IT4Innovations
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Fuzzy Sets and Fuzzy Classes in Universes of Sets

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UNIVERSITAS OSTRAVIENSIS
Institute of Geonics ASCR
BRNO UNIVERSITY OF TECHNOLOGY

Outline

A poor interest about cardinal theory of fuzzy sets

S. Gottwald.
Fuzzy uniqueness of fuzzy mappings.
Fuzzy Sets and Systems, 3:49–74, 1980.

M. Wygralak.
Vaguely defined objects. Representations, fuzzy sets and nonclassical cardinality theory.
Theory and Decision Library. Series B: Mathematical and Statistical Methods, Kluwer Academic Publisher, 1996.

M. Wygralak.
Cardinalities of Fuzzy Sets.
Kluwer Academic Publisher, Berlin, 2003.

Set and fuzzy set theories

- Zermelo–Fraenkel set theory with the axiom of choice (ZFC) - sets are introduced formally, classes are introduced informally;
- von Neumann–Bernays–Gödel axiomatic set theory (NBG) - classes are introduced formally, sets are special classes (difference between sets and proper classes is essential)
- type theory
- Gotwald cumulative system of fuzzy sets
- Novak axiomatic fuzzy type theory (FTT)
- Běhounek–Cintula axiomatic fuzzy class theory (FCT)

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Degrees of membership

Definition

A residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \rightarrow, \otimes, \bot, \top \rangle$ with four binary operations and two constants such that

1. $\langle L, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice,
2. $\langle L, \otimes, \top \rangle$ is a commutative monoid and
3. the adjointness property is satisfied, i.e.,

$$a \leq b \rightarrow c \quad \text{iff} \quad a \otimes b \leq c$$

holds for each $a, b, c \in L$.

Our prerequisite

In our theory, we assume that each residuated lattice is complete and linearly ordered.

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Example of linearly ordered residuated lattice

Example

Let T be a left continuous t-norm. Then

$$\mathbf{L} = \langle [0,1], \min, \max, T, \rightarrow_T, 0, 1 \rangle,$$

where $\alpha \rightarrow_T \beta = \bigvee \{\gamma \in [0,1] \mid T(\alpha, \gamma) \leq \beta\}$, is a complete linearly ordered residuated lattice.

E.g., Łukasiewicz algebra is determined by

$$T_{\text{L}}(a,b) = \max(0, a+b-1).$$

The residuum is then given by

$$a \rightarrow_{T_{\text{L}}} b = \min(1-a+b, 1).$$

Universe of sets motivated by Grothendieck

Definition

A universe of sets over $\mathbf{L}$ is a non-empty class $\mathfrak{U}$ of sets in ZFC satisfying the following properties:

(U1) $x \in y$ and $y \in \mathfrak{U}$, then $x \in \mathfrak{U}$,

(U2) $x, y \in \mathfrak{U}$, then $\{x, y\} \in \mathfrak{U}$,

(U3) $x \in \mathfrak{U}$, then $\mathrm{P}(x) \in \mathfrak{U}$,

(U4) $x \in \mathfrak{U}$ and $y_i \in \mathfrak{U}$ for any $i \in x$, then $\bigcup_{i \in x} y_i \in \mathfrak{U}$,

(U5) **$x \in \mathfrak{U}$ and $f : x \to L$, then $\mathscr{R}(f) \in \mathfrak{U}$,**

where L is the support of the residuated lattice $\mathbf{L}$.

Examples

Universes of sets over $\mathbf{L}$

- class of all sets,
- class of all finite sets,
- Grothendieck universes (suitable sets of sets).

Sets and classes in $\mathfrak{U}$

In ZFC, we have

- sets (introduced by axioms)
- classes (introduced informally as collections of sets)

Definition

Let $\mathfrak{U}$ be a universe of sets over $\mathbf{L}$. We say that

- a set x in ZFC is a set in $\mathfrak{U}$ if $x \in \mathfrak{U}$,
- a class x in ZFC is a class in $\mathfrak{U}$ if $x \subseteq \mathfrak{U}$,
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Basic properties

Theorem

Let $x, y \in \mathfrak{U}$ and $y_i \in \mathfrak{U}$ for any $i \in x$. Then we have

1. *$\emptyset$ and $\{x\}$ belong to $\mathfrak{U}$,*
2. *$x \times y$, $x \sqcup y$, $x \cap y$ and y^x belong to $\mathfrak{U}$,*
3. *if $z \in \mathfrak{U} \cup \{L\}$ and $f : x \to z$, then f and $\mathscr{R}(f)$ belong to $\mathfrak{U}$,*
4. *if $z \subseteq \mathfrak{U}$ and $|z| \leq |x|$, then z belongs to $\mathfrak{U}$,*
5. *$\prod_{i \in x} y_i$, $\bigsqcup_{i \in x} y_i$ and $\bigcap_{i \in x} y_i$ belong to $\mathfrak{U}$.*

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Theorem

Let $x, y \in \mathfrak{U}$ *and* $y_i \in \mathfrak{U}$ *for any* $i \in x$. *Then we have*

1. $\emptyset$ *and* $\{x\}$ *belong to* $\mathfrak{U}$,
2. $x \times y$, $x \sqcup y$, $x \cap y$ *and* y^x *belong to* $\mathfrak{U}$,
3. *if* $z \in \mathfrak{U} \cup \{L\}$ *and* $f : x \to z$, *then* f *and* $\mathscr{R}(f)$ *belong to* $\mathfrak{U}$,
4. *if* $z \subseteq \mathfrak{U}$ *and* $|z| \leq |x|$, *then* z *belongs to* $\mathfrak{U}$,
5. $\prod_{i \in x} y_i$, $\bigsqcup_{i \in x} y_i$ *and* $\bigcap_{i \in x} y_i$ *belong to* $\mathfrak{U}$.

Theorem (Extensibility of sets in $\mathfrak{U}$)

Let $x \in \mathfrak{U}$. *Then there exists* $y \in \mathfrak{U}$ *such that* $|x| \leq |y \setminus x|$.

Outline

Fuzzy sets in $\mathfrak{U}$

Definition

Let $\mathfrak{U}$ be a universe of sets over $\mathbf{L}$. A function $A : x \to L$ (in ZFC) is called a fuzzy set in $\mathfrak{U}$ if x is a set in $\mathfrak{U}$, i.e., $x \in \mathfrak{U}$.

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Denotation

- The domain $\mathscr{D}(A)$ is called a universe of discourse of A,
- $\mathcal{F}(\mathfrak{U})$ denotes the class of all fuzzy sets in $\mathfrak{U}$, clearly, $\mathcal{F}(\mathfrak{U})$ is a proper class in $\mathfrak{U}$,
- The set $\mathrm{Supp}(A) = \{x \in \mathscr{D}(A) \mid A(x) > \bot\}$ is called the support of A,

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Empty fuzzy set and singletons

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Fuzzy equivalence and fuzzy preordering relation

Definition

A fuzzy relation $R : z \times z \longrightarrow L$ is called the fuzzy equivalence provided that the following axioms hold for any $a, b, c \in z$:

(FE1) $R(a, a) = \top$,

(FE2) $R(a, b) = R(b, a)$,

(FE3) $R(a, b) \otimes R(b, c) \leq R(a, c)$.

Definition (Bodenhofer)

Let R be a fuzzy equivalence on z. A fuzzy relation $S : z \times z \to L$ is called the R–fuzzy partial ordering provided that the following axioms hold for any $a, b, c \in z$:

(FPO1) $S(a, a) = \top$,

(FPO2) $S(a, b) \otimes S(b, a) \leq R(a, b)$,

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Basic relations between fuzzy sets (equality relation)

Definition

We say that fuzzy sets A and B are identical (symbolically, $A = B$) provided that $\mathscr{D}(A) = \mathscr{D}(B)$ and $A(x) = B(x)$ for any $x \in \mathscr{D}(A)$.

Definition

We say that fuzzy sets A and B are identical up to negligibility (symbolically, $A \equiv B$) provided that $\mathrm{Supp}(A) = \mathrm{Supp}(B)$ and $A(x) = B(x)$ for any $x \in \mathrm{Supp}(A)$. We use $\mathrm{cls}(A)$ to denote the set of all fuzzy sets that are identical to A up to negligibility.

Example

Obviously $\emptyset \equiv \{0/a, 0/b\}$ or $\{0.9/a\} \equiv \{0.9/a, 0/b\}$ and $\{0/a, 0/b\} \in \mathrm{cls}(\emptyset)$.

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Lemma

Let $A, B \in \mathfrak{F}(\mathfrak{U})$. Then,

(i) $A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$,

(ii) $A \equiv B$ *if and only if* $A \subseteqq B$ *and* $B \subseteqq A$.

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Operations on $\mathfrak{F}(\mathfrak{U})$

Definition

Let $A, B \in \mathcal{F}(\mathfrak{U})$, $x = \mathscr{D}(A) \cup \mathscr{D}(B)$ and $A' \in \mathrm{cls}(A)$, $B' \in \mathrm{cls}(B)$ such that $\mathscr{D}(A') = \mathscr{D}(B') = x$. Then

- the union of A and B is a mapping $A \cup B : x \to L$ defined by
$$(A \cup B)(a) = A'(a) \vee B'(a),$$
- the intersection of A and B is a mapping $A \cap B : x \to L$ defined by
$$(A \cap B)(a) = A'(a) \wedge B'(a),$$
- the difference of A and B is a mapping $A \setminus B : x \to L$ defined by
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Let $A, B \in \mathcal{F}(\mathfrak{U})$, $x = \mathscr{D}(A) \times \mathscr{D}(B)$ and $y = \mathscr{D}(A) \sqcup \mathscr{D}(B)$ (the disjoint union). Then

- the product of A, B is a mapping $A \times B : x \to L$ defined by

$$(A \times B)(a, b) = A(a) \wedge B(b),$$

- the strong product of A, B is a mapping $A \times B : x \to L$ defined by

$$(A \otimes B)(a, b) = A(a) \otimes B(b),$$

- the disjoint union of A, B is a mapping $A \sqcup B : y \to L$ defined by

$$(A \sqcup B)(a, i) = \begin{cases} A(a, i), & \text{if } i = 1, \\ B(a, i), & \text{if } i = 2. \end{cases}$$

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Operations on $\mathfrak{F}(\mathfrak{U})$

Definition

Let $A \in \mathcal{F}(\mathfrak{U})$ and $A : x \to L$. Then, the complement of A is a mapping $\overline{A} : x \to L$ defined by

$$\overline{A}(a) = A(a) \to \bot \quad (or\ \overline{A} = \chi_x \setminus A)$$

for any $a \in x$.

Theorem

Let $A, B, C, D \in \mathcal{F}(\mathfrak{U})$, and let $\circledast \in \{\cap, \cup, \setminus, \times, \otimes, \sqcup\}$. If $A \equiv C$ and $B \equiv D$, then

$$A \circledast B \equiv C \circledast D,$$

i.e., $\equiv$ is a congruence w.r.t. $\circledast$.

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Example of operations on fuzzy sets

Example

Consider the Łukasiewicz algebra $\mathbf{L}$. For

$$A = \{1/a, 0.4/b\} \quad \text{and} \quad B = \{0.6/a, 0.2/c\}$$

we have

$$\begin{aligned}
A \cup B &= \{1/a, 0.4/b, 0.2/c\}, \\
A \cap B &= \{0.6/a, 0/b, 0/c\}, \\
A \setminus B &= \{0.4/a, 0.4/b, 0/c\}, \\
A \times B &= \{0.6/(a,a), 0.2/(a,c), 0.4/(b,a), 0.2/(b,c)\}, \\
A \otimes B &= \{0.6/(a,a), 0.2/(a,c), 0/(b,a), 0/(b,c)\}, \\
A \sqcup B &= \{1/(a,1), 0.4/(b,1), 0.6/(a,2), 0.2/(c,2)\}, \\
\overline{A} &= \{0/a, 0.6/b\}.
\end{aligned}$$

Function between fuzzy sets

- $\mathfrak{Func}$ denotes the class of all function in ZFC that belong to $\mathfrak{U}$
- $\mathfrak{Func}(x, y)$ denotes the set of all functions of a set x to a set y

Definition

Let $A, B \in \mathfrak{F}(\mathfrak{U})$, and let $f \in \mathfrak{Func}$. We say that f is a function of A to B (symbolically $f : A \longrightarrow B$) provided that $f \in \mathfrak{Func}(\mathscr{D}(A), \mathscr{D}(B))$ and

$$A(a) \leq B(f(a)) \quad \text{(or equivalently } A(a) \to B(f(a)) = \top)$$

for any $a \in \mathscr{D}(A)$.

1-1 correspondence between fuzzy sets

Definition

We say that $f : A \longrightarrow B$ is a 1-1 correspondence (symbolically $f : A \xrightarrow[\text{corr}]{\text{1-1}} B$) provided that there exists $f^{-1} : B \longrightarrow A$ such that $f^{-1} \circ f = 1_{\mathscr{D}(A)}$ and $f \circ f^{-1} = 1_{\mathscr{D}(B)}$.

Theorem

Let $A, B \in \mathfrak{F}(\mathfrak{U})$. Then, $f : A \xrightarrow[corr]{1\text{-}1} B$ if and only if $f : \mathscr{D}(A) \xrightarrow[corr]{1\text{-}1} \mathscr{D}(B)$ and $A(a) = B(f(a))$ for any $a \in \mathscr{D}(A)$.

1-1 correspondence between fuzzy sets

Definition

We say that $f : A \longrightarrow B$ is a 1-1 correspondence (symbolically $f : A \xrightarrow[\text{corr}]{\text{1-1}} B$) provided that there exists $f^{-1} : B \longrightarrow A$ such that $f^{-1} \circ f = 1_{\mathscr{D}(A)}$ and $f \circ f^{-1} = 1_{\mathscr{D}(B)}$.

Theorem

Let $A, B \in \mathfrak{F}(\mathfrak{U})$. Then, $f : A \xrightarrow[\textit{corr}]{\textit{1-1}} B$ if and only if $f : \mathscr{D}(A) \xrightarrow[\textit{corr}]{\textit{1-1}} \mathscr{D}(B)$ and $A(a) = B(f(a))$ for any $a \in \mathscr{D}(A)$.

Generalization of power set

Definition

Let $A \in \mathfrak{F}(\mathfrak{U})$, and let $x = \{y \mid y \subseteq \mathscr{D}(A)\}$. Then, the fuzzy set $\mathbf{P}(A) : x \longrightarrow L$ defined by

$$\mathbf{P}(A)(y) = \bigwedge_{z \in \mathscr{D}(A)} (\chi_y(z) \rightarrow A(z))$$

is called the fuzzy power set of A, where χ_y is the characteristic function of y on $\mathscr{D}(A)$.

Theorem

Let $A, B \in \mathfrak{F}(\mathfrak{U})$. Then,

1. $\mathbf{P}(A) \in \mathfrak{F}(\mathfrak{U})$;
2. *if $A \equiv B$, then $\mathbf{P}(A) \equiv \mathbf{P}(B)$.*

Fuzzy power set and image function

Theorem

Let $A, B \in \mathfrak{F}(\mathfrak{U})$, and let $f : A \longrightarrow B$ be a function between fuzzy sets. Then, the following diagram commutes

$$
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \\
{\scriptstyle i_1}\big\downarrow & & \big\downarrow{\scriptstyle i_2} \\
\mathbf{P}(A) & \xrightarrow[\quad f^{\rightarrow} \quad]{} & \mathbf{P}(B),
\end{array}
$$

where i_1, i_2 are the inclusion functions, i.e., $i_1(a) = \{a\}$ for any $a \in \mathscr{D}(A)$ and similarly i_2, and $f^{\rightarrow}$ is the image function of sets.

Example of fuzzy power set

It is easy to verify that

$$\mathbf{P}(A)(y) = \bigwedge_{z \in y} A(z),$$

Example

Let $\mathbf{L}$ be the Łukasiewicz algebra and $A = \{1/a, 0.4/b\}$. Then,

$$\mathbf{P}(A) = \{1/\emptyset, 1/\{a\}, 0.4/\{b\}, 0.4/\{a, b\}\}.$$

Fuzzy power set and cardinality of its support

Example

Let $\mathbf{L}$ be an arbitrary residuated lattice on $[0,1]$, let the set of all natural numbers $\mathbb{N}$ belongs to $\mathfrak{U}$, and let $A : \mathbb{N} \longrightarrow L$ be defined by

$$A(n) = \frac{1}{n}, \quad n \in \mathbb{N}.$$

Then, it holds $|\mathrm{Supp}(A)| = |\mathrm{Supp}(\mathbf{P}(A))|$.

Exponentiation for fuzzy sets

Motivation:

1. If $f \in B^A$, then $x \in A$ implies $f(x) \in B$, which is naturally in the degree $A(x) \to B(f(x))$.
2. It can be proved $hom(A \otimes B, C) \cong hom(A, C^B)$.

Definition

Let $A, B \in \mathfrak{F}(\mathfrak{U})$, and let $x = \mathscr{D}(A)$ and $y = \mathscr{D}(B)$. Then, the fuzzy set $B^A : y^x \longrightarrow L$ defined by

$$B^A(f) = \bigwedge_{z \in x} (A(z) \to B(f(z)))$$

is called the exponentiation of A to B.

Remark

If $A \equiv B$ and $C \equiv D$, then it is not true $A^C \equiv B^D$.

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If $A \equiv B$ and $C \equiv D$, then it is not true $A^C \equiv B^D$.

Definition of fuzzy classes

Definition

Let $\mathfrak{U}$ be a universe of sets over $\mathbf{L}$. A class function $\mathcal{A} : \mathfrak{X} \longrightarrow L$ (in ZFC) is called the fuzzy class in $\mathfrak{U}$ if $\mathfrak{X}$ is a class in $\mathfrak{U}$, i.e., $\mathfrak{X} \subseteq \mathfrak{U}$.

Remark

Since each set is a class in $\mathfrak{U}$, we obtain that each fuzzy set is a fuzzy class.

Definition

We say that a fuzzy class $\mathcal{A}$ in $\mathfrak{U}$ is a fuzzy set if there exists a fuzzy set A in $\mathfrak{U}$ such that $A \in \mathrm{cls}(\mathcal{A})$. Otherwise, we say that it is proper.

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Generalization of “to be identical up to negligibility”

Remark

Fuzzy class equivalence and fuzzy class class partial ordering is defined in the same way as for fuzzy sets.

Definition

Let $A, B \in \mathfrak{F}(\mathfrak{U})$. We say that fuzzy sets A and B are approximately identical up to negligibility in the degree α (symbolically, $[A \eqsim B] = \alpha$) if

$$\alpha = \bigwedge_{a \in \mathscr{D}(A) \cup \mathscr{D}(B)} A'(a) \leftrightarrow B'(a),$$

where $A' \in \mathrm{cls}(A)$, $B' \in \mathrm{cls}(B)$ such that $\mathscr{D}(A') = \mathscr{D}(B') = \mathscr{D}(A) \cup \mathscr{D}(B)$.

Properties of fuzzy class relation $\eqsim$

Theorem

Let $A, B, C, D \in \mathfrak{F}(\mathfrak{U})$*. Then,*

(i) $[A \eqsim B] = \top$ *if and only if* $A \equiv B$*;*

(ii) *if* $A \equiv C$ *and* $B \equiv D$*, then*

$$[A \eqsim B] = [C \eqsim D];$$

(iii) *the fuzzy class relation* $\eqsim$ *is a fuzzy equivalence.*

Generalization of "to be a fuzzy subset"

Definition

Let $A, B \in \mathfrak{F}(\mathfrak{U})$. We say that A is approximately a fuzzy subset of B in the degree α (symbolically, $[A \subsetneqq\!\!\sim B] = \alpha$) if

$$\alpha = \bigwedge_{a \in \mathscr{D}(A) \cup \mathscr{D}(B)} (A'(a) \to B'(a)),$$

where $A' \in \mathrm{cls}(A)$, $B' \in \mathrm{cls}(B)$ such that $\mathscr{D}(A') = \mathscr{D}(B') = \mathscr{D}(A) \cup \mathscr{D}(B)$.

Properties of fuzzy class relation $\underset{\sim}{\subseteq}$

Theorem

Let $A, B \in \mathfrak{F}(\mathfrak{U})$. *Then,*

(i) $[A \underset{\sim}{\subseteq} B] = [A' \underset{\sim}{\subseteq} B']$ *for any* $A' \in \mathrm{cls}(A)$ *and* $B' \in \mathrm{cls}(B)$;

(ii) $[A \underset{\sim}{\subseteq} B] \wedge [B \underset{\sim}{\subseteq} A] = [B \underset{\sim}{\equiv} A]$;

(iii) *the fuzzy class relation* $\underset{\sim}{\subseteq}$ *is a* $\underset{\sim}{\equiv}$*–fuzzy partial ordering;*

(iv) *the fuzzy power set* $\mathbf{P}(A)$ *is a fuzzy class in* $\mathfrak{U}$, *which is defined by*

$$\mathbf{P}(A)(x) := [\chi_x \underset{\sim}{\subseteq} A].$$

for any $x \in \mathfrak{U}$.

Properties of fuzzy class relation $\subsetapprox$

Theorem

Let $A, B \in \mathfrak{F}(\mathfrak{U})$. *Then,*

(i) $[A \subsetapprox B] = [A' \subsetapprox B']$ *for any* $A' \in \mathrm{cls}(A)$ *and* $B' \in \mathrm{cls}(B)$;

(ii) $[A \subsetapprox B] \wedge [B \subsetapprox A] = [B \approxeq A]$;

(iii) *the fuzzy class relation* $\subsetapprox$ *is a* $\approxeq$*–fuzzy partial ordering;*

(iv) *the fuzzy power set* $\mathbf{P}(A)$ *is a fuzzy class in* $\mathfrak{U}$, *which is defined by*

$$\mathbf{P}(A)(x) := [\chi_x \subsetapprox A].$$

for any $x \in \mathfrak{U}$.

Function between fuzzy sets in a certain degree

Definition

Let $A, B \in \mathfrak{F}(\mathfrak{U})$, and let $f \in \mathfrak{Func}$. We say that f is approximately a function of A to B in the degree α (symbolically, $[f : A \longrightarrow B] = \alpha$) if

$$\alpha = [f \in \mathfrak{Func}(\mathscr{D}(A), \mathscr{D}(B))] \otimes \bigwedge_{a \in \mathscr{D}(A)} (A(a) \to B(f(a))).$$

1-1 correspondence between fuzzy sets in a certain degree

Definition

Let $A, B \in \mathfrak{F}(\mathfrak{U})$, and let $f \in \mathfrak{Func}$. We say that f is approximately a 1-1 correspondence between A to B in the degree α (symbolically, $[f : A \xrightarrow[\text{corr}]{\text{1-1}} B] = \alpha$) if

$$\alpha = [f \in \mathfrak{Func}^{\text{1-1}}_{\text{corr}}(\mathscr{D}(A), \mathscr{D}(B))] \otimes \bigwedge_{a \in \mathscr{D}(A)} (A(a) \leftrightarrow B(f(a))).$$

Properties of 1-1 correspondence between fuzzy sets in a certain degree

Definition

Let $A, B, C \in \mathfrak{F}(\mathfrak{U})$ and $f, g \in \mathfrak{Func}$.

(i) $[\emptyset : \emptyset \xrightarrow[\text{corr}]{\text{1-1}} \emptyset] = \top$.

(ii) $[1_{\mathscr{D}(A)} : A \xrightarrow[\text{corr}]{\text{1-1}} A] = \top$.

(iii) If $g \circ f = 1_{\mathscr{D}(A)}$ and $f \circ g = 1_{\mathscr{D}(B)}$, then

$$[f : A \xrightarrow[\text{corr}]{\text{1-1}} B] = [g : B \xrightarrow[\text{corr}]{\text{1-1}} A].$$

(iv) If $g \circ f \in \mathfrak{Func}$, then

$$[f : A \xrightarrow[\text{corr}]{\text{1-1}} B] \otimes [g : B \xrightarrow[\text{corr}]{\text{1-1}} C] \leq [g \circ f : A \xrightarrow[\text{corr}]{\text{1-1}} B].$$

Bandler-Kohout (BK) fuzzy power set

Definition

Let $A \in \mathcal{F}(\mathfrak{U})$. Then a fuzzy class $\mathbf{F}(A) : \mathcal{F}(\mathfrak{U}) \to L$ defined by

$$\mathbf{F}(A)(B) = \begin{cases} [B \subsetapprox A], & \text{if } \mathscr{D}(B) = \mathscr{D}(A), \\ \bot, & \text{otherwise,} \end{cases}$$

is called a Bandler-Kohout (BK) fuzzy power class of A in $\mathfrak{U}$.

Lemma

- *A fuzzy class $\mathbf{F}(A)$ is a fuzzy set for any $A \in \mathfrak{F}(\mathfrak{U})$ if and only if $L \in \mathfrak{U}$.*
- *If $A \neq \emptyset$, then there is $B \in \operatorname{cls}(A)$ such that $\mathbf{F}(B) \neq \mathbf{F}(A)$.*

Definition

We say that $\mathbf{F}(A)$ is a Bandler-Kohout (BK) fuzzy power set of A in $\mathfrak{U}$ if $\mathbf{F}(A)$ is a fuzzy set in $\mathfrak{U}$.

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Definition

We say that $\mathbf{F}(A)$ is a Bandler-Kohout (BK) fuzzy power set of A in $\mathfrak{U}$ if $\mathbf{F}(A)$ is a fuzzy set in $\mathfrak{U}$.

$\mathbf{F}$ operator as a natural extension of $\mathbf{P}$ operator

Theorem

Let $A, B \in \mathfrak{F}(\mathfrak{U})$, and let $f : A \longrightarrow B$ be a function. Let $\mathbf{F}(A)$ and $\mathbf{F}(B)$ be BK fuzzy power sets. Then, the following diagram commutes

$$\begin{array}{ccc} \mathbf{P}(A) & \xrightarrow{\mathbf{P}(f)} & \mathbf{P}(B) \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} \\ \mathbf{F}(A) & \xrightarrow[\mathbf{F}(f)]{} & \mathbf{F}(B), \end{array}$$

where $\mathbf{P}(f)$ and $\mathbf{F}(f)$ are the image function for sets and fuzzy sets, resp., and i_1 and i_2 are the inclusion mappings.

Outline

Satisfactorily large fuzzy sets

Lemma

Let $A, B \in \mathfrak{F}(\mathfrak{U})$ *such that*

(i) $|\mathscr{D}(A)| = |\mathscr{D}(B)|$,

(ii) $|\mathrm{Supp}(A)| \leq |\mathscr{D}(B) \setminus \mathrm{Supp}(B)|$ *and* $|\mathrm{Supp}(B)| \leq |\mathscr{D}(A) \setminus \mathrm{Supp}(A)|$,

and let

$$\alpha := \bigvee_{f \in \mathfrak{Func}(\mathscr{D}(A), \mathscr{D}(B))} [f : A \xrightarrow[corr]{1\text{-}1} B].$$

Then, $[f : C \xrightarrow[corr]{1\text{-}1} D] \leq \alpha$ *for any* $C \in \mathrm{cls}(A)$, $D \in \mathrm{cls}(B)$, *and* $f \in \mathfrak{Func}$.

Definition

We say that fuzzy sets A and B form a pair of satisfactorily large fuzzy sets if they satisfies (i) and (ii) of the previous lemma.

Satisfactorily large fuzzy sets

Lemma

Let $A, B \in \mathfrak{F}(\mathfrak{U})$ *such that*

(i) $|\mathscr{D}(A)| = |\mathscr{D}(B)|$,

(ii) $|\mathrm{Supp}(A)| \leq |\mathscr{D}(B) \setminus \mathrm{Supp}(B)|$ *and* $|\mathrm{Supp}(B)| \leq |\mathscr{D}(A) \setminus \mathrm{Supp}(A)|$,

and let

$$\alpha := \bigvee_{f \in \mathfrak{Func}(\mathscr{D}(A), \mathscr{D}(B))} [f : A \xrightarrow[corr]{1\text{-}1} B].$$

Then, $[f : C \xrightarrow[corr]{1\text{-}1} D] \leq \alpha$ *for any* $C \in \mathrm{cls}(A)$, $D \in \mathrm{cls}(B)$, *and* $f \in \mathfrak{Func}$.

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Then, $[f : C \xrightarrow[corr]{1\text{-}1} D] \le \alpha$ *for any* $C \in \mathrm{cls}(A)$, $D \in \mathrm{cls}(B)$, *and* $f \in \mathfrak{Func}$.

Definition

We say that fuzzy sets A and B form a pair of satisfactorily large fuzzy sets if they satisfies (i) and (ii) of the previous lemma.

Graded Cantor's equipollence

Definition

Let $A, B \in \mathfrak{F}(\mathfrak{U})$. We say that A is approximately equipollent with B in the degree α (symbolically, $[A \overset{c}{\approx} B] = \alpha$) provided that there exists $A' \in \mathrm{cls}(A)$ and $B' \in \mathrm{cls}(B)$ such that

$$\alpha = \bigvee_{f \in \mathfrak{Func}(\mathscr{D}(A'), \mathscr{D}(B'))} [f : A' \xrightarrow[\text{corr}]{\text{1-1}} B']$$

and $[g : C \xrightarrow[\text{corr}]{\text{1-1}} D] \leq \alpha$ for any $C \in \mathrm{cls}(A)$, $D \in \mathrm{cls}(B)$, and $g \in \mathfrak{Func}$.

Theorem

Let $A, B, C, D \in \mathfrak{F}(\mathfrak{U})$ such that $A \equiv C$ and $B \equiv D$. Then $[A \overset{c}{\approx} B] = [C \overset{c}{\approx} D]$.

Graded Cantor's equipollence

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Let $A, B \in \mathfrak{F}(\mathfrak{U})$. We say that A is approximately equipollent with B in the degree α (symbolically, $[A \overset{c}{\approx} B] = \alpha$) provided that there exists $A' \in \mathrm{cls}(A)$ and $B' \in \mathrm{cls}(B)$ such that

$$\alpha = \bigvee_{f \in \mathfrak{Func}(\mathscr{D}(A'), \mathscr{D}(B'))} [f : A' \xrightarrow[\text{corr}]{\text{1-1}} B']$$

and $[g : C \xrightarrow[\text{corr}]{\text{1-1}} D] \leq \alpha$ for any $C \in \mathrm{cls}(A)$, $D \in \mathrm{cls}(B)$, and $g \in \mathfrak{Func}$. The fuzzy class relation $\overset{c}{\approx}$ is called the graded equipollence.

Theorem

Let $A, B, C, D \in \mathfrak{F}(\mathfrak{U})$ such that $A \equiv C$ and $B \equiv D$. Then $[A \overset{c}{\approx} B] = [C \overset{c}{\approx} D]$.

Theorem

Let $A, B, C, D \in \mathfrak{F}(\mathfrak{U})$ *such that* $A \equiv C$ *and* $B \equiv D$*. Then* $[A \overset{c}{\approx} B] = [C \overset{c}{\approx} D]$*.*

Graded equipollence of two finite fuzzy sets

Example

Let $A, B \in \mathfrak{F}(\mathfrak{U})$ be fuzzy sets given by

$$A(x) = \begin{cases} 0.9, & \text{if } x = a, \\ 0.5, & \text{if } x = b, \\ 0, & \text{if } x = c, \end{cases} \quad \text{and} \quad B(x) = \begin{cases} 0.5, & \text{if } x = 1, \\ 1, & \text{if } x = 2, \\ 0.2, & \text{if } x = 3, \\ 0, & \text{if } x = 4. \end{cases}$$

Let us put $z = \{a, b, c, 1, 2, 3, 4\}$ and consider $C \equiv A$ and $D \equiv B$ such that $\mathscr{D}(C) = \mathscr{D}(D) = z$. It is easy to see that C and D form a pair of satisfactorily large fuzzy sets. By Lemma, we find that

$$[A \overset{c}{\approx} B] = \bigvee_{f \in \mathfrak{Func}(z,z)} [f : C \xrightarrow[\text{corr}]{\text{1-1}} D] = [f_0 : C \xrightarrow[\text{corr}]{\text{1-1}} D] = 0.8,$$

where $f_0 : z \longrightarrow z$ is defined by $f_0(a) = 2, f_0(b) = 1, f_0(c) = 3, f_0(1) = 4,$ $f_0(2) = a, f_0(3) = b$, and $f_0(4) = c$.

Fuzzy class equivalence on $\mathcal{F}(\mathfrak{U})$

Theorem

The fuzzy class relation $\overset{c}{\approx}$ is a fuzzy class equivalence on $\mathfrak{F}(\mathfrak{U})$, i.e.,

(i) $[A \overset{c}{\approx} A] = \top$,

(ii) $[A \overset{c}{\approx} B] = [B \overset{c}{\approx} A]$,

(iii) $[A \overset{c}{\approx} B] \otimes [B \overset{c}{\approx} C] \leq [A \overset{c}{\approx} C]$

hold for arbitrary fuzzy sets $A, B, C \in \mathfrak{F}(\mathfrak{U})$.

Relations for perations with fuzzy sets

Let a, b, c, d be sets such that $a \sim c$ and $b \sim d$. Then, it is well-known that

$$a \cup b \sim c \cup d, \quad \text{whenever } a \cap b = \emptyset \text{ and } c \cap d = \emptyset,$$
$$a \times b \sim c \times d,$$
$$a \sqcup b \sim c \sqcup d.$$

Theorem

Let $A, B, C, D \in \mathfrak{F}(\mathfrak{U})$. *Then, it holds*

(i) $[A \overset{c}{\approx} C] \otimes [B \overset{c}{\approx} D] \leq [A \otimes B \overset{c}{\approx} C \otimes D]$,

(ii) $[A \overset{c}{\approx} C] \otimes [B \overset{c}{\approx} D] \leq [A \times B \overset{c}{\approx} D \times D]$,

(iii) *if* $\mathrm{Supp}(A \cap B) = \mathrm{Supp}(C \cap D) = \emptyset$, *then*

$$[A \overset{c}{\approx} C] \otimes [B \overset{c}{\approx} D] \leq [A \cup B \overset{c}{\approx} C \cup D],$$

(iv) $[A \overset{c}{\approx} C] \otimes [B \overset{c}{\approx} D] \leq [A \sqcup B \overset{c}{\approx} C \sqcup D]$.

Relations for fuzzy power sets

It is known in set theory that

$$a \overset{c}{\approx} b \text{ implies } \mathrm{P}(a) \overset{c}{\approx} \mathrm{P}(b)$$
$$a \not\approx \mathrm{P}(a)$$

Theorem

Let $A, B \in \mathfrak{F}(\mathfrak{U})$. *Then,*

(i) $[A \overset{c}{\approx} B] \leq [\mathbf{P}(A) \overset{c}{\approx} \mathbf{P}(B)]$

(ii) $[A \overset{c}{\approx} \mathbf{P}(A)] < \top$.

Is $[A \overset{c}{\approx} \mathbf{P}(A)] = \bot$ true?

Example

Let $A = \{1/a, 0.4/b\}$. Then,

$$\mathbf{P}(A) = \{1/\emptyset, 1/\{a\}, 0.4/\{b\}, 0.4/\{a,b\}\}.$$

Consider $C = \{1/a, 0.4/b, 0/c, 0/d\}$. Then

$$[A \overset{c}{\approx} \mathbf{P}(A)] = [C \approx \mathbf{P}(A)] = (1 \leftrightarrow 1) \wedge (1 \leftrightarrow 0.4) \wedge (0.4 \leftrightarrow 0) \wedge (0.4 \leftrightarrow 0) = 1 \wedge 0.4 \wedge 0.6 \wedge 0.6 = 0.4.$$

Hence, we obtain $0 < [A \overset{c}{\approx} \mathbf{P}(A)] < 1$.

Relations for Bandler-Kohout fuzzy power sets

Theorem

Let $A, B \in \mathfrak{F}(\mathfrak{U})$*, and let us assume that* $\mathbf{F}(A), \mathbf{F}(B) \in \mathfrak{F}(\mathfrak{U})$*.*

1. *If* A, B *form satisfactorily large pair of fuzzy sets, then*

$$[A \overset{c}{\approx} B] \leq [\mathbf{F}(A) \overset{c}{\approx} \mathbf{F}(B)],$$

2. $[A \overset{c}{\approx} \mathbf{F}(A)] = \bot$*.*

Relation for exponentiation of fuzzy sets

Theorem

Let $A, B, C, D \in \mathfrak{F}(\mathfrak{U})$ *such that* $|\mathscr{D}(A)| = |\mathscr{D}(C)|$ *and* $|\mathscr{D}(B)| = |\mathscr{D}(D)|$*. Then,*

$$[A \overset{c}{\approx} C] \otimes [B \overset{c}{\approx} D] \leq [B^A \overset{c}{\approx} C^D].$$

Theorem

Let $A, B, C \in \mathcal{F}(\mathfrak{U})$*. Then,*

$$[C^{A \otimes B} \overset{c}{\approx} (C^B)^A] = \top.$$

Remark

Note that an analogous relation to $\mathrm{P}(a) \overset{c}{\approx} \mathbf{2}^a$ cannot be proved for fuzzy sets.

Relation for exponentiation of fuzzy sets

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Note that an analogous relation to $\mathrm{P}(a) \overset{c}{\approx} \mathbf{2}^a$ cannot be proved for fuzzy sets.

Cantor-Bernstein theorem for fuzzy sets

One of tjhe forms of Cantor-Bernstein theorem states that if a, b, c, d are sets such that $b \subseteq a$ and $d \subseteq c$ and $a \sim d$ and $b \sim c$, then $a \sim c$.

Theorem

Let $A, B, C, D \in \mathfrak{F}(\mathfrak{U})$ be fuzzy sets with finite supports such that $B \subseteq A$ and $D \subseteq C$. Then, we have

$$[A \overset{c}{\approx} D] \wedge [C \overset{c}{\approx} B] \leq [A \overset{c}{\approx} C].$$

Corollary (Cantor-Bernstein theorem)

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Graded Cantor-Bernstein's equipollence

Remark

It can be demonstrated that Cantor-Bernstein theorem is not true for infinite fuzzy sets, and we need a stronger concept of graded equipollence – graded Cantor-Bernstein's equipollence, which is defined using a graded Cantor's dominance

$$[A \overset{c}{\precsim} B] := \bigvee_{C \subseteq B} [A \overset{c}{\approx} C].$$

Thus, the graded Cantor-Bernstein's equipollence is given by

$$[A \overset{cb}{\approx} B] := [A \overset{c}{\precsim} B] \wedge [B \overset{c}{\precsim} A].$$

Outline

A future work

- To build theory of fuzzy sets and fuzzy classes in the universe of sets.
- To define finiteness and infiniteness of fuzzy sets.
- To introduce ordinal and cardinal numbers (it is not so easy, if we want to follow the standard approach).
- To investigate relations between functional approach (based on graded Cantor's and Cantor-Bernstein's equipollences) to cardinality of fuzzy sets and the cardinality describe by cardinal or ordinal numbers (some results are done for fuzzy sets with finite universes).

$\vdots$

Thank you for your attention.