Models of set theory in Łukasiewicz logic

Zuzana Haniková

Institute of Computer Science
Academy of Sciences of the Czech Republic

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(joint work with Petr Hájek)

Why fuzzy set theory?

- try to capture a mathematical world: develop fuzzy mathematics (indicate a direction)
- study the notion of a set, and rudimentary notions of set theory (some properties may be available on a limited scale; classically equivalent notions need not be available in a weak setting)
- wider set-theoretic universe: recast the classical universe of sets as a subuniverse of the universe of fuzzy sets
- Explore the limits of (relative) consistency. (Which logics allow for an interpretation of classical ZF? Which logics give a consistent system?)

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Programme

Work with classical metamathematics.

Consider a logic L, magenta weaker than classical logic.
(Also, with well-developed algebraic semantics.)

Consider an axiomatic set theory T, governed by L.

The theory T should:

- generate a cumulative universe of sets
- be provably distinct from the classical set theory
- be reasonably strong
- be consistent (relative to ZF)

Between classical and non-classical:
classical set-theoretic universe is a sub-universe of the non-classical one

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Plan for talk

1. Logics without the contraction rule
2. Łukasiewicz logic
3. A set theory can strengthen its logic
4. **A**-valued universes
5. the theory FST (over Ł)
6. generalizations

A family of substructural logics: $\mathrm{FL_{ew}}$ and extensions

Consider propositional language $\mathcal{F}$.
($\mathrm{FL_{ew}}$-language: $\{\cdot, \rightarrow, \wedge, \vee, 0, 1\}$.)

A logic in a language $\mathcal{F}$ is a set of formulas closed under substitution and deduction.

"Substructural" — absence of some structural rules (of the Gentzen calculus for INT).
In particular, $\mathrm{FL_{ew}}$ is contraction free.

Structural rules:

$$\frac{\Gamma, \varphi, \psi, \Delta \Rightarrow \chi}{\Gamma, \psi, \varphi, \Delta \Rightarrow \chi}\,(e) \qquad \frac{\Gamma, \Delta \Rightarrow \chi}{\Gamma, \varphi, \Delta \Rightarrow \chi}\,(w) \qquad \frac{\Gamma, \varphi, \varphi, \Delta \Rightarrow \chi}{\Gamma, \varphi, \Delta \Rightarrow \chi}\,(c)$$

Removal of these rules calls for some changes:

- splitting of connectives
- changes to interpretation of a sequent

NB: $\mathrm{FL_{ew}}$ is equivalent to Höhle's monoidal logic (ML).

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Algebraic semantics for FL_{ew}

A FL_{ew}-algebra is an algebra $\mathbf{A} = \langle A, \cdot, \rightarrow, \wedge, \vee, 0, 1 \rangle$ such that:

1. $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice, 1 is the greatest and 0 the least element
2. $\langle A, \cdot, 1 \rangle$ is a commutative monoid
3. for all $x, y, z \in A$, $z \leq (x \rightarrow y)$ iff $x \cdot z \leq y$

FL_{ew} is the logic of FL_{ew}-algebras.

FL_{ew}-algebras form a variety;

the subvarieties correspond to axiomatic extensions of FL_{ew}.

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FL$_{ew}$ and some extensions

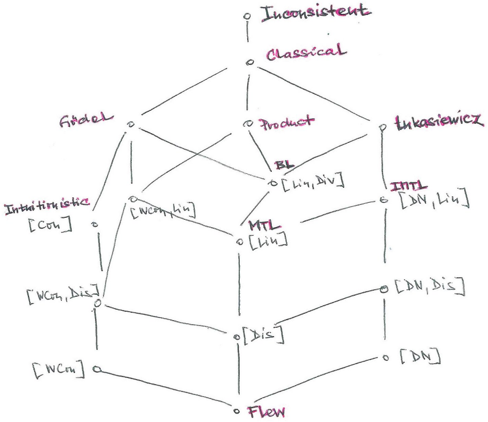

Figure:

Łukasiewicz logic

(More precisely, Łukasiewicz's infinite-valued logic, ca. 1920.
Denoted Ł.)

Usually conceived in a narrower language, such as:

- $\{+, \neg\}$
- $\{\rightarrow, \neg\}$ or $\{\rightarrow, 0\}$
- $\{\cdot, \rightarrow, 0\}$
- ...

Propositionally, the logic is given by the algebra

$$[0,1]_{Ł} = \langle [0,1], \cdot_{Ł}, \rightarrow_{Ł}, \min, \max, 0, 1 \rangle$$

with the natural order of the reals on $[0,1]$, and
$x \cdot_{Ł} y = \max(x + y - 1, 0)$
$x \rightarrow_{Ł} y = \min(1, 1 - x + y)$

NB: all operations of $[0,1]_{Ł}$ are continuous.
Hence, no two-valued operator is term-definable.

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Łukasiewicz logic — propositional axioms, completeness

Axioms:

- (Ł1) $\varphi \to (\psi \to \varphi)$
- (Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
- (Ł3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
- (Ł4) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

Deduction rule: modus ponens.

General algebraic semantics: MV-algebras.

Propositional Łukasiewicz logic is

- strongly complete w.r.t. MV-algebras
- finitely strongly complete w.r.t. $[0, 1]_{Ł}$

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Łukasiewicz logic with the Δ-projection

Semantics of Δ in a linearly ordered algebra **A**:

- $\Delta(x) = 1$ if $x = 1$
- $\Delta(x) = 0$ otherwise

Axioms:

- (Δ1) $\Delta\varphi \vee \neg\Delta\varphi$
- (Δ2) $\Delta(\varphi \vee \psi) \rightarrow (\Delta\varphi \vee \Delta\psi)$
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Łukasiewicz logic — first-order semantics

Assume the language $\{\in, =\}$.

Let $\mathbf{A}$ be an MV-chain.

Tarski-style definition of the value $\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v}$ of a formula φ in an $\mathbf{A}$-structure $\mathbf{M}$ and evaluation v in $\mathbf{M}$; in particular,

- ...
- $\|\forall x \varphi\|^{\mathbf{A}}_{\mathbf{M},v} = \bigwedge_{v \equiv_x v'} \|\varphi\|^{\mathbf{A}}_{\mathbf{M},v'}$
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An $\mathbf{A}$-structure $\mathbf{M}$ is safe if $\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v}$ is defined for each φ and v.

The truth value of a formula φ of a predicate language $\mathcal{L}$ in a safe $\mathbf{A}$-structure $\mathbf{M}$ for $\mathcal{L}$ is

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Łukasiewicz logic — first-order axioms

Axioms for quantifiers $\forall$, $\exists$:

($\forall$1) $\forall x \varphi(x) \to \varphi(t)$ (t substitutable for x in φ)
($\exists$1) $\varphi(t) \to \exists x \varphi(x)$ (t substitutable for x in φ)
($\forall$2) $\forall x(\chi \to \varphi) \to (\chi \to \forall x \varphi)$ (x not free in χ)
($\exists$2) $\forall x(\varphi \to \chi) \to (\exists x \varphi \to \chi)$ (x not free in χ)
($\forall$3) $\forall x(\varphi \vee \chi) \to (\forall x \varphi \vee \chi)$ (x not free in χ)

The rule of generalization: from φ entail $\forall x \varphi$.

NB: the two quantifiers are interdefinable in Ł.

Łukasiewicz logic — equality

Equality axioms for set-theoretic language:

- reflexivity
- symmetry
- transitivity
- congruence $\forall x, y, z(x = y \,\&\, z \in x \rightarrow z \in y)$
- congruence $\forall x, y, z(x = y \,\&\, y \in z \rightarrow x \in z)$

Moreover (for reasons given below), we postulate the law of the excluded middle for equality:

- $\forall x, y(x = y \vee \neg(x = y))$

Theorem

Let $T \cup \{\varphi\}$ be a set of sentences. Then $T \vdash_{Ł} \varphi$ iff for each MV-chain **A** *and each safe* **A***-model* **M** *of T, φ holds in* **M**.

NB: for a general language $\mathcal{L}$, the truths of $[0,1]_{Ł}$ are not recursively axiomatizable (in fact, they are Π_2-complete).

Analogous completeness for the expansion with Δ.

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Analogous completeness for the expansion with Δ.

Strengthening the logic

Let L be a consistent FL_{ew}-extension.
Let T be a theory over L.
If T proves $\varphi \vee \neg\varphi$ for an arbitrary φ, then
T is a theory over classical logic.

In other words,
adding the law of excluded middle (LEM): $\varphi \vee \neg\varphi$ to FL_{ew} yields classical logic.

Example: Grayson's proof of LEM from axiom of regularity:
Let $\{\emptyset \restriction \varphi\}$ stand for $\{x \mid x = \emptyset \wedge \varphi\}$.
Consider $z = \{\emptyset \restriction \varphi, 1\}$ (where $1 = \{\emptyset\}$)
Then z is nonempty, and consequently has a $\in$-minimal element.
If $\emptyset$ is minimal then φ holds,
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Thus, from regularity, one proves LEM for any formula.

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Thus, from regularity, one proves LEM for any formula.

Strengthening the logic

Lemma (Hájek ca. 2000)

Let Ł *be such that it proves the propositional formula* $(p \to p \,\&\, p) \to (p \vee \neg p)$. *Then, a set theory with*

- *separation (for open formulas),*
- *pairing (or singletons),*
- *congruence axiom for* $\in$

proves $\forall xy(x = y \vee \neg(x = y))$ *over* Ł .

Proof: take x, y.
Let $z = \{u \in \{x\} \mid u = x\}$, whence $u \in z \equiv (u = x)^2$.
Since $(x = x)^2$, we have $x \in z$.
If $y = x$ then $y \in z$ by congruence. Then $(y = x)^2$.
We proved $y = x \to (y = x)^2$, thus (by assumption on the logic) $x = y \vee \neg(x = y)$.

Strengthening the logic

Lemma (Grishin 1999)

In a theory with

- *extensionality,*
- *successors,*
- *congruence,*

LEM for $=$ *implies LEM for* $\in$*.*

Axioms of FST

- (ext.) $\forall xy(x = y \equiv (\Delta(x \subseteq y) \& \Delta(y \subseteq x)))$
- (empty) $\exists x \Delta \forall y \neg(y \in x)$
- (pair) $\forall x \forall y \exists z \Delta \forall u(u \in z \equiv (u = x \vee u = y))$
- (union) $\forall x \exists z \Delta \forall u(u \in z \equiv \exists y(u \in y \,\&\, y \in x))$
- (weak power) $\forall x \exists z \Delta \forall u(u \in z \equiv \Delta(u \subseteq x))$
- (inf.) $\exists z \Delta(\emptyset \in z \,\&\, \forall x \in z(x \cup \{x\} \in z))$
- (sep.) $\forall x \exists z \Delta \forall u(u \in z \equiv (u \in x \& \varphi(u, x)))$
  for any φ not containing free z
- (coll.) $\forall x \exists z \Delta[\forall u \in x \exists v \, \varphi(u, v) \to \forall u \in x \exists v \in z \varphi(u, v)]$
  for any φ not containing free z
- ($\in$-ind.) $\Delta \forall x(\Delta \forall y(y \in x \to \varphi(y)) \to \varphi(x)) \to \Delta \forall x \varphi(x)$
  for any φ
- (support) $\forall x \exists z(\mathrm{Crisp}(z) \& \Delta(x \subseteq z)))$

An **A**-valued universe

Work in classical ZFC.
Assume **A** is a complete (MV-)algebra.

Define $V^{\mathbf{A}}$ by ordinal induction.
$A^+ = A \setminus \{0^{\mathbf{A}}\}$.

- $V_0^{\mathbf{A}} = \{\emptyset\}$
- $V_{\alpha+1}^{\mathbf{A}} = \{f : \mathrm{Fnc}(f) \,\&\, \mathrm{Dom}(f) \subseteq V_\alpha^{\mathbf{A}} \,\&\, \mathrm{Rng}(f) \subseteq A^+\}$ for any ordinal α
- $V_\lambda^{\mathbf{A}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathbf{A}}$ for limit ordinals λ

$V^{\mathbf{A}} = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha^{\mathbf{A}}$

Define two binary functions from $V^{\mathbf{A}}$ into L, assigning to any $u, v \in V^{\mathbf{A}}$ the values $\|u \in v\|$ and $\|u = v\|$

$$\|u \in v\| = v(u) \text{ if } u \in D(v), \text{ otherwise } 0$$
$$\|u = v\| = 1 \text{ if } u = v, \text{ otherwise } 0$$

By induction on the complexity of formulas, define for any $\varphi(x_1, \ldots, x_n)$ an n-ary function from $(V^{\mathbf{A}})^n$ into L, assigning to an n-tuple $u_1, \ldots, u_n$ the value $\|\varphi(u_1, \ldots, u_n)\|$.

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Theorem

Let φ be a closed formula provable in FST. Then φ is valid in $V^{\mathbf{A}}$, i. e., ZF proves $\|\varphi\| = 1$.

We have obtained an interpretation of FST in ZFC.
FST is distinct from ZFC unless **A** is a Boolean algebra.

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An Inner Model of ZF in FST

Definition

(i) In a theory T, we say that a formula $\varphi(x_1, \ldots, x_n)$ in the language of T is *crisp* iff $T \vdash \forall x_1, \ldots, x_n \bowtie \varphi(x_1, \ldots, x_n)$.
(ii) In a (set) theory with language containing $\in$ we define $\mathrm{Crisp}(x) \equiv \forall u \bowtie (u \in x)$.

(Hereditarily crisp transitive set)

$$HCT(x) \equiv \mathrm{Crisp}(x) \& \forall u \in x(\mathrm{Crisp}(u) \& u \subseteq x)$$

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$$\mathrm{H}(x) \equiv \mathrm{Crisp}(x) \,\&\, \exists x' \in \mathrm{HCT}(x \subseteq x')$$

Lemma

The class H is both crisp and transitive in FST:

- $FST \vdash \forall x(x \in H \vee \neg(x \in H))$
- $FST \vdash \forall x, y(y \in x \,\&\, x \in H \to y \in H)$

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For φ a formula in the language of ZF, define φ^H inductively:
$\varphi^H = \varphi$ for φ atomic;
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Theorem

Let φ be a theorem of ZF. Then FST $\vdash \varphi^H$.

So H is an inner model of ZF in FST and ZF is consistent relative to FST.

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Ordinals and rank in FST

Let $\mathrm{Ord}_0(x)$ define ordinal numbers in classical ZFC.

The inner model H provides a suitable notion of ordinal numbers in FST:
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- $\mathrm{Ord}_0(x) \equiv \mathrm{Ord}_0^{\mathrm{H}}(x)$,
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Define ordinal numbers in FST:

$$\mathrm{Ord}(x) \equiv x \in \mathrm{H} \,\&\, \mathrm{Ord}_0(x)$$

Define:

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= WP(V_\alpha) \text{ for } \alpha \in \mathrm{Ord} \\ V_\alpha &= \bigcup_{\beta\in\alpha} V_\beta \text{ for a limit } \alpha \in \mathrm{Ord} \\ V &= \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha \end{aligned}$$

Then $\forall x \exists \alpha (x \in V_\alpha)$.

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Extensions and further work

- Work with an arbitrary MV-algebra (Chang's algebra).
  Can one get "nearly classical"?

  Lemma. Let $\mathbf{A}$ be an algebra, and let $\mathbf{M}$ be a model over $\mathbf{A}$.
  Let $\sim$ be a congruence on $\mathbf{A}$. Then $\mathbf{M}$ is a model over $\mathbf{A}/_{\sim}$.

- Work without Δ.

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