What is an inconsistent truth table?

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Joint work with G Badia (Otago) and P Girard (Auckland)

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This is evidence for a more general claim:

Metatheory determines object theory.

When we write down the orthodox clauses for a logic, whatever logic we presuppose in the background will be the object-level logic that obtains.

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But the syntax and semantics of paraconsistent and paracomplete logics—their grammar and truth tables—are always taken to be 'classically behaved', from Kripke 1974 to Field 2008.

When talking *about* a logic, must we be working in a classical metatheory?

How far can a logician who professes to hold that [paraconsistency] is the correct criterion of a valid argument, but who freely accepts and offers standard mathematical proofs, in particular for theorems about [paraconsistent] logic itself, be regarded as sincere or serious in objecting to classical logic? [Burgess]

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Okay ... then what is the plan for once everyone is converted to the One True (paraconsistent) Logic?

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And so the main reason for this paper is pragmatic, too—just to show the answer.

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Here, a paraconsistent set theory naturally generates a paraconsistent semantics.

Logic (Propositional Fragment)

Axioms

$$\vdash \varphi \to \varphi$$
$$\vdash (\varphi \to \psi) \wedge (\psi \to \chi) \to (\varphi \to \chi)$$

$$\vdash \varphi \vee \neg\varphi$$
$$\vdash \neg\neg\varphi \to \varphi$$
$$\vdash (\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$$
$$\vdash \varphi \wedge \psi \to \varphi$$
$$\vdash \varphi \wedge \psi \to \psi \wedge \varphi$$
$$\vdash \varphi \vee \psi \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$$
$$\vdash \varphi \wedge (\psi \vee \chi) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$

$$\vdash (\varphi \to \psi) \Rightarrow (\varphi \Rightarrow \psi)$$
$$\vdash \neg(\varphi \Rightarrow \psi) \Rightarrow \neg(\varphi \to \psi)$$
$$\vdash (\varphi \Rightarrow \psi) \wedge (\chi \Rightarrow \psi) \Rightarrow (\varphi \vee \chi \Rightarrow \psi)$$

$$\vdash x = y \Rightarrow (\varphi(x) \to \varphi(y))$$

Rules

$$\varphi, \varphi \Rightarrow \psi \vdash \psi$$
$$\varphi, \neg\psi \vdash \neg(\varphi \Rightarrow \psi)$$

$$\begin{array}{c} \Gamma, \varphi \vdash \psi \\ \hline\hline \Gamma \vdash \varphi \Rightarrow \psi \end{array} \qquad \begin{array}{c} \Gamma, \varphi, \chi \vdash \psi \\ \hline\hline \Gamma, \chi, \varphi \vdash \psi \end{array} \qquad \frac{\Gamma, \varphi, \chi \vdash \psi}{\Gamma, \varphi \wedge \chi \vdash \psi}$$

$$\frac{\Gamma \vdash \psi \qquad \Delta \vdash \varphi}{\Gamma, \Delta \vdash \varphi \wedge \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma, \varphi \vdash \psi} \qquad \frac{\Gamma \vdash \varphi \qquad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi}$$

Axiom (Ext)

$\forall z((z \in x \leftrightarrow z \in y) \leftrightarrow x = y$

Axiom (Abs)

$x \in \{z : \varphi\} \leftrightarrow \varphi x$

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Axiom (Choice)

A unique object can be picked out from any non-empty set.

Axiom (Induction)

Proofs by induction work for any recursively defined structure.

Two relations are added: syntactic validity $\vdash$ and semantic consequence $\models$.

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Definition

With Γ *a set of premises,*

$$\Gamma \vdash \varphi$$

iff φ *follows from some subset of* Γ *by valid rules.*

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If this sounds (comfortingly? suspiciously?) familiar, this is prelude for what is to come.

True vs true only: two values or three?

Tarski's theorem:

An exclusive and exhaustive partitioning of all the propositions into all-and-only the truths, versus all-and-only the non-truths, is **impossible**.

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Choose: *untruth-avoidance* or *truth-seeking*.

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If the original Tarski problem was insoluble, this new, three-tiered approach will be no less intractable.

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A dialetheic paraconsistentist should lead the discussion away from pre-Tarskian ideation, and use a formalism that does not invite or suggest such criticism.

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Relations can approximate the otherwise-desirable functional three valued semantics;

To reiterate, this is not really a *decision* on our part,

but rather a *requirement* for any logic that can express its own metatheory.

There are two truth values, t and f, which are duals,

$$\bar{\mathsf{t}} = \mathsf{f}$$

$$\bar{\bar{\mathsf{t}}} = \mathsf{t}$$

They are also exclusive, on pain of absurdity:

$$\mathsf{t} = \mathsf{f} \Rightarrow \varphi$$

for any φ.

Relational Truth Conditions

A truth-value assignment on $\mathtt{PROP}$ is any relation

$$R^0 \subseteq \mathtt{PROP} \times \{\mathrm{t}, \mathrm{f}\}$$

such that $x \in \mathtt{PROP} \Leftrightarrow \exists y(\langle x, y\rangle \in R^0)$, and

$$\langle p, \mathrm{t}\rangle \in R^0 \Leftrightarrow \langle p, \mathrm{f}\rangle \notin R^0 \qquad \langle p, \mathrm{f}\rangle \in R^0 \Leftrightarrow \langle p, \mathrm{t}\rangle \notin R^0$$

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By the law of excluded middle, R^0 is not empty:

either $\langle p, \mathrm{f}\rangle \in R^0$, or else $\langle p, \mathrm{f}\rangle \notin R^0$, in which case $\langle p, \mathrm{t}\rangle \in R^0$.

Definition of a model

Extend R^0 to $R \subseteq \texttt{FMLA} \times \{\mathrm{t}, \mathrm{f}\}$:

$$
\begin{array}{rcl}
\neg\varphi R\mathsf{t} & \Leftrightarrow & \varphi R\mathsf{f} \\
\neg\varphi R\mathsf{f} & \Leftrightarrow & \varphi R\mathsf{t}
\end{array}
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\begin{array}{rcl}
(\varphi \wedge \psi) R\mathsf{t} & \Leftrightarrow & \varphi R\mathsf{t} \wedge \psi R\mathsf{t} \\
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Satisfies $\varphi R\mathrm{t} \Leftrightarrow \neg(\varphi R\mathrm{f})$ and $\varphi R\mathrm{f} \Leftrightarrow \neg(\varphi R\mathrm{t})$

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Example

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i.e. φ is both satisfied and not in the model R.
This will be the situation with any contradiction.

Definition

A sentence ψ *is a* valid consequence *of* $\varphi_0, \ldots, \varphi_n$,

$$\varphi_0, \ldots, \varphi_n \vDash \psi$$

iff $\varphi_0 R \mathrm{t} \wedge \ldots \wedge \varphi_n R \mathrm{t} \Rightarrow \psi R \mathrm{t}$ *for all models* R*.*

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This is as usual.

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Any truth-value assignment R^0 *on* $\mathtt{PROP}$ *can be extended to a model* R *for propositional logic.*

Let $R^0 \subseteq \mathtt{PROP} \times \{\mathrm{t}, \mathrm{f}\}$ be an assignment on propositional variables. This means that

$$\langle p, \mathrm{t}\rangle \in R^0 \Leftrightarrow \langle p, \mathrm{f}\rangle \notin R^0 \qquad \langle p, \mathrm{f}\rangle \in R^0 \Leftrightarrow \langle p, \mathrm{t}\rangle \notin R^0$$

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One exists: let $R^0 = \{\langle p, \mathrm{t}\rangle, \langle p, \mathrm{f}\rangle\}$.

Extend R^0 with the lift R (which exists by comprehension):

$$
\begin{aligned}
R = \quad & \{\langle p, \mathsf{t}\rangle : pR^0\mathsf{t}\} \\
\cup \quad & \{\langle p, \mathsf{f}\rangle : pR^0\mathsf{f}\} \\
\\
\cup \quad & \{\langle \neg\varphi, \mathsf{t}\rangle : \langle \varphi, \mathsf{f}\rangle \in R\} \\
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\cup \quad & \{\langle \varphi \wedge \psi, \mathsf{t}\rangle : \langle \varphi, \mathsf{t}\rangle \in R \wedge \langle \psi, \mathsf{t}\rangle \in R\} \\
\cup \quad & \{\langle \varphi \wedge \psi, \mathsf{f}\rangle : \langle \varphi, \mathsf{f}\rangle \in R \vee \langle \psi, \mathsf{f}\rangle \in R\} \\
\\
\cup \quad & \{\langle \varphi \vee \psi, \mathsf{t}\rangle : \langle \varphi, \mathsf{t}\rangle \in R \vee \langle \psi, \mathsf{t}\rangle \in R\} \\
\cup \quad & \{\langle \varphi \vee \psi, \mathsf{f}\rangle : \langle \varphi, \mathsf{f}\rangle \in R \wedge \langle \psi, \mathsf{f}\rangle \in R\}
\end{aligned}
$$

Proof that R is a model

Show by induction that for any formula

$$(\star) \qquad \begin{aligned} \langle \varphi, \mathrm{t} \rangle \in R &\Leftrightarrow \langle \varphi, \mathrm{f} \rangle \notin R \\ \langle \varphi, \mathrm{f} \rangle \in R &\Leftrightarrow \langle \varphi, \mathrm{t} \rangle \notin R \end{aligned}$$

For the base case, for any proposition p and $x \in \{\mathrm{t}, \mathrm{f}\}$, by definition $pRx \Leftrightarrow pR^0x$ and $\neg(pRx) \Leftrightarrow \neg(pR^0x)$.

Induction: assume $(\star)$ as the inductive hypothesis.

To avoid contraction, we don't use the very same hypothesis for each inductive case. They are rather *hypothesis schemata*, each instance used once.

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- Such two-dimensional displays are often implicitly assumed to be functional look-up tables.
- No such assumption on the page.
- It is simply presupposing classicality to do so.
- Diagrams must be used with great care in mathematics!

	$\neg$
t	f
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The tables are read as, 'if t is *among* the values of φ, then f is *among* the values of $\neg\varphi$'.

Or more concisely, 'if φ is true, then $\neg\varphi$ is false'.

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The copula—the 'is' of predication—is not univocal in general, and it is not here.

Soundness

Theorem

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Corollary

For some φ, *it is the case that* $\vdash \varphi \Rightarrow \models \varphi$ *and* $\neg(\vdash \varphi \Rightarrow \models \varphi)$.

Corollary

$(\models \varphi \Rightarrow \bot) \Rightarrow (\vdash \varphi \Rightarrow \bot)$.

Completeness

Notation: given R,

$$\Theta_R^{\varphi} = \{p_i \ : \ \langle p_i, \mathrm{t}\rangle \in R\} \cup \{\neg p_i \ : \ \langle p_i, \mathrm{f}\rangle \in R\}$$

Θ contains as many copies of p_i as there are in φ.

Lemma

For any model R and formula φ,

1. $\langle \varphi, \mathrm{t}\rangle \in R \quad \Rightarrow \quad \Theta_R^{\varphi} \vdash \varphi$
2. $\langle \varphi, \mathrm{f}\rangle \in R \quad \Rightarrow \quad \Theta_R^{\varphi} \vdash \neg\varphi$

Proof: If $\langle \neg\psi, \mathrm{t}\rangle \in R$, then $\langle \psi, \mathrm{f}\rangle \in R$, so $\Theta \vdash \neg\psi$. If $\langle \neg\psi, \mathrm{f}\rangle \in R$, then $\langle \neg\neg\psi, \mathrm{f}\rangle \in R$, so $\Theta \vdash \neg\neg\psi$, so $\Theta \vdash \psi$. Etc. $\square$

Theorem

$\models \varphi \Rightarrow \vdash \varphi$

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Theorem Naive set theory is not trivial.

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Either naive set theory is trivial or not. If not, we are done. If trivial, then, since this very proof is in naive set theory, it follows that the system is not trivial—since, after all, anything follows. □

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"You can trust me."

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Using a paraconsistent meatheory constitutes no expressive loss. Rather it makes clearer and more explicit the hard facts.

Indeed, in a paraconsistent system, one *can* prove consistency and non-triviality.

This is the closest one can get to a guarantee that the proof methods themselves are reliable, by methods that are equally reliable.

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A paraconsistent substructural approach can at least match the classical textbook presentation of semantics, and may eventually be uniquely able to carry its own weight.