Prague seminar on
Non-Classical Mathematics

Naive Modal Set Theory

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Naive Set Theory

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Ext. $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$

Comp. $\exists y \forall x[x \in y \leftrightarrow \Phi]$

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Comp. $\exists y \forall x [x \in y \leftrightarrow \Phi]$

Russell $\Phi : x \notin x$

Curry $\Phi : x \in x \rightarrow \Psi$

Naive Modal Set Theory

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$$w \Vdash \Diamond \exists y \forall x [x \in y \leftrightarrow \Phi]$$

Naive Modal Set Theory

$$w \Vdash \Diamond \exists y \forall x [x \in y \leftrightarrow \Phi]$$

$$wRv$$

$$v \Vdash \exists y \forall x [x \in y \leftrightarrow \Phi]$$

Parsons

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A multiplicity of objects that exist together *can* constitute a set, but it is not necessary that they *do*. Given the elements of a set, it is not necessary that the set exists together with them.

C. Parsons, *What Is the Iterative Conception of Set?*, 1977.

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$$\Diamond\exists y\forall x[x \in y \leftrightarrow (\Phi)_0]$$

$$\Box\Diamond\exists y\forall x[x \in y \leftrightarrow (\Phi)_1]$$

Studd

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$w \Vdash \Diamond A$ iff there exists a v, wRv, and $v \Vdash A$
$w \Vdash \Box A$ iff for all v such that wRv, $v \Vdash A$

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$w \Vdash \Diamond A$ iff there exists a v, wRv, and $v \Vdash A$
$w \Vdash \Box A$ iff for all v such that wRv, $v \Vdash A$

$w \Vdash \blacklozenge A$ iff there exists a v, vRw and $v \Vdash A$
$w \Vdash \blacksquare A$ iff for all v such that vRw, $v \Vdash A$

Naive Modal Set Theory

Naive Modal Set Theory

Ext. $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$

MComp: $\Diamond \exists y \forall x[x \in y \leftrightarrow \blacklozenge \Phi]$

Russell

Russell

$$w \Vdash \Diamond \exists y \forall x [x \in y \leftrightarrow \blacklozenge (x \notin x)]$$

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$$w \Vdash \Diamond \exists y \forall x [x \in y \leftrightarrow \blacklozenge (x \notin x)]$$

$$wRv$$

Russell

$$w \Vdash \Diamond \exists y \forall x [x \in y \leftrightarrow \blacklozenge (x \notin x)]$$

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$$v \Vdash \exists y \forall x [x \in y \leftrightarrow \blacklozenge (x \notin x)]$$

Russell

$$w \Vdash \Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge(x \notin x)]$$

$$wRv$$

$$v \Vdash \exists y\forall x[x \in y \leftrightarrow \blacklozenge(x \notin x)]$$

$$v \Vdash \forall x[x \in r \leftrightarrow \blacklozenge(x \notin x)]$$

Russell

$$w \Vdash \Diamond\exists y \forall x[x \in y \leftrightarrow \blacklozenge(x \notin x)]$$

$$wRv$$

$$v \Vdash \exists y \forall x[x \in y \leftrightarrow \blacklozenge(x \notin x)]$$

$$v \Vdash \forall x[x \in r \leftrightarrow \blacklozenge(x \notin x)]$$

$$v \Vdash r \in r \leftrightarrow \blacklozenge(r \notin r)$$

Russell

$$v \Vdash r \in r \leftrightarrow \blacklozenge(r \notin r)$$

Russell

$$v \Vdash r \in r \leftrightarrow \blacklozenge(r \notin r)$$

$v \Vdash r \notin r$.

$v \Vdash \neg\blacklozenge(r \notin r)$

$v \Vdash \blacksquare(r \in r)$

$w \Vdash r \in r$

$v \Vdash r \in r$

$\times$

Russell

$$v \Vdash r \in r \leftrightarrow \blacklozenge(r \notin r)$$

$v \Vdash r \notin r.$	$v \Vdash r \in r$
$v \Vdash \neg\blacklozenge(r \notin r)$	$v \Vdash \blacklozenge(r \notin r)$
$v \Vdash \blacksquare(r \in r)$	$w' \Vdash r \notin r$
$w \Vdash r \in r$	
$v \Vdash r \in r$	
$\times$	

Curry

Curry

$$w \Vdash \Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge(x \in x \rightarrow A)]$$

Curry

$$w \Vdash \Diamond \exists y \forall x [x \in y \leftrightarrow \blacklozenge (x \in x \rightarrow A)]$$

$$wRv$$

Curry

$$w \Vdash \Diamond \exists y \forall x [x \in y \leftrightarrow \blacklozenge (x \in x \to A)]$$

$$wRv$$

$$v \Vdash \exists y \forall x [x \in y \leftrightarrow \blacklozenge (x \in x \to A)]$$

Curry

$$w \Vdash \Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge(x \in x \rightarrow A)]$$

$$wRv$$

$$v \Vdash \exists y\forall x[x \in y \leftrightarrow \blacklozenge(x \in x \rightarrow A)]$$

$$v \Vdash \forall x[x \in c \leftrightarrow \blacklozenge(x \in x \rightarrow A)]$$

Curry

$$w \Vdash \Diamond\exists y \forall x[x \in y \leftrightarrow \blacklozenge(x \in x \rightarrow A)]$$

$$wRv$$

$$v \Vdash \exists y \forall x[x \in y \leftrightarrow \blacklozenge(x \in x \rightarrow A)]$$

$$v \Vdash \forall x[x \in c \leftrightarrow \blacklozenge(x \in x \rightarrow A)]$$

$$v \Vdash c \in c \leftrightarrow \blacklozenge(c \in c \rightarrow A)$$

Curry

$$w \Vdash \Diamond\exists y \forall x[x \in y \leftrightarrow \blacklozenge(x \in x \to A)]$$

$$wRv$$

$$v \Vdash \exists y \forall x[x \in y \leftrightarrow \blacklozenge(x \in x \to A)]$$

$$v \Vdash \forall x[x \in c \leftrightarrow \blacklozenge(x \in x \to A)]$$

$$v \Vdash c \in c \leftrightarrow \blacklozenge(c \in c \to A)$$

$$v \Vdash c \in c \to \blacklozenge(c \in c \to A)$$

ZF Axioms

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Empty Set: $\Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge(x \neq x)]$

ZF Axioms

Empty Set: $\Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge(x \neq x)]$

Power Set: $\Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge(x \subseteq a)]$

ZF Axioms

Empty Set: $\Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge(x \neq x)]$

Power Set: $\Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge(x \subseteq a)]$

Infinity: $\Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge\Diamond(x \subseteq y)]$

Infinity

Infinity

$$w \Vdash \Diamond \exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

Infinity

$$w \Vdash \Diamond \exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

wRv and $\emptyset$ exists at v.

From MComp, $v \Vdash \Diamond \exists y \forall x[x \in y \leftrightarrow \blacklozenge\Diamond(x \subseteq y)]$

vRu and $u \Vdash \forall x[x \in i \leftrightarrow \blacklozenge\Diamond(x \subseteq i)]$

$u \Vdash \emptyset \subseteq i$

$u \Vdash \blacklozenge\Diamond(\emptyset \subseteq i)$

$u \Vdash \emptyset \in i$

Infinity

$$w \Vdash \Diamond \exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

Infinity

$$w \Vdash \Diamond \exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

$$u \Vdash z \in i$$

$$u \Vdash \{z\} \subseteq i$$

$$u \Vdash \blacklozenge \Diamond (\{z\} \subseteq i)$$

$$u \Vdash \forall x[x \in i \leftrightarrow \blacklozenge \Diamond (x \subseteq i)]$$

$$u \Vdash \{z\} \in i$$

$$u \Vdash \exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

$$v \Vdash \Diamond \exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

$$w \Vdash \Diamond \Diamond \exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

$$w \Vdash \Diamond \exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$$

Universal Set

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From MComp, $w \Vdash \Diamond \exists y \forall x[x \in y \leftrightarrow \blacklozenge(x \notin y)]$

Universal Set

From MComp, $w \Vdash \Diamond \exists y \forall x[x \in y \leftrightarrow \blacklozenge(x \notin y)]$

$v \Vdash \exists y \forall x[x \in y \leftrightarrow \blacklozenge(x \notin y)]$

$v \Vdash \forall x[x \in u \leftrightarrow \blacklozenge(x \notin u)]$

$v \Vdash a \in u \leftrightarrow \blacklozenge(a \notin u)$

Universal Set

From MComp, $w \Vdash \Diamond \exists y \forall x [x \in y \leftrightarrow \blacklozenge (x \notin y)]$

$v \Vdash \exists y \forall x [x \in y \leftrightarrow \blacklozenge (x \notin y)]$

$v \Vdash \forall x [x \in u \leftrightarrow \blacklozenge (x \notin u)]$

$v \Vdash a \in u \leftrightarrow \blacklozenge (a \notin u)$

$v \Vdash a \notin u$.	$v \Vdash a \in u$
$v \Vdash \neg \blacklozenge (a \notin u)$	$v \Vdash \blacklozenge (a \notin u)$
$v \Vdash \blacksquare (a \in u)$	$w' \Vdash a \notin u$
$w \Vdash a \in u$	
$v \Vdash a \in u$	
$\times$	

Modality

Consistency?

Consistency?

$$
\begin{array}{ccc}
a \in a & & a \notin a \\
w_0 & \longleftrightarrow & w_1 \\
\circlearrowright & & \circlearrowright
\end{array}
$$

Consistency?

$$\begin{array}{ccc} a \in a & & a \notin a \\ w_0 & \longleftrightarrow & w_1 \\ \circlearrowright & & \circlearrowright \end{array}$$

Let Φ be given
Two cases:

1. $w \Vdash \Phi$, for some w
2. $w \nVdash \Phi$, for every w

Consistency?

$$\begin{array}{ccc} a \in a & & a \notin a \\ w_0 & \longleftrightarrow & w_1 \\ \circlearrowright & & \circlearrowright \end{array}$$

Let Φ be given
Two cases:

1. $w \Vdash \Phi$, for some w

   $w_0 \Vdash \blacklozenge\Phi$

   So $w_0 \Vdash a \in a \leftrightarrow \blacklozenge\Phi$

2. $w \nVdash \Phi$, for every w

Consistency?

$$\begin{array}{ccc} a \in a & & a \notin a \\ w_0 & \longleftrightarrow & w_1 \\ \circlearrowright & & \circlearrowright \end{array}$$

Let Φ be given
Two cases:

1. $w \Vdash \Phi$, for some w

   $w_0 \Vdash \blacklozenge\Phi$

   So $w_0 \Vdash a \in a \leftrightarrow \blacklozenge\Phi$

2. $w \nVdash \Phi$, for every w

   $w_1 \nVdash \blacklozenge\Phi$

   So $w_1 \Vdash a \in a \leftrightarrow \blacklozenge\Phi$