How to prove conservativity by means of Kripke models

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Prague Seminar on Non-Classical Mathematics

Prague, June 11–13, 2015

Classical and intuitionistic theories

Let $\mathcal{L}$ be a first-order language. And let

$$\begin{array}{lll} \mathtt{T} & \text{be} & \text{a set of sentences of the language } \mathcal{L}, \\ \mathtt{T}^c & := & \{A \in \mathcal{L} : \mathtt{T} \vdash^c A\}, \\ \mathtt{T}^i & := & \{A \in \mathcal{L} : \mathtt{T} \vdash^i A\}. \end{array}$$

Classical and intuitionistic theories

- Embeddings of CQC into IQC.
  Negative translations — Gödel, Gentzen, Kolmogorov, ...
- Conservativity.
  $\mathtt{T}^c$ *is* Γ*-conservative over* $\mathtt{T}^i$ iff for all $A \in \Gamma$,
  if $\mathtt{T}^c \vdash A$, then $\mathtt{T}^i \vdash A$.

  Friedman's translation.

  Syntactic conditions on theories in intuitionistic logic sufficient for the negative translation and conditions sufficient for the Friedman's translation.

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Negative translation

For any formula A:

$$A \mapsto A^{-}$$

- $A^{-} = \neg\neg A$, for atomic formulae A,
- $(\cdot)^{-}$ commutes with $\wedge$, $\rightarrow$, $\forall$,
- $(A \vee B)^{-} = \neg(\neg A^{-} \wedge \neg B^{-})$,
- $(\exists x A)^{-} = \neg\forall x \neg A^{-}$

Properties of negative translation

Theorem

If $\Gamma \vdash^c A$ then $\Gamma^- \vdash^i A^-$.

Theorem

If Γ is closed under negative translation, then $\Gamma \vdash A$ implies $\Gamma \vdash A^-$.

$\mathtt{HA}$ is closed under negative translation.

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`HA` is closed under negative translation.

The Friedman translation

Let D be a fixed formula. For any formula A (with some restrictions on the variables),

$$A \mapsto A^D$$

- $A^D = A \vee D$, for atomic formulae A,
- $(\cdot)^D$ commutes with $\wedge$, $\vee$, $\rightarrow$, $\forall$, and $\exists$.

Properties of the Friedman translation

Theorem
If $\Gamma \vdash^i A$ then $\Gamma^D \vdash^i A^D$.

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Π_2-conservativity

Theorem

$\mathtt{PA}$ is Π_2-conservative over $\mathtt{HA}$.

Proof (sketch).

Let A be a bounded formula.

$\mathtt{PA} \vdash \exists x A$

$\mathtt{HA} \vdash (\exists x A)^{-}$

$\mathtt{HA} \vdash ((\exists x A)^{-})^{\exists y A}$

$\mathtt{HA} \vdash \forall x (A(x) \to \exists y A(y)) \to \exists y A(y)$

$\mathtt{HA} \vdash \exists x A.$

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Π_2-conservativity: a generalization

Theorem

Let $\mathtt{T}^i$ be such a theory that all atomic formulae are decidable in $\mathtt{T}^i$, and let $\mathtt{T}^i$ be closed under the Friedman and negative translations. Then $\mathtt{T}^c$ is conservative over $\mathtt{T}^i$ with respect to the class of formulae of the form

$$\forall x \exists y A,$$

where A is a quantifier-free formula.

Classical and intuitionistic theories

The formula A is called

spreading if $\mathtt{IQC} \vdash A(B_1^{\mathrm{neg}}, \ldots, B_n^{\mathrm{neg}}) \rightarrow A(B_1, \ldots, B_n)^{\mathrm{neg}}$

wiping if $\mathtt{IQC} \vdash A(B_1, \ldots, B_n)^{\mathrm{neg}} \rightarrow A(B_1^{\mathrm{neg}}, \ldots, B_n^{\mathrm{neg}})$

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essentially isolating if it is of the form $\forall x(A \rightarrow \forall y B)$, with A spreading and B isolating.

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Classical and intuitionistic theories: known results

Theorem

Let $\mathtt{T}$ is closed under the negative translation and let $\mathtt{T} \vdash^c A$. Then $\mathtt{T} \vdash^i A$, provided that

1. A is wiping, or
2. A is isolating and $\bot$ only occurs positively in $\mathtt{T}$ and negatively in A.

Theorem

If A is the negation of a prenex formula, then whenever $\mathtt{CQC} \vdash A$ then $\mathtt{IQC} \vdash A$.

Classical and intuitionistic theories: a syntactic approach

Syntactic translations *just work*!

The syntactic conditions lead us to ‘artificial’ classes of formulae.

Let's try another way...

A Kripke model $\mathcal{M}$ is a tuple $(I, \leqslant, \{\mathcal{M}(w) : w \in I\}, \Vdash)$ such that

- $I \neq \emptyset$ and $(I, \leqslant)$ is a poset,
- $\mathcal{M}(u) \subseteq \mathcal{M}(w)$ if $u \leqslant w$,
- $w \Vdash A$ iff $\mathcal{M}(w) \models A$, for all atomic formulae A.

The forcing relation is inductively extended to the set of all formulae.

Let's try another way: a semantic approach

Use semantics to prove conservativity results.

Exploit the coexistence of classical satisfiability and intuitionistic forcing within a Kripke model.

Exploit the interplay between classical and intuitionistic theories in Kripke models.

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The key idea

To prove that $\mathtt{T}^c \vdash A$ iff $\mathtt{T}^i \vdash A$ one can show that

- if $\mathtt{T}^i \nvdash A$ then $\mathtt{T}^c \nvdash A$
- assuming that $\mathtt{T}^i \nvdash A$, by the Completeness Theorem, we have a Kripke model $\mathcal{M}$ such that

$$\mathcal{M} \Vdash \mathtt{T}^i \quad \text{and} \quad \mathcal{M} \nVdash A$$

- we look for a classical counter-model M such that

$$M \models \mathtt{T}^c \quad \text{and} \quad M \not\models A$$

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Bad news

The interplay between classical and intuitionistic theories in Kripke models is a complex issue. In particular,

- In the given Kripke model $\mathcal{M} \Vdash \mathtt{T}^i$ there may be no worlds M with $M \models \mathtt{T}^c$.
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Forcing-stable, satisfaction-stable and stable formulae

Definition

We say that a formula A is

- *forcing-stable (f-stable* for short*) in a theory* $\mathtt{T}^i$ iff for every Kripke model $\mathcal{M}$ of $\mathtt{T}^i$ and every node w in $\mathcal{M}$ we have if $w \Vdash A$ then $\mathcal{M}(w) \models A$
- *satisfaction-stable (s-stable* for short*) in a theory* $\mathtt{T}^i$ iff for every Kripke model $\mathcal{M}$ of $\mathtt{T}^i$ and every node w in $\mathcal{M}$ we have if $\mathcal{M}(w) \models A$ then $w \Vdash A$,
- *stable in a theory* $\mathtt{T}^i$ iff A is f-stable and s-stable.

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Stable formulae

The class of stable formulae in IQC coincides with the set of positive formulae.

Definition

Let $\mathcal{P}(\mathtt{T}^i)$ be the smallest class such that

- $\{A : A \text{ is atomic}\} \subseteq \mathcal{P}(\mathtt{T}^i)$,
- $\{A : \mathtt{T}^i \vdash A \vee \neg A\} \subseteq \mathcal{P}(\mathtt{T}^i)$,
- if $A, B \in \mathcal{P}(\mathtt{T}^i)$ then $A \wedge B, A \vee B, \exists x A \in \mathcal{P}(\mathtt{T}^i)$.

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F-stable formulae

The class of f-stable formulae in $\mathtt{IQC}$ contains the formulae of the form $\forall x A$ where A is a positive formula.

Definition

Let $\mathtt{T}^i$ be an intuitionistic theory. The class $\mathcal{F}(\mathtt{T}^i)$ of *generalized semi-positive formulae in* $\mathtt{T}^i$ is the least class of formulae such that

- $\bot \in \mathcal{F}(\mathtt{T}^i)$,
- $\mathcal{P}(\mathtt{T}^i) \subseteq \mathcal{F}(\mathtt{T}^i)$,
- if $B, C \in \mathcal{F}(\mathtt{T}^i)$ then $B \wedge C, B \vee C, \exists x B, \forall x B \in \mathcal{F}(\mathtt{T}^i)$,
- if $B \in \mathcal{P}(\mathtt{T}^i)$ and $C \in \mathcal{F}(\mathtt{T}^i)$, then $B \to C \in \mathcal{F}(\mathtt{T}^i)$.

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A formula A is *semi-positive* if each subformula of A of the form $B \rightarrow C$ has B atomic.

The class of semi-positive formulae is exactly the class of formulae which are preserved under taking submodels of Kripke models resulting in restricting the frame of a given model (A. Visser)

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F-stable formulae

Classically, every first-order formula is equivalent to a semi-positive formula.

The semi-positive formulae are f-stable in any theory $\mathtt{T}^i$.

If $\mathtt{T}^i$ is an intuitionistic theory in which all atomic formulae are decidable then every prenex formula is f-stable in $\mathtt{T}^i$.

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F-stable formulae

Theorem

For every intuitionistic theory $\mathtt{T}^i$, every generalized semi-positive formula in $\mathtt{T}^i$ is forcing-stable in $\mathtt{T}^i$.

A Kripke model $\mathcal{M}$ is called $\mathtt{T}^c$-*normal* if for every $w \in W$, we have $\mathcal{M}(w) \models \mathtt{T}^c$.

Corollary

If $\mathtt{T}$ is a set of semi-positive sentences, then every Kripke model of $\mathtt{T}^i$ is $\mathtt{T}^c$-normal. In particular, $\mathtt{T}^i$ is complete with respect to a class of $\mathtt{T}^c$-normal Kripke models.

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S-stable formulae

Theorem

The class of s-stable formulae in $\mathtt{T}^i$ contains the class $\mathcal{P}(\mathtt{T}^i)$. Moreover, for any formula A such that there is a s-stable formula $B \in \mathcal{P}(\mathtt{T}^i)$ such that $\mathtt{CQC} \vdash A \leftrightarrow B$ and $\mathtt{IQC} \vdash B \to A$.

Conservatitvity via $\mathtt{T}^c$-normal models

Definition

For a given theory $\mathtt{T}^i$ we define a class $\mathcal{A}(\mathtt{T}^i)$ of formulae of the form

$$\forall x(C \to \forall y D)$$

where C is f-stable and D is s-stable in $\mathtt{T}^i$.

Theorem

Assume that $\mathtt{T}^i$ is complete with respect to a class of $\mathtt{T}^i$-normal Kripke models. Then $\mathtt{T}^c$ is conservative over $\mathtt{T}^i$ with respect to the class $\mathcal{A}(\mathtt{T}^i)$.

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Sketch of proof.

Assume that

$$\mathtt{T}^{i} \nvdash \forall x(C(x) \to \forall y D(x,y)),$$

where C is f-stable and D is s-stable over $\mathtt{T}^{i}$.

By completeness, there is a $\mathtt{T}^{c}$-normal Kripke model $\mathcal{M} \Vdash \mathtt{T}^{i}$ and a world u of $\mathcal{M}$ such that for some $a, b \in \mathcal{M}(u)$,

$$u \Vdash C(a) \qquad (1)$$

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Sketch of proof continued.

From (1) and f-stability of C we get $\mathcal{M}(u) \models C$.

From (2) we get $\mathcal{M}(u) \not\models D(a,b)$, since D is s-stable.

Hence $\mathcal{M}(u) \not\models C(a) \to \forall y D(a,y)$.

Since $\mathcal{M}$ is $\mathtt{T}$-normal, $\mathcal{M}(u) \models \mathtt{T}^c$. Hence

$$\mathtt{T}^c \nvdash \forall x(C \to \forall y D)$$

as required. □

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From (2) we get $\mathcal{M}(u) \not\models D(a, b)$, since D is s-stable.

Hence $\mathcal{M}(u) \not\models C(a) \to \forall y D(a, y)$.

Since $\mathcal{M}$ is T-normal, $\mathcal{M}(u) \models \mathtt{T}^c$. Hence

$$\mathtt{T}^c \nvdash \forall x(C \to \forall y D)$$

as required. □

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Conservativitvy via $\mathtt{T}^c$-normal models

Corollary (Π_2-completeness)

Let the theory $\mathtt{T}^i$ be complete with respect to a class of $\mathtt{T}^c$-normal Kripke models. Then $\mathtt{T}^c$ is conservative over $\mathtt{T}^i$ with respect to the class $\{\forall x A : A \in \mathcal{P}(\mathtt{T}^i)\}$.

NB If all atomic formulae are decidable in $\mathtt{T}^i$, then for any formula $A \in \mathcal{P}(\mathtt{T}^i)$, the formula $\forall x A$ is in Π_2.

Conservatitvity via $\mathtt{T}^c$-normal models

Corollary (Negations of prenex formulae)

Let the theory $\mathtt{T}^i$ be complete with respect to a class of $\mathtt{T}^c$-normal Kripke models. Then $\mathtt{T}^c$ is conservative over $\mathtt{T}^i$ with respect to the class $\{\neg A : A \in \mathcal{F}(\mathtt{T}^i)\}$.

NB The class $\mathcal{F}(\mathtt{T}^i)$ contains the class of generalized semi-positive formulae and the class of prenex formulae.

Conservatitvity via pruning

Definition (van Dalen, Mulder, Krabbe, Visser)

Let $\mathcal{M} = (W, \leqslant, \{\mathcal{M}(w) : w \in W\}, \Vdash)$ be a Kripke model and let $w \in W$. Assume that F is a sentence, possible with parameters from M_w, such that $(\mathcal{M}, w) \nVdash F$. We define the Kripke model

$$\mathcal{M}^F = (W^F, \leqslant^F, \{\mathcal{M}(v) : v \in W^F\}, \Vdash^F)$$

such that $W^F = \{v \in W : v \geqslant w \text{ and } (\mathcal{M}, v) \nVdash F\}$ and $\leqslant^F$ is the restriction of $\leqslant$ to the set W^F. The forcing relation of the model $\mathcal{M}^F$ is denoted by $\Vdash^F$.

Conservatitvity via pruning

First Pruning Lemma (van Dalen, Mulder, Krabbe, Visser)

Let $\mathcal{M}$ be a Kripke model and w be a node of $\mathcal{M}$ such that $(\mathcal{M}, w) \nVdash F$ for some sentence F with parameters from M_w. Then

$$(\mathcal{M}, w) \Vdash A^F \text{ iff } (\mathcal{M}^F, w) \Vdash^F A,$$

for every A.

Conservatitvity via pruning

Theorem

Consider a theory $\mathtt{T}^i$. Let a formula A be positive or decidable in $\mathtt{T}^i$. Then

$$\mathtt{T}^i \vdash \exists x A^{\exists x A} \to \exists x A.$$

Moreover, if additionally the theory $\mathtt{T}^i$ satisfies the formula

$$\mathrm{CD} = \forall x(C(x) \vee D) \to (\forall x C(x) \vee D),$$

where the variable x is not free in D, then for any sequence of quantifiers Q_i

$$\mathtt{T}^i \vdash Q_1 x_1 \ldots Q_n x_n A^{Q_1 x_1 \ldots Q_n x_n A} \to Q_1 x_1 \ldots Q_n x_n A.$$

Conservatitvity via pruning

Theorem

Assume that the theory $\mathtt{T}^i$ is closed under the Friedman translation with respect to the class of $\mathcal{P}(\mathtt{T}^i)$ and complete with respect to a class of conversely well-founded Kripke models. Then $\mathtt{T}^c$ is conservative over $\mathtt{T}^i$ with respect to the class of formulae of the form $\forall x \exists y A$ where A belongs to $\mathcal{P}(\mathtt{T}^i)$.

Sketch of proof.

Let $\mathtt{T}^i \nvdash \forall x \exists y A$.

We find a conversely well-founded Kripke model $\mathcal{M} \Vdash \mathtt{T}^i$ such that

$$\mathcal{M} \nVdash \forall x \exists y A.$$

There is w and $a \in \mathcal{M}(w)$ such that

$$(\mathcal{M}, w) \nVdash \exists y A(a, y).$$

Hence

$$(\mathcal{M}, w) \nVdash \exists y A(a, y)^{\exists y A(a,y)}.$$

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Sketch of proof continued.

Prune $\mathcal{M}$ with respect to the formula $F := \exists y A(a, y)$.

By the Pruning Lemma, $(\mathcal{M}^F, w) \nVdash^F \exists y A(a, y)$.

For some terminal world $v \geqslant w$ in $\mathcal{M}^F$,

$$(\mathcal{M}^F, v) \nVdash^F \exists y A(a, y) \text{ and } (\mathcal{M}^F, v) \Vdash^F \mathtt{T}^i T.$$

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Since v is a terminal node in $\mathcal{M}^F$,

$$\mathcal{M}(v) \not\models \forall x \exists y A(x,y) \text{ and } \mathcal{M}(v) \models \mathtt{T}^c$$

and hence, we get a desired counter-model.

In particular,

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Conservatitvity via pruning

Theorem

Assume that the theory $\mathtt{T}^i$ is closed under the Friedman translation and complete with respect to the class of conversely well-founded Kripke models with constant domains. Then $\mathtt{T}^c$ is conservative over $\mathtt{T}^i$ with respect to the class of prenex formulae with a positive formula as the matrix.

Characterization of $\mathtt{T}$-normal Kripke models

Let $\mathtt{T}$ be a set of sentences. We define

$$\mathcal{H}\mathtt{T} = \{(\neg B)^A : A \text{ is arbitrary, } B \text{ is semipositive, and } \mathtt{T}^c \vdash \neg B\}.$$

Theorem (S. Buss)

$(\mathcal{H}\mathtt{T})^i \vdash A$ iff A is true in the class of all $\mathtt{T}$-normal Kripke models.

Corollary

If $\mathtt{T}^i \vdash (\mathcal{H}\mathtt{T})^i$ then $\mathtt{T}^i$ is complete with respect to $\mathtt{T}$-normal Kripke models.

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Completeness theorems for arithmetic

Theorem (S. Buss, K. Wehmeier)

$\mathtt{HA} \vdash \mathcal{H}\mathtt{PA}$.

Corollary

$\mathtt{HA}$ is complete with respect to the class of $\mathtt{PA}$-normal Kripke models.

However, $\mathtt{HA}$ is *not* sound with respect to the class of $\mathtt{PA}$-normal Kripke models.

We can verify directly that every Kripke model if $\mathtt{i}\Delta_0$ is $\mathtt{I}\Delta_0$-normal.

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Stable formulae in HA

For every Δ_0-formula of the language of arithmetic,

$$\mathtt{i}\Delta_0 \vdash \forall x(A(x) \vee \neg A(x)).$$

In particular,

- $\mathcal{P}(\mathtt{i}\Delta_0) = \mathcal{P}(\mathtt{HA}) = \Sigma_1$,
- $\mathcal{A}(\mathtt{i}\Delta_0) = \mathcal{A}(\mathtt{HA}) \supseteq \Pi_2$,
- $\mathcal{A}(\mathtt{i}\Delta_0) = \mathcal{A}(\mathtt{HA}) \supseteq \{\neg A : A \text{ is a prenex formula}\}$,
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Conservativity of arithmetic

Theorem

The theories $\mathtt{I}\Delta_0$ and $\mathtt{PA}$ are conservative over $\mathtt{i}\Delta_0$ and $\mathtt{HA}$ respectively, with respect to the class $\mathcal{A}(\mathtt{HA})$ which includes, in particular,

- Π_2,
- negations of prenex formulae,
- negations of semi-positive formulae.

Conclusion

- Our results apply to wide classes of theories.
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