 &: \forall k > 2, S_k \text{ is untrue,} \\
S_3 &: \forall k > 3, S_k \text{ is untrue,} \\
&\vdots
\end{aligned}
$$

Yablo shows that every sentence S_n is untrue. Then, "the sentences *subsequent* [his emphasis] to any given S_n are all untrue, whence S_n is true after all!" [ibid]. Yablo's paradox can be viewed as a non-self-referential liar's paradox.

Yablo-style Reformulation

Consider the following sequence of assumptions where numerals represent game theoretical agents.

$$A_1 : 1 \text{ believes that } \forall k > 1, k'\text{s assumption is untrue},$$
$$A_2 : 2 \text{ believes that } \forall k > 2, k'\text{s assumption is untrue},$$
$$A_3 : 3 \text{ believes that } \forall k > 3, k'\text{s assumption is untrue},$$
$$\vdots$$

Now, for a contradiction, assume A_n is true for some n. Therefore, n believes that $\forall k > n$, k's assumption is untrue. In particular, $n + 1$'s assumption is untrue. Then, $n + 1$ believes that $\forall k > n + 1$, k's assumption is true, which contradicts the initial assumption that A_n is true. The choice of n was arbitrary, so each A_n in the sequence is untrue.

Yablo-style Reformulation

$$A_1 : 1 \text{ believes that } \forall k > 1, k'\text{s assumption is untrue,}$$
$$A_2 : 2 \text{ believes that } \forall k > 2, k'\text{s assumption is untrue,}$$
$$A_3 : 3 \text{ believes that } \forall k > 3, k'\text{s assumption is untrue,}$$
$$\vdots$$

Now, similar to Yablo's reasoning, for any n, the sentences subsequent to A_n are all untrue rendering A_n true for each n. As the choice of n was random, each A_n turns out to be true.

Conclusion

I consider this work as a step towards paraconsistent / non-classical game theory.

Our long term goal is to give a broader theory of (non-classical, non-utilitarian) rationality via games and logic.

Thank you for your attention!

Talk slides and the papers are available at:

`www.CanBaskent.net/Logic`

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